Physics 711: Assignment 3

(to be submitted by Monday, November 14, 2016)

The Hamiltonian of the three-dimension quantum harmonic oscillator—describing a particle of mass m in an isotropic Hooke's law potential—can be decomposed into three one-dimensional harmonic oscillators:

$$\hat{H} = \frac{\hat{\boldsymbol{p}}^2}{2m} + \frac{m\omega^2\hat{\boldsymbol{r}}^2}{2} = \hbar\omega\left(a_x^\dagger a_x + a_y^\dagger a_y + a_z^\dagger a_z + \frac{3}{2}\right) \equiv \hbar\omega\left(\hat{N} + \frac{3}{2}\right).$$

Here, a_x, a_y, and a_z are the annihilation operators for a quantum of vibrational motion in the three orthogonal directions. It is convenient to define a basis of Cartesian states

$$|n_x, n_y, n_z\rangle = \frac{1}{\sqrt{n_x! n_y! n_z!}} (a_x^\dagger)^{n_x} (a_y^\dagger)^{n_y} (a_z^\dagger)^{n_z} |0,0,0\rangle$$

that satisfy $\hat{H}|n_x, n_y, n_z\rangle = \hbar\omega(n_x + n_y + n_z + 3/2)|n_x, n_y, n_z\rangle$.

1. List all the Cartesian states that have energy eigenvalues below $4\hbar\omega$. Indicate the degeneracies.

2. Find a general expression for the degeneracy of the states with total occupation N.

3. Show that the orbital angular momentum operator $\hat{\boldsymbol{L}} = (\hat{L}_x, \hat{L}_y, \hat{L}_z)$ and its magnitude can be expressed as follows:
$$\hat{L}_i = i\hbar \sum_{jk} \epsilon_{ijk} a_j a_k^\dagger, \quad \hat{\boldsymbol{L}}^2 = \hbar^2\Big[\hat{N}(\hat{N}+1) - \sum_{jk} a_j^\dagger a_j^\dagger a_k a_k\Big].$$

4. Find states $|n, l, m\rangle$ such that $\hat{\boldsymbol{L}}^2|n, l, m\rangle = \hbar l(l+1)$ and $\hat{L}_z|n, l, m\rangle = \hbar m$ with $m = -l, -l+1, \ldots, l-1, l$. Work this out for $l = 0, 1, 2$. Explain why the $l \geq 2$ cases are different from $l = 0$ and $l = 1$.

The final four questions are concerned with the connection between SU(2) and SO(3). Recall that for every $U \in$ SU(2), there is a corresponding 3×3 rotation matrix $R^{ab} = \frac{1}{2}\operatorname{Tr}\sigma^a U \sigma^b U^\dagger$.

5. For a unit vector $\boldsymbol{n}$ in three-dimensional space, demonstrate that
$$U(\phi\boldsymbol{n}) = e^{-i(\phi/2)\boldsymbol{n}\cdot\boldsymbol{\sigma}} = I\cos\frac{\phi}{2} - i\boldsymbol{n}\cdot\boldsymbol{\sigma}\sin\frac{\phi}{2}$$
and that $U(\phi\boldsymbol{n}) \in$ SU(2).

6. Prove the following four identities involving traces of the Pauli matrices: $\operatorname{Tr}\sigma^a = 0$, $\operatorname{Tr}\sigma^a\sigma^b = 2\delta^{ab}$, $\operatorname{Tr}\sigma^a\sigma^b\sigma^c = 2i\epsilon^{abc}$, $\operatorname{Tr}\sigma^a\sigma^b\sigma^c\sigma^d = 2(\delta^{ab}\delta^{cd} - \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc})$.

7. Show that the 3×3 rotation matrix arising from $U(\phi\boldsymbol{n})$ has the form
$$R^{ab} = \frac{1}{2}\operatorname{Tr}\sigma^a U \sigma^b U^\dagger = \delta^{ab}\cos\phi + (1-\cos\phi)n^a n^b - \epsilon^{abc} n^c \sin\phi.$$
Compute $R(\phi\boldsymbol{e}_x)$ and $R(\phi\boldsymbol{e}_z)$.

8. Show that the action of two rotations $R_1 = R(\phi\boldsymbol{n})$ and $R_2 = R(\theta\boldsymbol{m})$, applied in succession, is characterized by product of the corresponding unitary matrices $U_2 = U(\theta\boldsymbol{m}) = e^{i(\theta/2)\boldsymbol{m}\cdot\boldsymbol{\sigma}}$ and $U_1 = U(\phi\boldsymbol{n}) = e^{i(\phi/2)\boldsymbol{n}\cdot\boldsymbol{\sigma}}$:
$$\sum_b R_2^{ab} R_1^{bc} = \frac{1}{2}\operatorname{Tr}\sigma^a U_2 U_1 \sigma^c U_1^\dagger U_2^\dagger.$$