Phys 729 – General Relativity and Gravitation – Fall 2015
Homework Assignment #5, due Wednesday, October 28

In your solutions to the following problems, please include full sentences explaining all of the steps in your reasoning. A solution that includes only equations or very few words will not be considered complete.

Problem #1 (Phys 503)

Show that if K^μ is a Killing vector field on a manifold with metric $(M, g_{\mu\nu})$ and a particle moves along a geodesic $x^\mu(\tau)$ with 4-velocity $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$, then the component of the particle's 4-momentum mu^μ along K^μ is constant along the worldline.

Problem #1 (Phys 729)

Let M be a manifold with Lorentzian metric $g_{\mu\nu}$ and associated derivative operator ∇_μ. A function on M which is a solution of the equation $\nabla_\mu \nabla^\mu \alpha = 0$ is called a *harmonic function*. In the case where M is a 2-dimensional manifold, let α be harmonic and let $\epsilon_{\mu\nu}$ be an antisymmetric tensor satisfying $\epsilon_{\mu\nu}\,\epsilon^{\mu\nu} = -2$. Consider the equation $\nabla_\mu \beta = \epsilon_{\mu\nu} \nabla^\nu \alpha$ for a function β.
(a) Show that the integrability conditions for this equation are satisfied, and thus, locally, there exists a solution, β. Show that β is also harmonic, $\nabla_\mu \nabla^\mu \beta = 0$. ($\beta$ is called the harmonic function *conjugate* to α.)
(b) By choosing α and β as coordinates on M, show that the metric can be put in the form

$$\mathrm{d}s^2 = \pm\Omega^2(\alpha, \beta)\,[-\mathrm{d}\alpha^2 + \mathrm{d}\beta^2]\,.$$

Reminder: A pair of coupled partial differential equations of the form $\partial f/\partial x^\mu = F_\mu$ for $\mu = 1,\, 2$, have a solution f iff the integrability conditions $\partial F_\mu/\partial x^\nu = \partial F_\nu/\partial x^\mu$ are satisfied.

Problem #2

Show that in two dimensions the Riemann tensor of any metric takes the form $R_{abcd} = R\, g_{a[c}\, g_{d]b}$.

(Hint: Using the fact that in n dimensions the Riemann tensor has $\frac{1}{12}\, n^2\,(n^2 - 1)$ independent components, show that $R\, g_{a[c} g_{d]b}$ has the right number of independent components and all of the symmetries of the Riemann tensor.)

Problem #3

Carroll, Chapter 5, Exercise 1.