SCIENTIFIC COMPUTING: LECTURE 16

- ODE of vectors
- Example: Projectile motion with air drag
- Systems of coupled ODEs
- Example: Spread of an epidemic

CLASS NOTES

- Midterm project due date extended until next Tuesday (3/30)
- No HW this week – finish your projects!
- Reading for Differential Equations in Appendix B.

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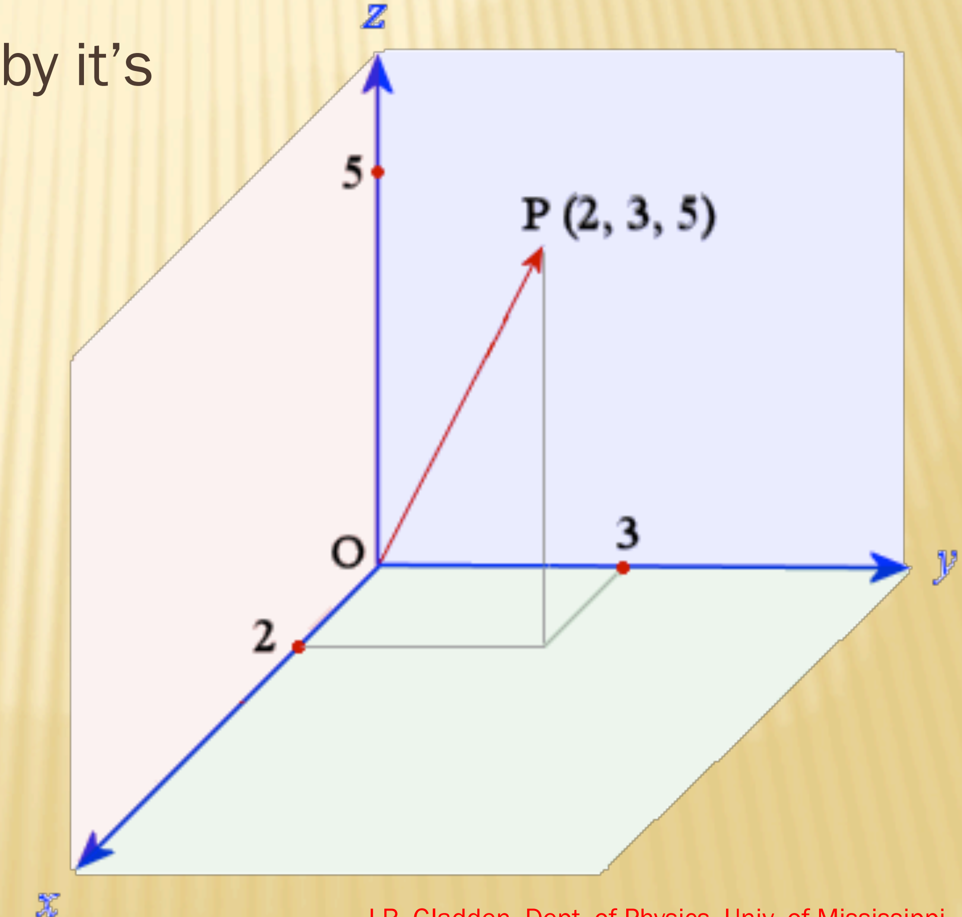
VECTOR QUANTITIES

- Often in science the quantities we are interested in are <u>vectors</u> which require a magnitude and direction.
- Typically we express a vector by it's individual components:

$$\vec{v} = (v_x, v_y, v_z)$$

- Computationally, our 1D problems now become 3 1-D problems. We just perform the calculations on each of the components individually.



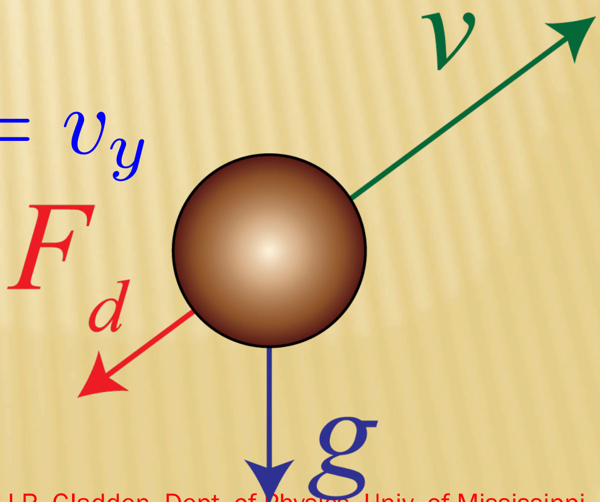
2-D EXAMPLE: PROJECTILE MOTION W/ DRAG

- Restrict to 2-D, so only need x and y components of position (r) and velocity (v).
- Force of gravity is only in y direction, drag force is always opposite to velocity (x and y components constantly change).

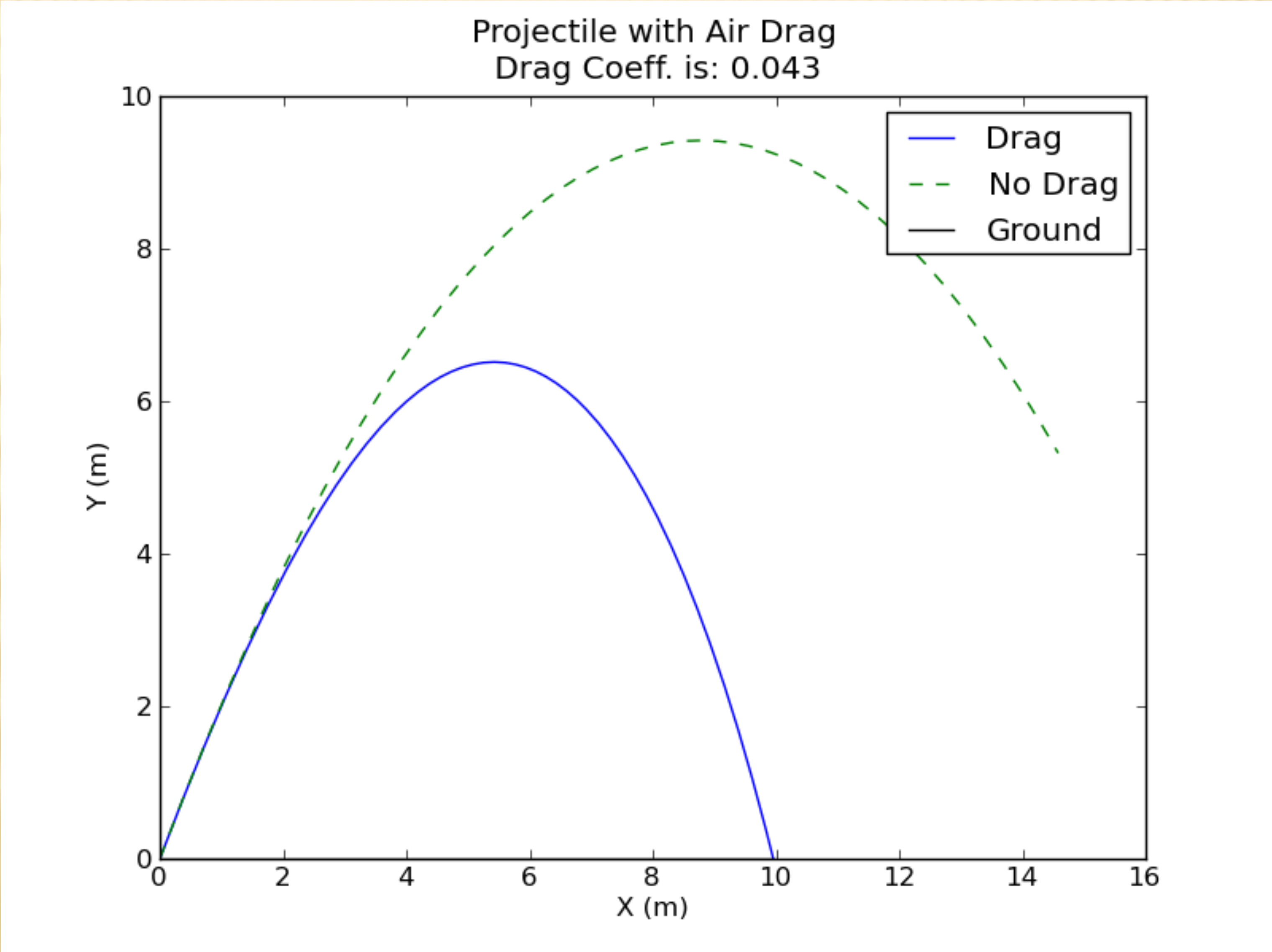
$$\frac{dv_x}{dt} = \frac{1}{m}F_{dx}(v_x) \qquad \frac{dr_x}{dt} = v_x$$

$$\frac{dv_y}{dt} = \frac{1}{m}F_{dy}(v_y) - g \qquad \frac{dr_y}{dt} = v_y$$

$$\vec{F}_d = -\frac{1}{2}C_d\rho A|\vec{v}|\vec{v}$$

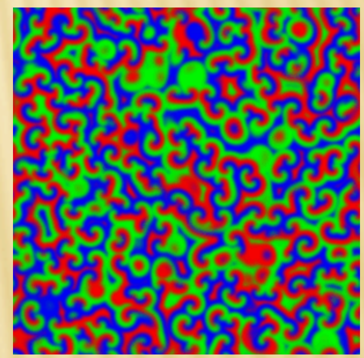


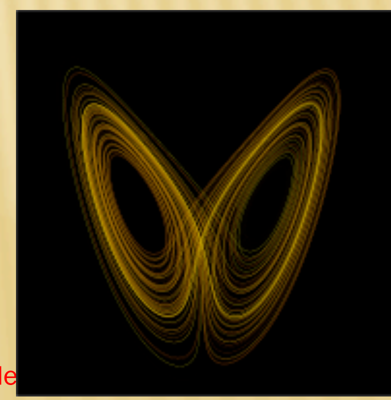
PROJECTILE WITH DRAG



SYSTEMS OF ODES

- In many ‘systems’ we encounter in science, the evolution of a variable depends on the values of other variables, … which in turn depend on the value of the original variable.
- We say ODEs which describe the evolution of these variables are coupled.
- Examples:
  - Predator – prey models (Lotka-Volterra)
  - Spread of infectious diseases
  - Chemical reactions (Belousov-Zhabotinski)
  - Climate models (Lorentz)





EXAMPLE: SPREAD OF A DISEASE

- Simple model of how an infectious disease (like the flu) might spread in a 'closed' population.
- Two groups of people: number of susceptibles (S) who can catch the disease and infectives (I) which have the disease.

$$\frac{dS}{dt} = -rSI \qquad \frac{dI}{dt} = rSI - aI$$

- The parameters r and a are characteristic of the disease (i.e. how infectious the disease is and rate of recovery respectively)
- Think about the logic behind these equations!
- Also need initial (t=0) conditions:

$$S(0) = S_0 \qquad I(0) = I_0$$

SOLUTION USING EULER

- Need to discretize using forward Euler method.
- For time = 0 to T in n time steps:

$$\Delta t = \frac{T}{n}$$

- For a time step k:

$$\frac{S_{k+1} - S_k}{\Delta t} = -rS_k I_k$$

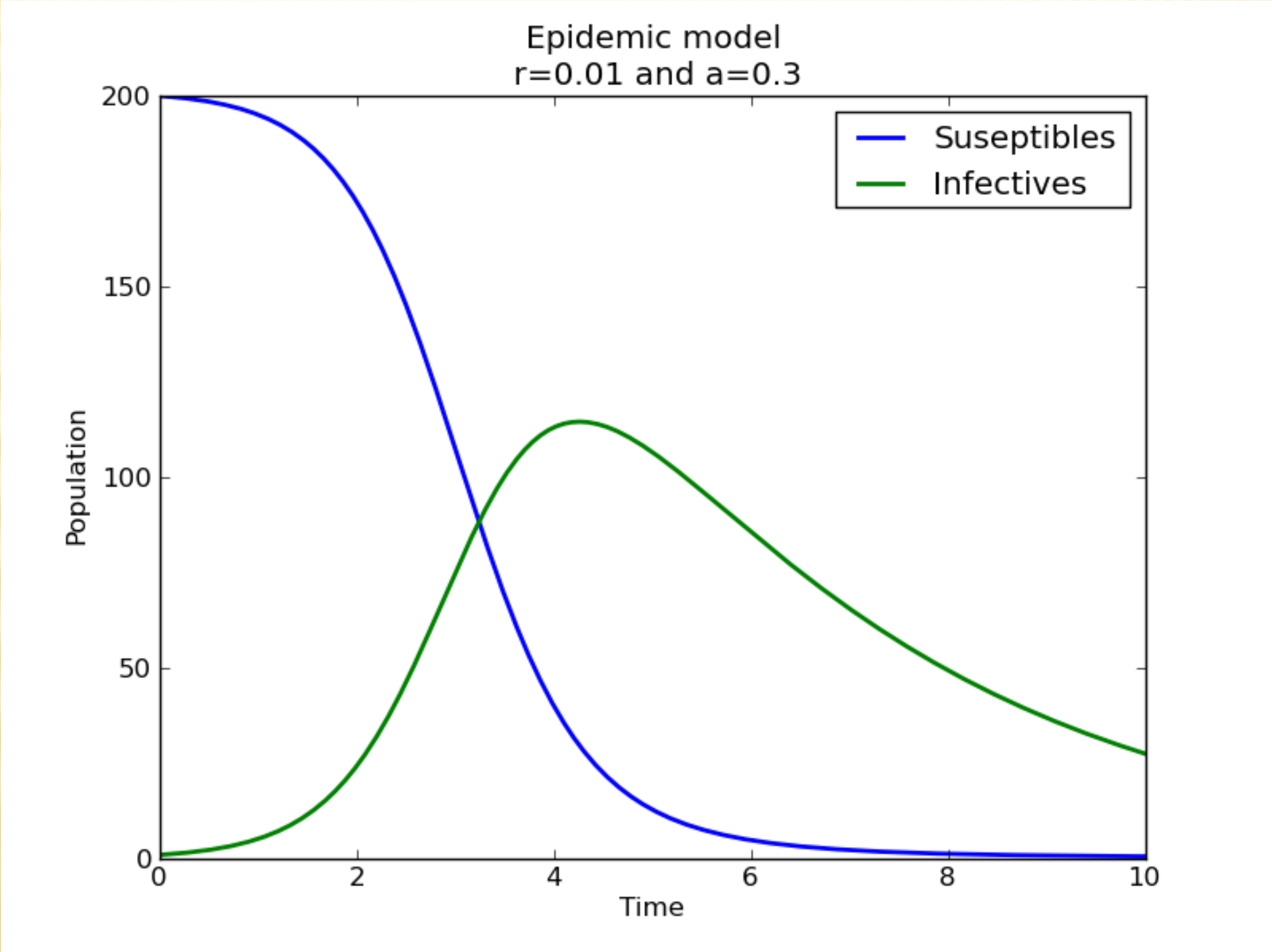
$$\frac{I_{k+1} - I_k}{\Delta t} = rS_k I_k - aI_k$$

- So to compute the S and I values for the next time step (k+1):

$$S_{k+1} = S_k - \Delta t r S_k I_k$$

$$I_{k+1} = I_k + \Delta t (r S_k I_k - a I_k)$$

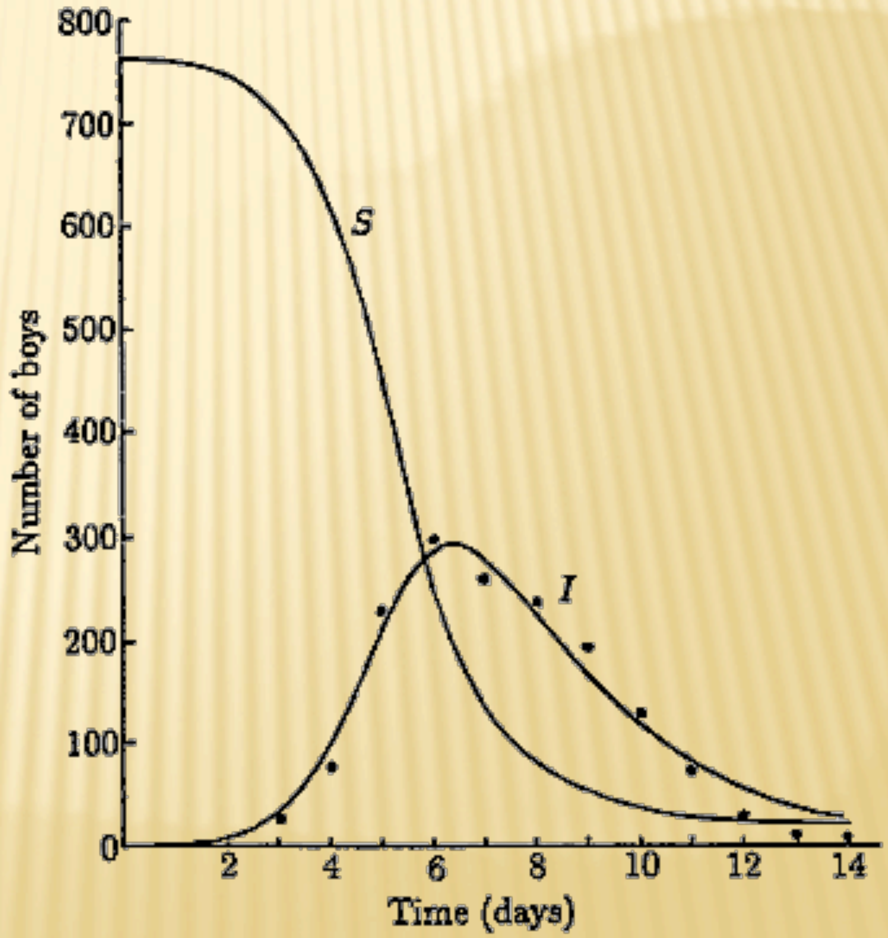
RESULTS



Epidemic model
r=0.01 and a=0.3
Suseptibles
Infectives
Population
Time
200
150
100
50
0
0
2
4
6
8
10

DOES IT WORK?

- Data was collected during a flu outbreak in a British boarding school (closed population) in 1978.
- This simple ODE system was used to successfully model the data by adjusting r and a values (non-linear fitting using a model rather than an analytic function).



Data from *Mathematical Biology* by J.D. Murray – excellent book!

STEADY-STATE SOLUTIONS

- An important question to ask is "Are there values for the variables for which the system does NOT evolve?"
- These values define steady-state solutions and define "fixed points" for the equations.
- Remember, variables are not changing if their derivatives are 0.
- There is one fixed point for the epidemic model: I = 0 since I is in every term on the RHS of all equations.
- Let's take a look at a model which has a more interesting fixed point.

LOTKA-VOLTERRA PREDATOR-PREY MODEL

- System of ODEs describing the change in population of predators (y) and prey (x).

$$\frac{dx}{dt} = x(a - cy - bx) \qquad \frac{dy}{dt} = -y(d - ex)$$

- Interpretation of parameters: a - prey birth rate, b – prey death rate (other than being eaten), c – rate of consumption, d – predator death rate, e – predator birth rate
- Find fixed points by setting both derivatives to 0 and solving for x and y:
- There are others, see them?

$$x_0 = \frac{d}{e} \qquad y_0 = \frac{a}{c} - \frac{bd}{ce}$$

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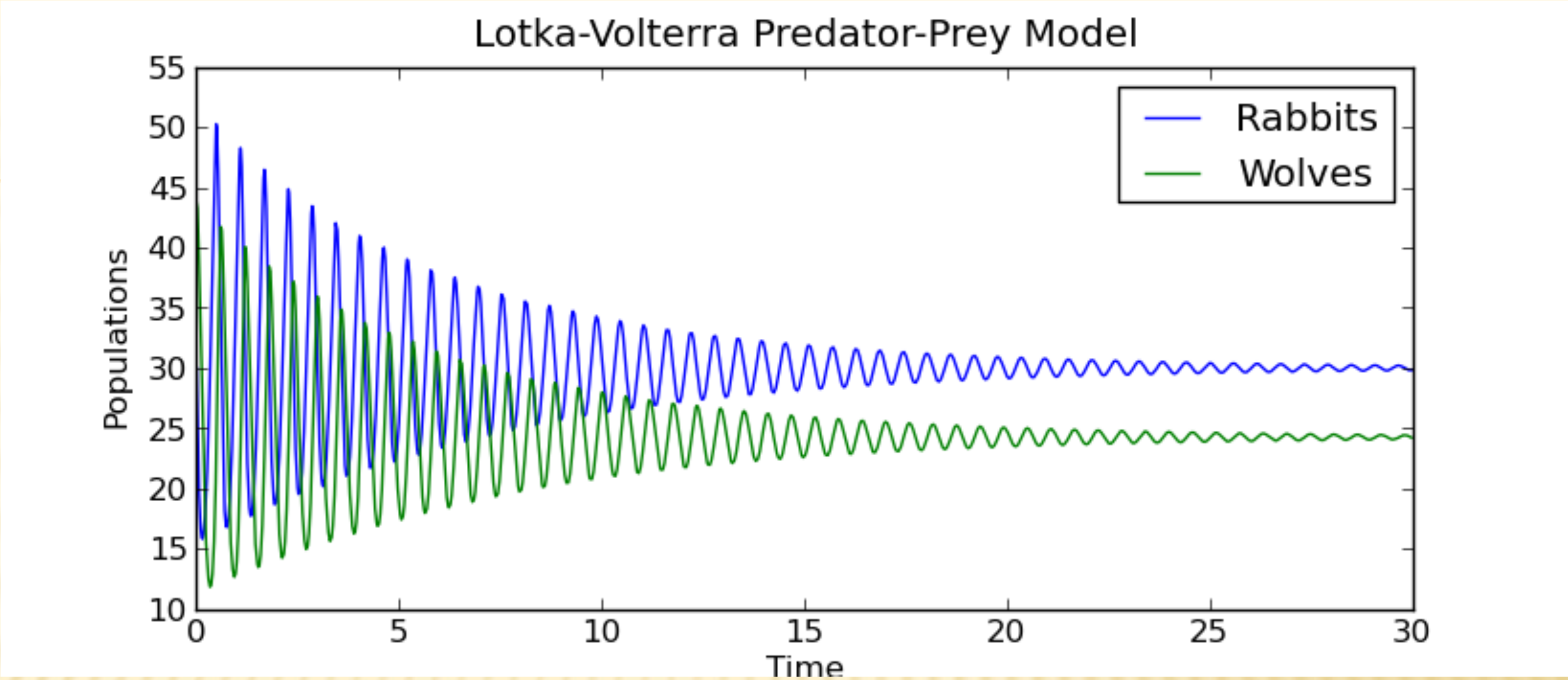
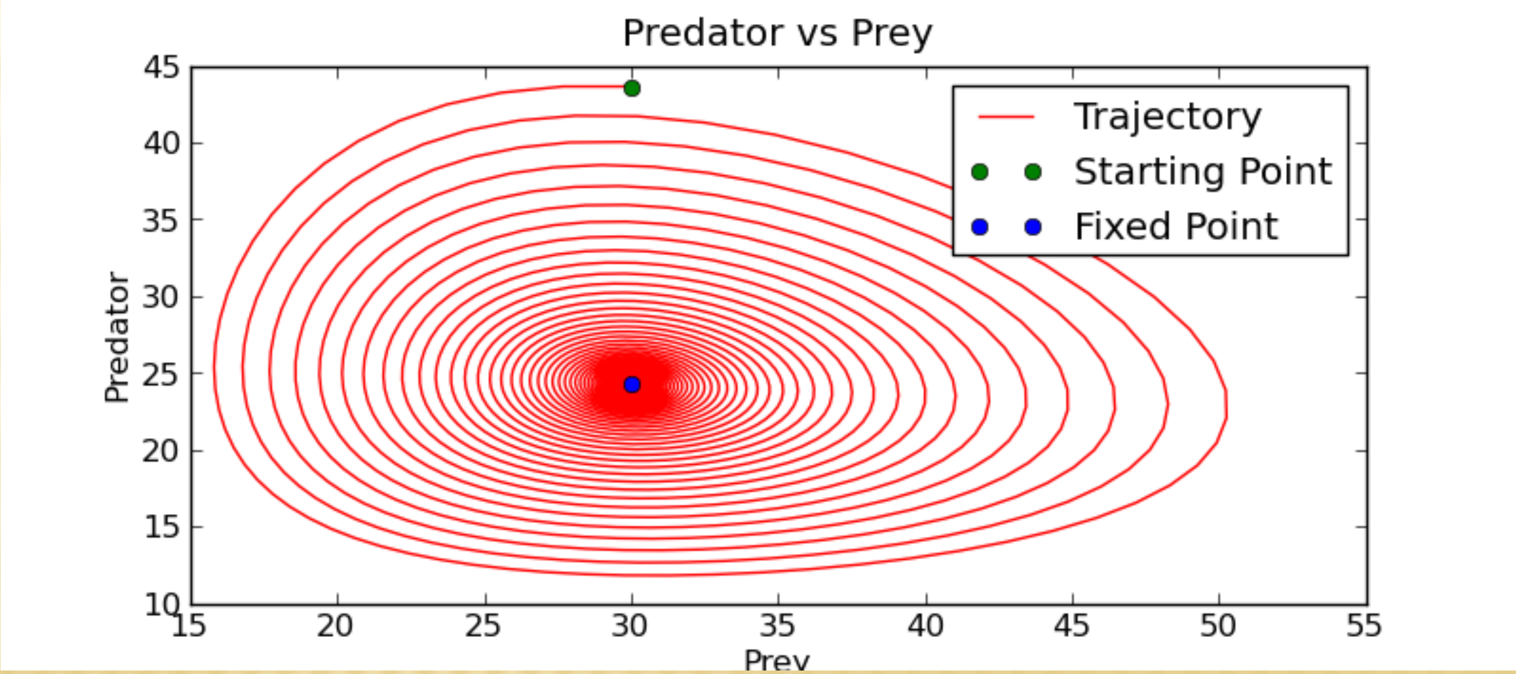

Lotka-Volterra Predator-Prey Model
Rabbits
Wolves
Populations
Time
55
50
45
40
35
30
25
20
15
10
0
5
10
15
20
25
30


Predator vs Prey
Trajectory
Starting Point
Fixed Point
Predator
Prey
45
40
35
30
25
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30
35
40
45
50
55

ssissippi