Electron-phonon interaction

* The physical mechanism for superconductivity took nearly 50 years to sort out.

→ huge theoretical challenge to explain the tiny energy scales involved
  \( \approx 10^{-5} \text{ eV per electron} \) compared to a Coulomb scale \( \approx 1 \text{ eV per electron} \)

→ first hint as to the origin of superconductivity was the discovery of the isotope effect: i.e. different isotopes of the same metal have different critical temperatures \( T_c \) below which they show zero resistance and \( T_c \) obeys the relation

\[
T_c M^x = \text{const with } x \approx 0.5
\]
Assuming a value of 0.32080 for the ratio $\nu(B^3)/\nu(H^3)$, one obtains

$$\nu(Nb^{99})/\nu(H^3) = 0.24441 \pm 0.0003.$$  

This value should be increased by a Lamb correction of 0.38 percent because of the diamagnetic effects of the atomic electrons. Use of this correction and of the value 5.8804 for the $g$-factor of the proton leads to the following value for the nuclear $g$-factor of Nb$^{99}$:

$$g(Nb^{99}) = 1.3702 \pm 0.0007.$$  

Since the spin of Nb$^{99}$ is known from hyperfine structure data to be 9/2, the nuclear magnetic moment of Nb$^{99}$ is

$$\mu(Nb^{99}) = 6.1659 \pm 0.0032 \text{ nuclear magnetons.}$$

These values are to be compared with the earlier values $g = 1.18$ and $\mu = 5.31$ obtained by Meeks and Fisher from hyperfine structure studies. The value obtained in the present work lies close to the upper Schmidt limit value of $+0.79$ for a $\frac{9}{2}$ spin odd proton, which Nb$^{99}$ has according to Mayer's theory.

We wish to express our appreciation to the Ohio State University Development Fund and Research Foundation for grants that made this work possible.

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Isotope Effect in the Superconductivity of Mercury

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March 24, 1950

THE existence of a small quantity of Hg$^{198}$ at the National Bureau of Standards prompted us to investigate its properties as a superconductor. The sample available to us had a high degree of isotopic separation and was approximately 98 percent pure Hg$^{198}$. The average atomic weight of natural mercury is 200.6. The mercury had been produced by the transmutation of gold and had been prepared by distilling it off the bombarded gold foil.

Preliminary results are now available on the critical field behavior and transition temperature. The destruction of superconductivity was detected magnetically by a ballistic galvanometer method and the zero-field transition temperature determined by extrapolation of the critical field curve to zero field. (Further details of the experimental method will appear elsewhere.) Tests were made with natural mercury and with two specimens of Hg$^{198}$ about 400 mg each (both derived from the original sample). The natural mercury was prepared by our Chemistry Division and is presumed to contain less than 0.001 percent impurity. The specimens of Hg$^{198}$ were separately redistilled in vacuum (following the original distillation from the gold foil) and were enclosed in Pyrex capillaries which were evacuated and sealed off. Temperatures were measured with a helium vapor pressure thermometer using the tables prepared at the Royal Society Mond Laboratory (June 4, 1949) for reducing the pressures to degrees Kelvin.

The results are indicated in Fig. 1 which is a plot of current in the Helmholtz coils (at critical field) vs. the absolute temperature for both the natural mercury and Hg$^{198}$. Any small difference in slope is not significant since the demagnetization factors of the samples and the exact disposition in the field may have been different. The intercepts are significant and give the transition temperatures as $4.156^\circ \text{K}$ for natural mercury and $4.177^\circ \text{K}$ for Hg$^{198}$.

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Magic Numbers and Elements with No Stable Isotopes

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March 20, 1950

THAT the abnormal instability of all isotopes of $^{43}$Tc and $^{152}$Pm may have something to do with the occurrence, close by, of magic neutron numbers 50 and 82, is an obvious enough suggestion. It becomes even more plausible if we consider that only the absence of odd-mass isotopes requires a special explanation (even-mass isotopes are, for $Z=43$ and 61, excluded by parity rules). Two other elements, A and Ce, then appear to share the same anomaly and in both the latter cases the connection with the magic neutron numbers (20 and 82, respectively) is even more evident.
the observation that 70 percent of the closely spaced double pulses are followed by a third pulse interpretable as a $\mu$-decay while only 30 percent of all the sweeps show a second pulse.

(d) By extrapolation of the decay curve of Fig. 2 back to zero time, we conclude that about 1.5 $\pi^+$-mesons were stopped in each gram of crystal per hour. A Kodak NTB nuclear emulsion placed near the crystal indicated that about 1.2 $\pi^-$-mesons were stopped in each gram of emulsion per hour.

The circled points in Fig. 2 show the number of delayed pulses corrected for accidental. By fitting an exponential curve to these points one finds for the mean life of $\pi^+$-mesons the value $\tau = (1.65 \pm 0.33) \times 10^{-10}$ sec. Within the rather large statistical errors involved, this value of the mean life agrees with the value $(1.97 \pm 0.14) \times 10^{-10}$ sec. obtained by Martinelli and Panofsky on the decay of $\pi^+$-mesons in flight, and with the value $(1.11 \pm 0.31) \times 10^{-10}$ sec. obtained by Richardson on the decay of $\pi^-$-mesons in flight.

We wish to thank the members of the M.I.T. synchrotron staff who have cooperated in making these measurements. Professors Bruno Rossi and Matthew Sands have contributed much encouragement and sound advice. We thank Mr. W. B. Smith who constructed and assisted in the testing of most of the equipment, and Mr. A. Grubman who grew the stilbene crystals.

1 This work was supported in part by the joint program of the AEC and ONR.
2 Physics Department and Laboratory for Nuclear Science and Engineering.
3 Electrical Engineering Department and Laboratory for Nuclear Science and Engineering.

Superconductivity of Isotopes of Mercury*

C. A. Reynolds, B. Serin, W. H. Wright, and L. B. Newitt
Rutgers University, New Brunswick, New Jersey
March 24, 1950

The superconducting transition temperatures of natural lead and the lead isotope obtained from the decay of uranium have been determined from resistance measurements by Kamerlingh Onnes and Tsun2 and by Justi3. In both instances, no detectable change in transition temperature between the natural element and the isotope was found.

In the measurements to be described, critical field vs. temperature curves for mercury samples enriched in various isotopes were determined by susceptibility measurements. The samples were obtained from the U.S. Atomic Energy Commission, and were prepared by electromagnetic separation. The average mass numbers for the enriched samples (samples 1, 2, 4) are shown in Table 1, sample 3 is natural mercury.

The samples varying in mass from 50 to 100 mg, were sealed under helium in capillaries about 0.5 mm id., giving samples about 3 cm long. A coil of No. 41 copper wire was wound around each capillary. The samples were placed in a Dewar vessel of liquid helium, and formed the secondary of a mutual inductance with a solenoid placed outside the shield Dewar. The solenoid was excited by a 1000 c/sec. oscillator, and produced a magnetic field at the sample of less than 0.1 oersted. The signal picked up by the sample coil was amplified and detected. The signal was balanced out with the sample superconducting, and then the detected signal number. The values obtained by Maxwell for Hg$^{198}$ and natural Hg are included. The slope of the line in Fig. 2 is 0.009 °K/(mass number).

![Fig. 1. The critical magnetic field as a function of the absolute temperature.](image1)

![Fig. 2. The transition temperature vs. the average mass numbers of the isotopic mixtures.](image2)

<table>
<thead>
<tr>
<th>Sample</th>
<th>Average mass number</th>
<th>Transition temperature °K</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>203.4</td>
<td>4.126</td>
</tr>
<tr>
<td>2</td>
<td>202.0</td>
<td>4.143</td>
</tr>
<tr>
<td>3</td>
<td>201.7</td>
<td>4.150</td>
</tr>
<tr>
<td>4</td>
<td>199.7</td>
<td>4.161</td>
</tr>
</tbody>
</table>

* This work was supported by the ONR, by the Research Corporation, by the Rutgers University Research Council, and by the Radio Corporation of America.

That result suggests that the lattice is an active participant in creating the superconducting state.

* How can electrons interact with each other via phonons (quantized vibrational excitations of the lattice)?

→ Phonons are characterized by crystal momentum \( \mathbf{q} \) and energy \( \hbar \omega \).

→ An electron with wave vector \( \mathbf{k}_1 \) can interact with another electron with wave vector \( \mathbf{k}_2 \) by exchanging a phonon.

\[
\begin{align*}
\mathbf{k}_1 & \quad \leftrightarrow \quad \mathbf{k}_1' \\
\mathbf{k}_2 & \quad \leftrightarrow \quad \mathbf{k}_2'
\end{align*}
\]

→ By conservation of momentum:

\[
\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_1' + \mathbf{k}_2'
\]

and

\[
\mathbf{k}_1 = \mathbf{k}_1' + \mathbf{q}
\]

\[
\mathbf{k}_2 = \mathbf{k}_2' + \mathbf{q}
\]
What is the sign of this effective electron-electron interaction?

Here's a rough argument:

1. When electron \( \vec{E}_1 \) interacts with the lattice and is scattered into state \( \vec{E}_1' \), it causes local oscillations of the electron density at frequency \( \omega = (\epsilon_{\vec{k}_1} - \epsilon_{\vec{k}_1'})/\hbar \).

2. If at some instance the local electron density is enhanced at some position in the crystal (more negative charge), the surrounding ions will start moving toward the negative excess charge in order to compensate it.

3. But the ion mass is large and the motion is slow, so there is a tendency to overcompensate and create net positive charge.
The second electron with wave vector $\mathbf{k}_2$ can interact with the positive charge excess; and the interaction can be attractive if the lattice vibrations are in phase with the "driving force" at frequency $\omega = (3\varepsilon_i - 3\varepsilon_i')/4$ and if $\omega < \omega_D$, the Debye frequency which is the maximum frequency of oscillation for the ions.

Consider a model of a simple harmonic oscillator of mass $m$, natural frequency $\omega_0$, subject to a driving force $Fe^{i\omega t}$.

Then the equation of motion is

$$\ddot{x} + \omega_0^2 x = \frac{F}{m} e^{i\omega t}$$

A solution of the form $x(t) = x_0 e^{i\omega t}$ gives

$$x_0 (i\omega)^2 + \omega_0^2 x_0 = \frac{F}{m}$$
or

\[ x_0 = \frac{f}{\omega_0^2 - \omega^2} \]

→ tells us that the oscillations \( x = x_0 e^{i\omega t} \)
are in phase with the external time
when \( \omega^2 < \omega_0^2 \) but out of phase when
\( \omega^2 > \omega_0^2 \)

* Proposed law for phonon-mediated attraction

→ attractive potential \(-V_0\) if \( |\varepsilon_{k_1} - \varepsilon_{k_1'}| < \hbar \omega_0 \)

→ scattering only possible if \( k_1' \) is unoccupied

\[ V(\varepsilon_1 - \varepsilon_1') = \begin{cases} -V_0 & \text{if } |\varepsilon_1 - \varepsilon_3| \leq \hbar \omega_0 \text{ and } |\varepsilon_3 - \varepsilon_{k_1'}| \leq \hbar \omega_0 \\ 0 & \text{otherwise} \end{cases} \]

2\( \Delta k \) range of interaction

\[ \Delta k \sim \frac{\hbar \omega_0}{\varepsilon_F} \quad , \quad \varepsilon_F = \frac{\hbar^2 k_F^2}{2m} \]
In our previous look at the BCS Hamiltonian

\[ \hat{H} = \sum_k \left( c_{k+} c_{-k} \right) \begin{pmatrix} \varepsilon_k - \mu & \Delta \\ \Delta^* & \varepsilon_k - \mu \end{pmatrix} \left( c_{k+} \right)^* \]

\[ = \sum_k \left( \delta_{k+}^+ \delta_{k+}^* \right) \begin{pmatrix} \sqrt{(\varepsilon_k - \mu)^2 + \Delta^2} & 0 \\ 0 & -\sqrt{(\varepsilon_k - \mu)^2 + \Delta^2} \end{pmatrix} \left( \delta_{k+} \right)^* \]

\[ = \sum_k \sum_{n=\pm} n \frac{\delta_{kn}^+ \delta_{kn}}{\sqrt{2(\varepsilon_k - \mu)^2 + \Delta^2}} \]

\[ \overset{\text{Bogoliubov transform}}{\rightarrow} \]

\[ \overset{\text{diagonalized}}{\rightarrow} \]

\[ \overset{\text{energy eigenvalues}}{\rightarrow} \]

\[ E_{k_2} = +E_k \]

\[ E_{k_1} = -E_k \]

\[ \text{gap equation} \]

\[ \Delta(\varepsilon_k) = \sum_{k'} V(k-k') \Delta(k') \frac{f(E_{k_2}) - f(E_{k_1})}{E_{k_2} - E_{k_1}} \]
\[ A = \sum \left( \frac{(-V_c)}{2E_k} \left( f(E_k) - f(-E_k) \right) \right) \]

\[ 1 = V_0 \sum \left( \frac{f(E_k) - C(-f(E_k))}{2E_k} \right) \]

\[ = V_0 \sum \left( 1 - \frac{2f(E_k)}{2E_k} \right) \]

\[ \int d\varepsilon \frac{D(G_{EF})}{\varepsilon^2 + \Delta^2} \left( 1 - \frac{2}{e^{E_{k}kB_1} + 1} \right) \]

\[ = \int d\varepsilon \frac{\delta \varepsilon}{\sqrt{\varepsilon^2 + \Delta^2}} \tanh \left( \frac{\sqrt{\varepsilon^2 + \Delta^2}}{2kB_1} \right) \]

Critical \( T_c, \Delta \rightarrow 0 \)

\[ \frac{1}{D(G_{EF})V_0} = \int \delta \varepsilon \frac{d\varepsilon}{\varepsilon} \tan \frac{\varepsilon}{2kB_1} \]
$k_B T_c = 1.14 + \omega_D e^{-1/\Delta_0 V_0}$

We already showed that

$\Delta_0 = 2 \pi \omega_D e^{-1/\Delta_0 V_0}$

So $2\Delta_0 = 3.52 k_B T_c$

Also, this provides an explanation for the isotope effect since $\omega_D \sim m^{-1/2}$