1. A system of independent electrons is described by the purely bilinear Hamiltonian

\[ \hat{H} = \sum_{\alpha} \int d^3 r \ \hat{\psi}_\alpha(r)^\dagger T(r)\hat{\psi}_\alpha(r). \]

Here, \( T \) is some differential operator, not necessarily \(-\hbar^2 \nabla^2 / 2m\), and \( \alpha \) ranges over the two spin projections. For concreteness, suppose that the Hamiltonian describes electrons bound to a diatomic molecule.

(a) Consider a single-particle basis \( \{ \phi_1, \phi_2 \} \) consisting of two states that are not eigenfunctions of \( T \). We’ll think of \( \phi_1(r) \) as a wave function localized on atom 1 and \( \phi_2(r) \) as a wave function localized on atom 2. Use the field operator expansion

\[ \hat{\psi}_\alpha(r) = \sum_{j=1,2} \phi_j(r) \chi_{\alpha} c_{j,\alpha} \]

to show that

\[ \hat{H} = \sum_{\alpha} (c_{1,\alpha}^\dagger - t) \left( \begin{array}{c} \varepsilon_1 \\ -t^* \\ \varepsilon_2 \end{array} \right) \left( \begin{array}{c} c_{1,\alpha} \\ c_{2,\alpha} \end{array} \right), \]

where

\[ \varepsilon_j = \int d^3 r \ \phi_j(r)^* T(r) \phi_j(r) \quad \text{and} \quad -t = \int d^3 r \ \phi_1(r)^* T(r) \phi_2(r). \]

(b) For the case of a homonuclear atom (\( \varepsilon_1 = \varepsilon_2 = \varepsilon \)), solve for the one- and two-particle ground states.

(c) Now consider the case where the atoms in the molecule are different (\( \varepsilon_1 = \varepsilon - \Delta \) and \( \varepsilon_2 = \varepsilon + \Delta \)). What is the average number of electrons on atom 1 and on atom 2 in the one-particle ground state?

2. The Hamiltonian

\[ \hat{H} = \varepsilon a^\dagger a + \Delta \left[ a^2 + (a^\dagger)^2 \right] \]

is built from bosonic creation and annihilation operators, \( a^\dagger \) and \( a \).

(a) Show that there is an operator \( \hat{A} \sim a + \lambda a^\dagger \) that obeys the bosonic commutation relation \([\hat{A}, \hat{A}^\dagger] = 1\). Determine the constant of proportionality.

(b) Show that, up to a constant, \( \hat{H} = \varepsilon \hat{A}^\dagger \hat{A} \), so long as you can find values \( \varepsilon \) and \( \lambda \) such that

\[ \varepsilon = \frac{\varepsilon(1 + \lambda^2)}{1 - \lambda^2} \quad \text{and} \quad \Delta = \frac{2\varepsilon\lambda}{1 - \lambda^2}. \]

(c) Explicitly construct the ground state and first three excited states. \textit{Hint:} consider repeated applications of \( \hat{A}^\dagger \) on the bosonic vacuum.