The Ising Model

• *General idea:* The Ising model is a lattice system proposed to model the behavior of a ferromagnet. It goes beyond the mean-field approximation and assigns separate degrees of freedom to different spins, but makes other drastic approximations which imply that quantitatively its results do not really reproduce the behavior of a ferromagnet. However, its features are very instructive, and there are several other fields (including, in physics, the liquid-gas transition) in which it has proved to be very useful.

• *The model:* We can simplify the treatment of a lattice of N spins by replacing $H = -\sum_{i,j\neq i} J^{ij}\, \vec{s}_i \cdot \vec{s}_j$ with $-J \sum'_{i,j} \vec{s}_i \cdot \vec{s}_j$, where the prime indicates a sum over nearest neighbors and $J = J^{ij}$ is related to the $\tilde{J} = \sum_{j\neq i} J^{ij}$ we introduced in the treatment of mean-field ferromagnetism by $\tilde{J} = \nu J$; this defines the Heisenberg model. If we further simplify the model and replace $\vec{s}_i$ with $s_i = \pm 1$, we obtain the Ising model Hamiltonian

$$H = -J \sum\nolimits'_{i,j} s_i s_j \;.$$

The lattice is often a square one, with $\nu = 2D$ nearest neighbors per site in D spatial dimensions; the number of nearest neighbors each lattice site has is a crucially important parameter.

Overview of Results

• *Mean-field approximation:* There is a phase transition at $T = T_\mathrm{c} = \tilde{J}\,\hbar^2/4k_\mathrm{B}$ (here, i is any fixed site) in any dimensionality, with $4\pi\chi = (T - T_\mathrm{c})^{-1}$ (critical exponent -1).

• *1D Ising model:* It is equivalent to an easily solvable, non-interacting model. There is no spontaneous magnetization or phase transition, because each lattice site has too few neighbors.

• *2D Ising model:* It is still solvable, but not nearly as simple. There is a phase transition with spontaneous magnetization below $T_\mathrm{c} = 2.269\, J/k_\mathrm{B}$, with $\bar{M} \propto (T - T_\mathrm{crit})^\gamma$, $\gamma = 1/8$ (different from the mean-field result).

• *3D Ising model:* It is most likely not solvable, but the known results are just based on numerical simulations. The latter show that there is a phase transition, with $\bar{M} \propto (T - T_\mathrm{crit})^\gamma$, $\gamma = 0.313$.

• *Infinite-dimensional limit:* Each spin is influenced by so many neighbors that fluctuations are not important, and the mean-field treatment gives good results in this case.

The 1D Ising Model

• *Partition function:* With open-ended boundary conditions, in which each s_i has two neighbors, except for s_1 and s_N which only have one neighbor, the partition function corresponding to the Hamiltonian above is

$$Z = \sum_{\text{states } \alpha} \mathrm{e}^{-\beta E_\alpha} = \sum_{\{s_i=\pm 1\}} \mathrm{e}^{\beta J \Sigma'_{\langle ij\rangle} s_i s_j} = \sum_{\{s_i=\pm 1\}} \mathrm{e}^{\beta J \Sigma_{i=1}^{N-1} s_i s_{i+1}} \;,$$

since in 1D all nearest-neighbor pairs are of the form (s_i, s_{i+1}). Therefore, switching to "transition" variables $t_i := s_i s_{i+1}$, with values $t_i = \pm 1$ and replacing $\{s_i \mid i = 1, ..., N\}$ by $\{s_1,\ t_i \mid i = 1, ..., N-1\}$, we get

$$Z = \sum_{\{s_1, t_i = \pm 1\}} \mathrm{e}^{\beta J \Sigma_{i=1}^{N-1} t_i} = 2 \prod_{i=1}^{N-1} \sum_{t_i=\pm 1} \mathrm{e}^{\beta J t_i} = 2\, (2\cosh \beta J)^{N-1} \;.$$

• *Helmholtz free energy:* From Z we easily find $F = -k_\mathrm{B} T \ln Z = -k_\mathrm{B} T \left[\ln 2 + (N-1)\ln(\mathrm{e}^{\beta J} + \mathrm{e}^{-\beta J})\right]$, which is an analytic function in the whole range $0 < T < \infty$, so there will be no phase transition, as we can also show calculating $\langle s\rangle$ with $B \neq 0$ and $\partial \bar{M}/\partial B$, or looking directly at the alignment of the spins.

• *Spin alignment:* Given any site i, the probability of a spin flip ($t_i = -1$) at that site is

$$q = \frac{\mathrm{e}^{-\beta J}}{\mathrm{e}^{\beta J} + \mathrm{e}^{-\beta J}} = \frac{\mathrm{e}^{-2\beta J}}{1 + \mathrm{e}^{-2\beta J}} \;.$$

Thus, on average we expect a spin flip every $1/q = 1 + \mathrm{e}^{2\beta J}$ sites, and the line splits into domains of mean length $1 + \mathrm{e}^{2\beta J}$, which is finite for all T; there is never a complete alignment or infinite correlation length.

• *Transfer matrix method:* A method by which one calculates the partition function $Z_N(B, T)$ of the Ising model with an external magnetic field B as $\mathrm{tr}\, P^N = \lambda_1^N + \lambda_2^N$, where P is the transfer matrix, with eigenvalues λ_i that can be explicitly calculated. For details, see the Pathria & Beale textbook.

Renormalization-Group Approach to a Lattice Model

• *Goal:* Develop a way to find out information about a lattice system such as the Ising model by applying to it a sequence of mappings to equivalent systems with fewer lattice sites, which are either simple enough that the partition function can be calculated for them, or such that useful information can be obtained simply from the transformation properties of Z, without having to actually calculate it.

• *Step 1:* Define a coarse-graining process or a decimation process for the system.

• *Step 2:* Find a (Kadanoff) transformation from a theory with N lattice variables s_i, coupling constants K_n, and Hamiltonian $H(K, N; \{s_i\})$, to one with fewer lattice sites and Hamiltonian $H(K', N'; \{s_I\}) = H(K, N; \{s_i\}) + C(K, N, r)$ of the same form except for the values of the parameters $N \mapsto N' = r\, N$ and $K_i \mapsto K'_i(K, N)$, and an additive extensive function C, which must therefore be proportional to N' and does not depend on the lattice variables. Then the relation between the partition functions is of the form

$$Z(K, N) = f(K)^{N'}\, Z(K', N')\ .$$

• *Step 3:* Iterate the transformation, and find out whether either f and Z eventually become easy to calculate, or the iterated map has different asymptotic behaviors depending on the initial values of K and N, with fixed points at certain values K^*. Because critical values of the system parameters correspond to situations in which the correlation length of the system becomes infinite (in the $N \to \infty$ limit, at least) and the state of the system is scale-invariant, critical values K^* occur at unstable fixed points of the mapping $K \mapsto K'$, and the rate of change of K' near critical points gives approximate values of the critical exponents.

• *Remark:* A good quantity to look at is the intensive function $g(K) := \frac{1}{N} \ln Z(K, N) = \frac{N'}{N} \big[\ln f(K) + g(K') \big]$.

Renormalization-Group Approach to the 1D Ising Model

• *Setup:* We Apply the method to the (solvable) 1D theory as a warmup for the 2D case. Start with N spins $s_i = \pm 1$ at the sites of a 1D lattice, this time with periodic boundary conditions, and define $K := \beta J$. Then

$$Z(K, N) = \sum\nolimits_{\{s_i\}} \mathrm{e}^{K \Sigma'_{\langle ij \rangle} s_i s_j} = \sum\nolimits_{\{s_i\}} \mathrm{e}^{K \Sigma_{i=1}^N s_i s_{i+1}}\ ,$$

where a prime denotes a summation over nearest neighbors. Define a “decimation” process of summing over all possible values of half of the spins, for example the even ones. Then

$$Z(K, N) = \sum\nolimits_{\text{odd } s_i} \dots \big(\mathrm{e}^{K(s_1+s_3)} + \mathrm{e}^{-K(s_1+s_3)}\big) \big(\mathrm{e}^{K(s_3+s_5)} + \mathrm{e}^{-K(s_3+s_5)}\big) \dots$$

• *Kadanoff transformation:* Look for a K' such that Z has the form of an Ising model partition function, by imposing that $\mathrm{e}^{K(s+s')} + \mathrm{e}^{-K(s+s')} = f(K)\, \mathrm{e}^{K' s s'}$ for all s and s'. If we write down and solve the two equations obained by setting $s = s'$ and $s = -s'$, respectively, we find that the transformation is given by

$$K' = \tfrac{1}{2} \ln \cosh(2K)\ , \qquad f(K) = 2\, (\cosh 2K)^{1/2}\ .$$

Therefore (using $K = \frac{1}{2} \cosh^{-1} \mathrm{e}^{2K'}$),

$$g(K') = 2\, g(K) - \ln \big[2\, (\cosh 2K)^{1/2}\big]\ , \qquad g(K) = \tfrac{1}{2}\, g(K') + \tfrac{1}{2}\, K' + \tfrac{1}{2} \ln 2\ .$$

• *Result:* If we plot $K(K')$ or $K'(K)$ and iterations of $g(K')$ and $g(K)$, we see that there are no fixed points (except for $K = 0$ and ∞, where the system, respectively totally disordered and totally ordered, is scale-free; notice that at $T = 0$ an arbitrarily small magnetic field would align the spins).

Reading

• *Course textbook:* Kennett, §§ 10.1–10.4.
• *Other books:* Pathria & Beale, § 13.2 (transfer matrix approach) and § 14.2.A (renormalization group); Chandler, Ch 5, pp 119 ff; Halley, pp 150–160; Mattis & Swendsen, §§ 2.7–2.8 and 8.5–8.10;
Plischke & Bergersen, § 3.6, Chapter 6; Reif, Mentioned in p 429; Schwabl, §§ 6.5 and 7.3.