# Physics 451: Introduction to Quantum Mechanics

First In-class Test

Thursday, September 25, 2014 / 09:30–10:45 / Room 2-228, Lewis Hall

Student's Name: ___________________

## Instructions

There are 12 questions. You should attempt all of them. Mark your response on the test paper in the space provided. **Please use a pen**. If in answering a question you sketch a diagram, please provide meaningful labels. Aids of any kind—including class notes, textbooks, cheat sheets, and calculators—are not permitted.

Good luck!

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|---|---|---|---|---|
| 4 | points | short answer | questions | 1–4 |
| 1 | | long answer | | 5 |
| 1 | | ⋮ | | 6 |
| 3 | | | | 7 |
| 3 | | | | 8 |
| 3 | | | | 9 |
| 4 | | | | 10 |
| 3 | | | | 11 |
| 3 | | | | 12 |
| **25** | **points** | | | |

## Integrals

$$
\begin{aligned}
\int_0^a dx\,(p+qx+rx^2) &= pa + \frac{qa^2}{2} + \frac{ra^3}{3} \\
\int_a^\infty dx\;\exp(-|k|x) &= \frac{1}{a}\exp(-|k|a) \\
\int_{a_1}^{a_2} dx\;\exp(-kx)(p+qx+rx^2) &= \frac{1}{k}\left[e^{-ka_1}(p+qa_1+ra_1^2) - e^{-ka_2}(p+qa_2+ra_2^2)\right] \\
&\quad + \frac{1}{k^2}\left[e^{-ka_1}(q+2ra_1) - e^{-ka_2}(q+2ra_2)\right] \\
&\quad + \frac{1}{k^3}\left[e^{-ka_1}(2r) - e^{-ka_2}(2r)\right]
\end{aligned}
$$

## Trigonometric identities

$$
\begin{aligned}
2\cos\theta &= e^{i\theta} + e^{-i\theta} \\
2i\sin\theta &= e^{i\theta} - e^{-i\theta}
\end{aligned}
$$

## Short answer questions (4 points)

1. In the following equation, cross out the two terms that are not part of the *time-dependent* Schrödinger equation and identify the *potential* by circling it.

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} + E\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t) + \frac{4\pi\hbar^2\ell}{m}|\psi(x,t)|^2\psi(x,t)$$

2. We studied the case of a single particle in one dimension, constrained by an infinite square well potential. Which of the following is an incorrect statement.
   (a) The eigenstates $\{|\phi_n\rangle : n = 1, 2, 3, \ldots\}$ of the system are states of definite parity.
   (b) The eigenstates are normalizable.
   (c) The eigenstates are (countably) infinite in number.
   (d) All the eigenstate wave functions $\{\phi_n(x)\}$ are localized in space, even those at arbitrarily high energy.
   (e) The eigenstates are sinusoidal in shape.
   (f) According to the customary labelling, the number of nodes (i.e., points where $\phi_n(x) = 0$ with $x$ strictly inside the well) is equal to $n - 1$.
   (g) The energy $E_n$ of the state $|\phi_n\rangle$ scales inversely with $n$.

3. The overlap between infinite square well eigenstates satisfies

$$\langle\phi_m|\phi_n\rangle = \frac{2}{L}\int_0^L dx\, \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

   This is a statement of what?
   (a) orthogonality
   (b) orthonormality
   (c) completeness
   (d) duality
   (e) criticality

4. Which of the following is an incorrect statement about the wave function $\psi(\mathbf{r}) \sim \exp(i\mathbf{k}\cdot\mathbf{r})$.
   (a) $\psi(\mathbf{r})$ is a plane wave state.
   (b) $\psi(\mathbf{r})$ represents a state of definite momentum $\mathbf{p} = \hbar\mathbf{k}$.
   (c) The probability density $|\psi(\mathbf{r})|^2$ of finding the particle in the vicinity of $\mathbf{r}$ is uniform in space.
   (d) The form $\psi(\mathbf{r}) \sim \exp(i\mathbf{k}\cdot\mathbf{r})$ emerges only in regions where the potential is exactly zero.

## Long answer questions (21 points)

Consider an electron living in a double well potential. $|L\rangle$ and $|R\rangle$ denote occupation in the left and right positions. The shape of the wavefunction in each state is identical but translated by a distance $\ell$. In other words, $\psi_L(x) = \langle x|L\rangle = \phi(x)$ and $\psi_R(x) = \langle x|R\rangle = \phi(x - \ell)$.

5. Write down the integral expressions for the overlaps $\langle L|R\rangle$ and $\langle R|L\rangle$, taking into account that $\phi(x)$ may be complex.

6. Show explicitly that $\langle L|R\rangle = \langle R|L\rangle^*$

7. Suppose that $\phi(x)$ is real-valued and defined piecewise as follows.

$$\phi(x) = C \begin{cases} 2 - x^2/a^2 & \text{if } |x| < a \\ \tau \exp(-\sigma(|x|/a)) & \text{if } |x| \geq a \end{cases}$$

Here, $a$ is a constant with units of length. Determine the values of $\tau$ and $\sigma$ (dimensionless parameters) that lead to a physically realistic wave function. Determine the correct normalization constant $C$.

8. Show that the overlap goes as $e^{-2\ell/a}$ in the limit of large well separation. A rough (but convincing) sketch of a proof is fine.

9. The Hamiltonian of the system is

$$\hat{H} = E_L|L\rangle\langle L| + E_R|R\rangle\langle R|.$$

Formulate the *generalized eigenvalue problem* that follows from representing the *time-independent* Schrödinger equation in the $\{|L\rangle, |R\rangle\}$ basis. You should find that it has the (linear algebra) form $H\Psi = ES\Psi$, where

$$H = \begin{pmatrix} H_{LL} & H_{LR} \\ H_{RL} & H_{RR} \end{pmatrix} = \begin{pmatrix} E_L + s^2 E_R & ? \\ ? & ? \end{pmatrix}$$

is the $2 \times 2$ collection of matrix elements of the Hamiltonian;

$$S = \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}$$

is the matrix of overlaps (with $s = \langle L|R\rangle = \langle R|L\rangle$); and

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

is the column vector of coefficients from the expansion $|\psi\rangle = \psi_L|L\rangle + \psi_R|R\rangle$. Be sure to compute all the elements of $H$ in terms of $E_L$, $E_R$, and $s$.

10. In the special case where $E_L = E_R \equiv E_0$, show that $|\psi_+\rangle = |L\rangle + |R\rangle$ and $|\psi_-\rangle = |L\rangle - |R\rangle$ are eigenstates. (Don't worry that neither state is properly normalized.) Determine the corresponding eigenvalues $E_+$ and $E_-$, expressed in terms of $E_0$ and $s$.

11. Continue under the assumption that $E_L = E_R \equiv E_0$. The system is prepared in the initial state $|\psi(0)\rangle = \frac{3}{5}|L\rangle + \frac{4}{5}|R\rangle$ (not properly normalized, but don't worry about that yet). Compute the full time dependence of the state $|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle$. You should find that

$$|\psi(t)\rangle = \frac{1}{10}e^{-iE_0 t/\hbar}\left(7e^{-isE_0 t/\hbar}|\psi_+\rangle - e^{+isE_0 t/\hbar}|\psi_-\rangle\right).$$

12. Building on the result in question 11, you must now show that the probability of measuring the particle in the right-hand well is

$$P_R = \frac{|\langle R|\psi(t)\rangle|^2}{\langle\psi(0)|\psi(0)\rangle} = \frac{1}{100 + 96s}\left[50(1 + s^2) + 96s + 14(1 - s^2)\cos(sE_0 t/\hbar)\right].$$

What happens to the period of oscillation as the well separation becomes large? Also comment on the opposite limit in which the well separation becomes so small that the two wells coincide.