The 2-Dimensional Ising Model

Overview of Analytical Results

• *Reminder:* The standard 2D Ising model consists of a lattice with N lattice sites, at each site i of which is a spin s_i. The partition function, in the absence of an external magnetic field, is

$$Z(N,T) = \sum\nolimits_{\{s\}} \mathrm{e}^{\beta J \Sigma'_{\langle ij\rangle} s_i s_j} \;.$$

A treatment using the Weiß mean-field theory approximation, with the above definition of J and $s_i = \pm 1$, gives a second-order phase transition in all dimensions D, at a critical temperature or value of $K := \beta J$

$$T_\mathrm{c} = \frac{2DJ}{k_\mathrm{B}} \;, \qquad \text{or} \qquad K^* = \frac{1}{2D} \;.$$

• *Exact treatment:* An exact expression for the partition function was found by Lars Onsager in 1944. The proof that the model is solvable is not a simple one, and is considered a major achievement. Using a somewhat simplified approximate treatment, one finds that

$$Z(N,T) = [2\cosh(\beta J)\,\mathrm{e}^{I(\beta J)}]^N \;, \quad \text{with}$$
$$I(K) = \frac{1}{2\pi}\int_0^\pi \mathrm{d}\phi \ln\left\{\tfrac{1}{2}\left[1 + (1 - \xi^2(K)\sin^2\phi)^{1/2}\right]\right\} \;, \quad \xi = 2\sinh(2\,\beta J)/\cosh^2(2\,\beta J) \;,$$

and the system exhibits spontaneous magnetization below a critical temperature

$$T_\mathrm{c} = 2.269\, J/k_\mathrm{B} \;, \quad \text{or} \quad K^* = 0.44069 \;, \quad \text{at which} \quad \sinh(2\,\beta_\mathrm{c} J) = 1 \;.$$

For temperatures just below the critical point, consistently with a second-order phase transition,

$$C = \frac{\partial \bar E}{\partial T}\Big|_{B=0} \sim -\frac{8\,k_\mathrm{B} N}{\pi}\,(\beta J)^2 \ln|T - T_\mathrm{c}| \;, \qquad \bar M \sim \text{const} \times N\,|T - T_\mathrm{c}|^{1/8} \;.$$

Renormalization Group: Block-Spin Approach

• *Setup:* In this approach, for simplicity it is convenient to start with N spins $s_i = \pm 1$ on a triangular lattice with periodic boundary conditions. After grouping blocks of 6 triangles to form hexagons that tile the plane, pick one triangle in each block I and introduce a coarse graining procedure by which its 3 spins s_1^I, s_2^I and s_3^I are replaced by block variables $\mu_I := \mathrm{sgn}(s_1^I + s_2^I + s_3^I)$ (the majority sign) and $\sigma_I := |s_1^I + s_2^I + s_3^I|$. Then the Hamiltonian can be written as

$$H(K, \{\mu_I, \sigma_I\}) = H_0(K, \{\mu_I, \sigma_I\}) + V(K, \{\mu_I, \sigma_I\}) \;,$$

where H_0 is the term coming from interactions within each block and V represents the interactions between blocks, and a corresponding partition function can be defined for the coarse-grained lattice of spin blocks (for more details and results, see the treatment in the books by Reichl and Gould & Tobochnik).

Renormalization Group: Decimation Approach

• *Setup:* Again start with N spins $s_i = \pm 1$, this time on an $n \times n$ square lattice with periodic boundary conditions, and call $K := \beta J$. Then the partition function is

$$Z_{(1)} = \sum\nolimits_{\{s\}} \mathrm{e}^{K \Sigma'_{\langle ij\rangle} s_i s_j} \;.$$

Again define the decimation process as summing over the values of half of the spins, this time chosen to form a checkerboard pattern.

• *Kadanoff transformation:* The partially summed partition function cannot be seen as a Z of the same form as the original one. What we can do instead is consider $Z_{(1)}$ as a special case ($K_2 = K_3 = 0$) of a partition function that includes two more coupling constants,

$$Z_{(3)}(K_1, K_2, K_3, N) = f(K)^N \sum_{\{s\}} \mathrm{e}^{K_1 \Sigma'_{\langle ij\rangle} s_i s_j + K_2 \Sigma''_{\langle lm\rangle} s_l s_m + K_3 \Sigma'''_{\langle pqrt\rangle} s_p s_q s_r s_t} ,$$

where a double prime denotes a sum over next-nearest neighbors, and a triple prime a sum over squares made of nearest-neighbor pairs. Then, for example, the expression one gets by summing over s_5,

$$\sum_{s_5} \mathrm{e}^{K\, s_5(s_1+s_2+s_3+s_4)} = \mathrm{e}^{K\,(s_1+s_2+s_3+s_4)} + \mathrm{e}^{-K\,(s_1+s_2+s_3+s_4)} ,$$

can be identified with a term in a partition function of the form $Z_{(3)}$,

$$\begin{aligned} &\mathrm{e}^{K\,(s_1+s_2+s_3+s_4)} + \mathrm{e}^{-K\,(s_1+s_2+s_3+s_4)} \\ &= f(K)\,\mathrm{e}^{(K_1/2)\,(s_1s_2+s_2s_3+s_3s_4+s_4s_1) + K_2(s_1s_3+s_2s_4) + K_3\, s_1s_2s_3s_4} \end{aligned}$$

if we define the following mapping:

$$\begin{aligned} &K_1 = \tfrac14 \ln\cosh(4K)\ , \quad K_2 = \tfrac18 \ln\cosh(4K)\ , \quad K_3 = \tfrac18 \ln\cosh(4K) - \tfrac12 \ln\cosh(2K) \\ &f(K) = 2\,[\cosh(2K)]^{1/2}\,[\cosh(4K)]^{1/8}\ . \end{aligned}$$

The partially summed partition function at a larger scale now has next-nearest-neighbor terms and terms with four spins around a square; because of the high number of neighbors of each site, the resulting interactions are more complicated. They are also not of a form for which an exact renormalization group calculation can be performed.

• *Result:* We introduce an approximation in which we neglect K_3 and replace K_2 by a modified, effective nearest-neighbor coupling constant $K'(K_1, K_2)$ that takes into account next-nearest neighbors; one gets

$$K' = \tfrac38 \ln\cosh(4K)\ ,$$

which has a non-trivial, unstable fixed point K^* satisfying $K = \frac38 \ln\cosh(4K)$, i.e., $K^* = 0.50698$, in addition to two stable ones at $K = 0$ and ∞.

• *Next:* Perform a numerical simulation of the system.

Reading

• *Pathria & Beale:* Sec 13.4 (exact results), and Secs 14.2, 14.4 (renormalization group).
• *Chandler:* Ch 5, pp 119 ff.
• *Gould & Tobochnik:* Sec 9.6 (triangular lattice, block spins)
• *Halley:* pp 150–160.
• *Mattis & Swendsen:* Secs 2.7–2.8, 8.5–8.10.
• *Plischke & Bergersen:* Chapter 6, and Chapter 9 for simulations.
• *Reichl:* Sec 5.8 (triangular lattice, block spins).
• *Reif:* Mentioned in p 429.
• *Schwabl:* Secs 6.5 (analytical) and 7.3 (renormalization group, square lattice).