Critical Phenomena and Universality

Introduction

• *Setting:* We want to develop a theory of systems which exhibit phase transitions, in which a system variable Ψ, the *order parameter* (e.g., magnetization), changes qualitatively as a *control parameter* T, a quantity that we will think of as an independent variable (e.g., temperature), varies past a *critical value* $T_{\rm c}$. When the system is in equilibrium at $T > T_{\rm c}$ it is in a disordered state $\Psi = 0$, while for $T < T_{\rm c}$ it can be in more than one ordered states, two in the case of a single real variable $\Psi = \pm\Psi_0$, an example of symmetry breaking.

• *Plan:* We will describe the Landau theory, which uses a mean-field approximation. Starting from a general form for the free energy of the system, written as an analytic function of T and Ψ, it allows us to find the behavior of quantities including Ψ and the susceptibility $\chi := \partial\Psi/\partial B$ to a change in an external *ordering field* B (e.g., a magnetic field) near the critical value $T = T_{\rm c}$, characterized by *critical exponents*,

$$C_V \sim \frac{1}{(T-T_{\rm c})^\alpha} \quad \text{or} \quad \frac{1}{(T_{\rm c}-T)^{\alpha'}}\;, \quad \Psi_0 \sim (T_{\rm c}-T)^\beta\;, \quad \chi_0 \sim \frac{\partial\Psi}{\partial B} \sim \frac{1}{(T-T_{\rm c})^\gamma} \quad \text{or} \quad \frac{1}{(T_{\rm c}-T)^{\gamma'}}$$

without having to calculate the partition function exactly. What one finds is that each type of phase transition has its characteristic set of critical exponents, independent of the actual system being considered, a phenomenon called *universality*.

Landau theory

• *Idea:* A mean field theory, originally developed by Lev Landau in the 1940s to describe superconductivity, which has become one of the basic theoretical tools for describing phase transitions. The starting point for the theory is the observation that a system with a phase transition in which temperature is the control parameter must have a Helmholtz free energy which is not an analytic function of temperature.

• *Singularities of F:* In statistical mechanics, the Helmholtz free energy for a system is of the form

$$F = -k_{\rm B}T \ln Z = -k_{\rm B}T \ln \textstyle\sum_s {\rm e}^{-\beta H(s)}\;,$$

where s labels the microstates of the system, H is its Hamiltonian and each term in the sum is analytic, so the singularities in F arise only in the thermodynamic limit in which the system has an infinite number of degrees of freedom. These singularities can arise in $\partial F/\partial T$ (first-order transitions), in $\partial^2F/\partial T^2$ (second-order or continuous transitions), etc.

• *Formalism:* Assume that T is the control parameter, call the order parameter field (e.g., magnetization in a ferromagnet) Ψ, defined so that its value is zero in the disordered phase. Write the free energy as

$$F(T,\Psi) = F_0(T) + F_{\rm L}(T,\Psi)\;, \quad \text{or} \quad f(T,\Psi) = f_0(T) + f_{\rm L}(T,\Psi)\;,$$

where F_0 is analytic, the Landau functional $F_{\rm L}$ contains the singularities in T but depends analytically on Ψ and satisfies all symmetries of the Hamiltonian associated with Ψ (such as translational and rotational invariance), and $f := F/N$. At a given temperature T, the value of Ψ minimizes the free energy.

• *Form of the free energy:* Because of the $\Psi \mapsto -\Psi$ symmetry, in the absence of an external ordering field the free energy can be written as a sum of even powers,

$$F(T,\Psi) = F_0(T) + a(T)\Psi^2 + b(T)\Psi^4 + O(\Psi^6)\;.$$

Minimizing F with respect to Ψ gives

$$0 = \frac{\partial F}{\partial\Psi} = 2\,a(T)\Psi + 4\,b(T)\Psi^3\;, \quad \text{i.e.,} \quad M = 0 \quad \text{or} \quad \Psi = \pm\sqrt{-a/2b(T)}\;.$$

The phase transition occurs where $a(T)$ changes sign, and we can parametrize $a(T) = a_0\,(T-T_{\rm c})/T_{\rm c}$, so that in the low-temperature, ordered phase

$$\Psi = \pm\left(\frac{a_0}{2b(T)}\,\frac{|T-T_{\rm c}|}{T_{\rm c}}\right)^{1/2}.$$

To find the behavior of the magnetic susceptibility near the critical temperature we add a small external field B, which breaks the symmetry and adds a term to F,

$$F(T, M) = F_0(T) - B\Psi + a(T)\Psi^2 + b(T)\Psi^4 + O(\Psi^6)\ .$$

Minimizing the free energy gives $0 = -B + 2\,a(T)\Psi + 4\,b\Psi^3$, which implies that at T_c the magnetization satisfies $4\,b\Psi^3 = B$, and taking a derivative with respect to B gives $0 = -1 + 2\,a(T)\chi(T) + 12\,b(T)\Psi^2\chi(T)$, which implies that

$$\chi = \begin{cases} T_\mathrm{c}/[2\,a(T)\,(T-T_\mathrm{c})] & \text{for } T > T_\mathrm{c} \\ T_\mathrm{c}/[4\,a(T)\,|T-T_\mathrm{c}|] & \text{for } T < T_\mathrm{c}\ . \end{cases}$$

Universality

• *Idea:* The critical exponents at a continuous phase transition depend only on the symmetry of the order parameter, the dimension of space and the range of interactions. Systems are divided into universality classes.

• *Example:* Two physically dissimilar transitions that lie in the same universality class are the ferromagnetic transition in the Ising model and the liquid-gas transition in a fluid.

• *Comment:* Because the Landau theory is a mean-field theory its quantitative predictions cannot always be trusted.

Landau-Ginzburg Theory

• *Idea:* A generalization, in which local fluctuations of the order parameter are taken into account by considering an order field ψ, in terms of which the free energy can be written as a polynomial expansion of the type

$$F_\mathrm{L}(T, \Psi) = \int_V \mathrm{d}^3r\,\left[\tfrac{1}{2}\,a_0(T-T_*)\,\Psi^2 + ...\right].$$

Reading

• *Pathria & Beale:* Chapter 12, in particular § 12.9.

• *Other textbooks:* Gould & Tobochnik, Chapter 9.

• *Online resources:*
http://www2.ph.ed.ac.uk/~mevans/sp/sp12.pdf.