Modal expansions for sound propagation in the nocturnal boundary layer

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A modal model is developed for the propagation of sound over an impedance ground plane in a stratified atmosphere which is downward refracting near the ground but upward refracting at high altitudes. The sound’s interaction with the ground is modeled by an impedance with both real and imaginary parts so that the ground is lossy as well as compliant. Such sound speed profiles are typical of the atmospheric boundary layer at night and, together with the ground impedance, have been used extensively to model ground to ground sound propagation in the nocturnal environment. Applications range from community noise and bioacoustics to meteorology. The downward refraction near the ground causes the propagation to be ducted, suggesting that the long range propagation is modal in nature. This duct is, however, leaky due to the upward refraction at high altitudes. The modal model presented here accounts for both the attenuation of sound by the ground as well as the leaky nature of the duct. © 2004 Acoustical Society of America.

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I. INTRODUCTION

Sound propagation outdoors at night is characterized by the duct that forms in the first few hundred meters of the atmosphere. Typically, due to cooling of the atmosphere by the earth, the temperature of the air increases as one moves upward to an inversion point, generally at 100 to 300 m elevation, above which it decreases slowly for the next few kilometers. This temperature inversion causes a part of the sound field to propagate along the ground in a ducted fashion.

It should be noted that the sound speed depends on humidity as well as temperature and that atmospheric winds can have a profound effect on propagation.2,3 For low angle propagation, however, these effects can be modeled by using the effective sound speed3,4

\[ c = \sqrt{\gamma RT(1+0.511q)} + \hat{n} \cdot \mathbf{v}. \]

Here \( c \) is the sound speed, \( \gamma \) is the ratio of the constant pressure to constant volume specific heats for dry air, \( R \) is the gas constant for dry air, \( q \) is the ratio of the mass of water vapor to dry air, \( \hat{n} \) is the horizontal propagation direction, and \( \mathbf{v} \) is the wind speed vector.

Some nocturnal profiles for the effective sound speed, determined by direct measurements of temperature, humidity, and wind speed, can be found in Fig. 16 of Ref. 5. The distinguishing feature of such sound speed profiles is the presence of an inversion: the sound speed increases with increasing altitude up to the inversion height, typically found between 100 to 300 m elevation, above which it slowly decreases. The result is an atmosphere which is downward refracting near the ground, causing a duct, but upward refracting at high altitudes.

Below the inversion the rate of increase of the sound speed can be large. Care will be taken that the techniques developed here do not depend on any assumptions about the magnitude of the sound speed gradient near the ground. At high altitudes, however, the decrease of the sound speed is gradual. It will be assumed here that at sufficiently high altitudes the rate of decrease of the acoustic wavelength is small. Some model sound speed profiles are presented in Fig. 1.

Propagation in such sound speed profiles, together with a ground impedance condition6 with both imaginary (compliant) and real (resistive) parts to model the sound’s interaction with the ground, has been widely studied with intended applications including physical meteorology,5 community noise modeling7,4 and bioacoustics.8 In modeling the sound propagation the parabolic equation and horizontal wave number integration techniques have been preferred.4,9 These are robust numerical schemes which work well regardless of the sound speed profile, however, they are “black box” routines which obscure the fact that the physics of ducted sound propagation is modal.

This paper continues the study begun in Ref. 10 of modal expansions for sound propagation in horizontally stratified downward refracting atmospheres over lossy and compliant ground planes. The main advantage of modal expansions over other methods is that they give a transparent decomposition of the sound field into physically different, independently propagating parts. In addition, once the modes have been computed, they are essentially analytical solutions requiring negligible computation times.

In Ref. 10 sound speed profiles which become constant at sufficiently high altitudes were studied. A modal expansion similar to those commonly used in ocean acoustics9 was developed and a technique for finding the corresponding horizontal wave numbers and attenuation coefficients was presented. It was shown that for asymptotically constant
sound speed profiles there is a clean separation into an upwardly propagating part and a ducted part, the ducted part of the sound field completely described by a sum over a small number of vertical modes whose amplitudes decrease exponentially with height. The upwardly propagating part is described by an integral whose computation can be intensive, but which is negligible at low angles and long ranges and thus need not be computed. The sum over modes will be referred to as the modal sum while the remaining integral will be referred to as the continuum integral.

In the asymptotically upward refracting case considered here the duct is imperfect in that sound trapped in the duct can leak upward to the region in which the sound speed profile is upward refracting and escape into the upper atmosphere. It follows that the separation into ducted and upwardly propagating parts cannot hold with complete rigor. In fact, in the lossless case, in which the real part of the ground impedance is zero, there is no modal sum at all. Rather the entire sound field is contained in the continuum integral, the ducted behavior manifesting itself in sharp peaks in the integrand.

In the lossy case, in which the real part of the ground impedance is not zero, it will be seen that there is a modal sum. This is somewhat astonishing since there must be some mechanism through which the ducted part of the sound field leaks into the upper atmosphere. The vertical modes will be seen to extend very high into the atmosphere, although typically with negligible amplitude, before they begin to decrease exponentially with height. The physical picture is that the power flux into the ground due to the attenuation of sound by the ground is greater than the power flux upward into the atmosphere due to the asymptotic upward refraction. Thus, although the ducted part of the field does leak into the upper atmosphere, the attenuation by interaction with the ground is sufficient to prevent the field from extending upward without bound. A similar phenomenon has been recently found in an ocean acoustics propagation model in which some of the classical leaky modes become trapped modes if the bulk attenuation in the acoustic medium is sufficient. It will be seen that for most frequencies the modal sum is sufficient to describe the ducted part of the sound field. There are, however, certain narrow frequency bands in which the continuum integral also contributes to the ducted part of the sound field. These are the frequency bands in which the number of terms in the modal sum changes. At these transition frequencies a mode emerges from the continuous spectrum, accompanied by a sharp peak in the integrand of the continuum integral, reminiscent of the peaks one finds in the lossless case. It will be shown that at such frequencies the continuum integral has a significant contribution which can, at low altitudes, be estimated by a discrete sum over terms of the same form as the terms in the modal sum, obtained from the residue theorem of complex analysis. These additional terms will be referred to as quasi-modes.

The resulting sum of modes and possible quasi-modes results in a model for sound propagation in the nocturnal boundary layer valid up to ranges of 10 km or more in a frequency range from a few Hz, the frequencies at which modes begin to appear, up to the kHz range, where the number of modes begins to become prohibitively large, making the method unwieldy. The method is particularly powerful at low frequencies for which the number of modes is small. Throughout, the method is illustrated by explicit computations for a specific model sound speed profile. These computations are restricted to frequencies below 100 Hz so that the number of modes is small and the required numerical techniques are straightforward.

II. FORMULATION OF THE PROBLEM

Consider the propagation of sound over a plane. Let \( x \) denote the horizontal coordinates (those parallel to the plane) and let \( z \) denote the vertical coordinate (the height above the plane). Let \( t \) denote time and let \( \omega = 2\pi f \) be the angular frequency corresponding to \( f \) cycles per second.

The sound speed, \( c(z) \), is assumed to depend only on \( z \). Further, it is assumed to be downward refracting near the ground but to be upward refracting for large \( z \). Explicitly, \( c(z) \) is assumed to have an absolute maximum at some height. Above or below the absolute maximum there may be local maxima, but generally speaking the sound speed is increasing at the ground and decreasing for sufficiently large \( z \). Let \( H \) be the height above which \( c(z) \) is strictly decreasing. The rate of decrease of \( c(z) \) at high altitudes is assumed to be slow in the sense that if \( z \gg H \), then

\[
0 < -c'(z) \ll \omega.
\]

This means that the relative rate of change of \( c(z) \), \( c'(z)/c(z) \), divided by \( 2\pi \), is small compared with an acoustic wavelength \( f/c \).

Recall the model sound speed profiles plotted in Fig. 1. In the numerical results presented in this paper sound speed profile 1 is used. The analysis presented here applies equally well to sound speed profiles of the form of profiles 2 and 3, however, the numerical analysis depends on the results of Ref. 10 which would require a slight generalization to handle sound speed profiles with multiple maxima like profile 2.
Introduce the notation
\[
k(z) = \frac{\omega}{c(z)}
\] (2)
for the height dependent wave number. Note that if \(z \gg H\), then \(0 < k'(z) \ll k(z)^2\). In addition it is assumed \(k'(z)\) is bounded, which is to say that, as \(z \to \infty\), \(k(z)\) grow no faster than linearly in \(z\). Finally, the mean density, \(\rho_0(z)\), is assumed to vary so slowly with \(z\) that \(c \rho_0 \partial \rho_0 \ll 1\), so that the relative change in density over an acoustic wavelength is negligible.

The acoustic pressure, \(P(x_H, z, t)\), is assumed to be time harmonic
\[
P(x_H, z, t) = \text{Re} \, \hat{P}(x_H, z) e^{-i \omega t},
\] (3)
where the amplitude \(\hat{P}(x_H, z)\) satisfies the Helmholtz equation
\[
(\nabla^2 + k(z)^2) \hat{P}(x_H, z) = 0.
\] (4)
To model the effect of the ground an impedance condition is assumed at \(z = 0\),
\[
\left. \frac{\partial \hat{P}(x_H, z)}{\partial z} \right|_{z = 0} = -C(\omega) \hat{P}(x_H, 0),
\] (5)
where \(C(\omega)\) is related to the impedance \(Z(\omega)\) through
\[
C(\omega) = \frac{i \omega \rho_0(0)}{C(\omega)}.
\]
In general, both the real and imaginary parts of \(C(\omega)\) are nonzero. In the explicit computations presented here \(C(\omega)\) is chosen as in Eq. (24) of Ref. 10.

Equation (3) will be solved by separating variables and using an eigenfunction expansion\(^{10,13}\) in the vertical coordinate. The vertical equation may be written
\[
\left( \frac{d^2}{dz^2} + k(z)^2 - \eta \right) \psi(z) = 0,
\] (6)
where \(\eta\) is the separation parameter. The impedance condition (5) leads to
\[
\psi'(0) = -C\psi(0).
\] (7)
Consider the large \(z\) behavior of the solutions \(\psi\) of the differential equation (6). If \(z\) is large enough so that (1) is valid, then the WKB approximation\(^{9,14}\) may be used. For real valued \(\eta\) one obtains the asymptotic form
\[
\psi(z) \approx (k(z)^2 - \eta)^{-1/4} \left( A e^{i f_{10}^{c, s} \sqrt{k(z)^2 - \eta} \, dz'} + B e^{-i f_{10}^{c, s} \sqrt{k(z)^2 - \eta} \, dz'} \right)
\] (8)
as \(z \to \infty\); here \(z_0\) is large enough so that \(k(z)^2 > \eta\) for \(z > z_0\) insuring that no turning points are encountered in the integration and \(A\) and \(B\) are constants. Choose the square roots in (8) so that they have non-negative real parts. Note that for real values of \(\eta\) the solutions are asymptotically oscillatory and remain bounded as \(z \to \infty\).

If \(\text{Im} \, \eta \neq 0\), then the WKB approximation provides two types of solutions: those that grow exponentially as \(z\) increases, and those that grow exponentially as \(z\) decreases, which is to say that they decrease exponentially as \(z\) increases. These later solutions play a prominent role in the analysis presented here and will require their own notation: let, for \(\text{Im} \, \eta \neq 0\), \(\psi_+(\eta, z)\) be the solution of (6) which has the large \(z\) asymptotic form
\[
\psi_+(\eta, z) \approx (k(z)^2 - \eta)^{-1/4} \left( A e^{i f_{10}^{c, s} \sqrt{k(z)^2 - \eta} \, dz'} + B e^{-i f_{10}^{c, s} \sqrt{k(z)^2 - \eta} \, dz'} \right)
\] (9)
for some \(z_0\) sufficiently large so that \(k(z)^2 > \text{Re} \, \eta\) for \(z > z_0\).

The solutions \(\psi_+(\eta, z)\) do not generally satisfy the boundary condition (7). However, there may be a discrete set of values of \(\eta\), \(\{\eta_1, \eta_2, \ldots, \eta_N\}\), for which the solution \(\psi_+(\eta, z)\) given above does satisfy (7).\(^{13}\) The values \(\{\eta_1, \eta_2, \ldots, \eta_N\}\) will be referred to as the point spectrum. Here \(N\) may be any whole number from 0 to \(\infty\).

It follows from the large \(z\) asymptotics (8) that the \(\eta_j\) must have nonzero imaginary parts. In particular, if \(C\) were real valued, then the boundary condition (7) would make the eigenvalue problem (6) self-adjoint forcing the point spectrum to be real valued.\(^{13}\) It follows that if \(C\) is real valued, then there is no point spectrum and \(N = 0\).

Given an \(\eta_j\) in the point spectrum, the mode corresponding to \(\eta_j\) is given by
\[
\phi_j(z) = \left( \int_0^\infty \psi_j(\eta_j, z)^2 \, dz \right)^{-1/2} \psi_+(\eta_j, z).
\] (10)
This normalization is chosen so that
\[
\int_0^\infty \phi_j(z)^2 \, dz = 1.
\] (11)
Note that in Eq. (11) the function squared, \(\phi_j(z)^2\), rather than the complex modulus squared, \(|\psi(z)|^2\), is integrated. This is the normalization appropriate to the eigenfunction expansion which arises from the non-self-adjoint boundary condition (7).\(^{10,13}\) In practice it can be difficult to compute the integral in (10). An alternative formula, (A8), is presented in the Appendix.

The decrease in the magnitude of \(\psi_+(\eta_j, z)\) as \(z \to \infty\) is not quite exponential. From the asymptotic form (9) one obtains
\[
\psi_+(\eta, z) \approx (k(z)^2 - \eta)^{-1/4} \times e^{-i f_{10}^{c, s} \sqrt{k(z)^2 - \eta} \, dz'} \left[ \frac{1}{2} \text{Im} \, f_{10}^{c, s} \left( \frac{1}{2} \sqrt{k(z)^2 - \eta} \right) \, dz' \right].
\] (A8)
Note that \(\sqrt{k(z)^2 - \eta}\) is an increasing function of \(z\) for large \(z\). It follows that the first integral in the exponential function above grows more rapidly with \(z\) than a linear function of \(z\).
\[ \lim_{z \to \infty} \frac{1}{z} \int_{z_0}^{\infty} \sqrt{k(z')}^2 - \Re \eta \, dz' = \infty, \]

while the second integral grows more slowly than a linear function of \( z \),

\[ \lim_{z \to \infty} \frac{1}{z} \int_{z_0}^{\infty} \frac{1}{\sqrt{k(z')}^2 - \Re \eta} \, dz' = 0 \]

[for example, if \( k(z)^2 \) is linear for large \( z \), then this last integral \( \sim \text{const.} \, z^{1/2} \)]. Thus, \( \psi_+ (\eta, z) \) asymptotically oscillates more rapidly than a sinusoid with an amplitude that decreases more slowly than an exponential. In addition, to reach this asymptotic form, \( z \) must be large enough so that \( k(z)^2 > \Re \eta \). Thus the asymptotic decrease in the magnitude of \( \psi_+ (\eta, z) \) does not generally begin until quite high up in the atmosphere.

The set of values of \( \eta \) for which the solutions of (6) subject to (7) are not exponentially decreasing but do remain bounded as \( z \to \infty \) is called the continuous spectrum. It follows from (8) that the continuous spectrum is the entire real line. The associated solutions of (6) subject to (7) are called continuum modes. Let \( \psi_0(z) \) be the continuum mode associated to the real number \( \epsilon \). If appropriately normalized, then one has biorthonormality in the sense that

\[ \int_{0}^{\infty} \psi_0(z) \psi_0^*(z') \, dz = \delta(\epsilon - \epsilon'), \]

\[ \int_{0}^{\infty} \phi_j(z) \phi_k(z) \, dz = \delta_{jk}, \]

and

\[ \int_{0}^{\infty} \psi_0(z) \phi_j(z) \, dz = 0, \]

and completeness\(^{15,13}\)

\[ \delta(z - z') = \sum_{j=1}^{\infty} \phi_j(z) \phi_j(z') + \int_{-\infty}^{\infty} \psi_0(z) \psi_0(z') \, d\epsilon. \]

(12)

In the Appendix it is shown how one may normalize \( \psi_0(z) \). Let \( \psi_0(C, \eta, z) \) be the solution of (6) satisfying the boundary condition (7) normalized by \( \psi_0(C, \eta, 0) = 1 \) and let \( \psi_+(\eta, z) \) be as above. Let \( \epsilon \) be a real number and let

\[ F_\pm(C, \epsilon) = \psi_0^*(\epsilon \mp i0^+, 0) + \sqrt{\pi} \psi_+(\epsilon \mp i0^+, 0). \]

(13)

Here the prime indicates the derivative with respect to \( z \) and \( \epsilon \pm i0^+ \) indicates the limiting value as \( \pm \text{Im } \epsilon \) approaches zero while remaining positive. One then has

\[ \psi_0(z) = \frac{1}{\sqrt{\pi} F_-(C, \epsilon) F_+(C, \epsilon)} \psi_-(C, \epsilon, z). \]

(14)

One can now express the pressure amplitude \( \hat{P}(x_H, z) \) in an eigenfunction expansion with respect to the eigenfunctions \( \phi_j \) and \( \psi_+ \). One obtains

\[ \hat{P}(x_H, z) = \sum_{j=1}^{\infty} p_j(x_H) \phi_j(z) + \int_{-\infty}^{\infty} p(\epsilon, x_H) \psi_+(z) \, d\epsilon \]

(15)

with expansion coefficients

\[ p_j(x_H) = \int_{0}^{\infty} \phi_j(z) \hat{P}(x_H, z) \, dz \]

and

\[ p(\epsilon, x_H) = \int_{0}^{\infty} \psi_+(z) \hat{P}(x_H, z) \, dz. \]

(16)

(17)

The expansion coefficients satisfy the two dimensional Helmholtz equations

\[ (\nabla_H^2 + \eta) p_j(x_H) = 0 \]

(18)

and

\[ (\nabla_H^2 + \epsilon) p(\epsilon, x_H) = 0 \]

(19)

for \( x_H \) in source free regions of space. Here

\[ \nabla_H = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}. \]

Given appropriate boundary conditions or sources, (18) and (19) are straightforward to solve.

III. FINDING THE POINT SPECTRUM

Once the spectrum is known, producing the associated modes is a straightforward numerical exercise. In principle, given \( \eta \), one solves (6) subject to (7). For the continuum modes this procedure works well since the continuous spectrum is known \( a \ priori \) and consists of real values of \( \eta \) for which the solutions of (6) are bounded.

The chief difficulty is in finding the point spectrum. Further, once the point spectrum is known, solving (6) subject to (7) is numerically unstable for complex valued \( \eta \) because of the exponential behavior of solutions to (6). It is preferable to solve (6) subject to the condition that

\[ \frac{\psi'(z)}{\psi(z)} = \begin{cases} -i \sqrt{k(z)^2 - \eta} & \text{if } \text{Im } \eta > 0, \\ i \sqrt{k(z)^2 - \eta} & \text{if } \text{Im } \eta < 0, \end{cases} \]

(20)

for large \( z \). This forces the solution to have the asymptotic form (9) and is numerically stable.\(^{9,14}\)

A. Solving the eigenvalue equation

Consider the eigenvalue problem given by (6) and (7). Recall the solution, \( \psi_+(\eta, z) \), of (6) chosen to have the asymptotic form (9) for large \( z \). If \( \psi_+(\eta, z) \) satisfies (7), then \( \eta \) is in the point spectrum and \( \psi_+(\eta, z) \) is, up to normalization, the eigenfunction corresponding to \( \eta \). Dividing both sides of (7) by \( \psi(0) \) it follows that the point spectrum is precisely the set of all \( \eta \) with \( \text{Im } \eta \neq 0 \) for which

\[ \frac{\psi'_+(\eta, 0)}{\psi_+(\eta, 0)} + C = 0. \]

(21)
To specify $\psi_+(\eta, z)$ completely it is necessary to fix $z_0$ in (9). However, with the eigenvalue condition written in the form (21) it becomes clear that specifying $z_0$ is not necessary since only the logarithmic derivative of $\psi_+(\eta, z)$ is required. One need only find any solution of (6) which satisfies (20) for large $z$, track its logarithmic derivative back to $z=0$, and then substitute into (21). In principle one could use this procedure to search for the roots of (21).

In practice such procedures are prone to failure. Not only is it difficult to search blindly for roots in the complex plane, but such a search is further complicated here by the exponential growth and decay of the solutions to (6) which can make numerical procedures unstable. To produce stable and robust algorithms for finding the point spectra one must have some means to analytically estimate the roots with sufficient accuracy to allow standard numerical routines (the Newton–Raphson method, for example) to converge rapidly from the estimate to the exact value.

The technique used in Ref. 10 to find the point spectrum, that of treating $\text{Im} \, \text{C}$ as a perturbation, cannot be applied here since if $\text{Im} \, \text{C} = 0$, then, as pointed out in the previous section, there is no point spectrum to perturb. Instead decompose $k(\zeta)^2$ as follows:

$$k(\zeta)^2 = k_0^2 + v_1(\zeta) + v_2(\zeta),$$

where

$$k_0 = k(H)$$

is the minimum value of $k$,

$$v_1(\zeta) = \begin{cases} k(\zeta)^2 - k_0^2 & \text{if } \zeta \leq H, \\ 0 & \text{if } \zeta \geq H, \end{cases}$$

and

$$v_2(\zeta) = \begin{cases} 0 & \text{if } \zeta \leq H, \\ k(\zeta)^2 - k_0^2 & \text{if } \zeta \geq H. \end{cases}$$

In Fig. 2 the decomposition (22) is depicted for the vertical sound speed profile $1$ from Fig. 1.

Since $k(\zeta)$ is slowly varying for large $\zeta$ it follows that $v_2(\zeta)$ is as well. Write Eq. (6) in the form

$$\frac{d^2}{dz^2} + v_1(z) + v_2(z) - (\eta - k_0^2) \psi(z) = 0.$$ 

It will be shown that the point spectrum can be obtained by treating $v_2(\zeta)$ as a perturbation.

Let

$$C = A + i B$$

be the decomposition of $C$ into its real and imaginary parts. Let $L_0$ denote the differential operator obtained from

$$L_0 = \frac{d^2}{dz^2} + v_1$$

by imposing (7) with $B = 0$,

$$\psi'(0) = -A \psi(0),$$

let $L_1$ denote the differential operator obtained from

$$L_1 = \frac{d^2}{dz^2} + v_1 + v_2,$$

by imposing (7) with $B \neq 0$, and let $L$ denote the differential operator obtained from

$$L = \frac{d^2}{dz^2} + v_1 + v_2$$

by again imposing (7) with $B \neq 0$.

The operator $L_0$ is self-adjoint and numerous techniques exist for finding its spectrum. The point spectrum of $L_0$ consists of $M$ real, positive eigenvalues $\eta_1^{(0)}, \ldots, \eta_M^{(0)}$. The continuous spectrum of $L_0$ is the negative real half-line, $(-\infty, 0)$. The techniques of Ref. 10 can then be applied to produce the point spectrum of $L_1$ from that of $L_0$; the continuous spectrum is unchanged. One finds that the point spectrum of $L_1$ consists again of $M$ values, $\eta_1^{(1)}, \ldots, \eta_M^{(1)}$, all with positive real and imaginary parts. Let $\phi_j^{(1)}(\zeta)$ be the eigenfunction [normalized as in (11)] of $L_1$ corresponding to $\eta_j^{(1)}$.

It remains to be shown that the point spectrum of $L = L_1 + v_2$ can be obtained from that of $L_1$. Note that $v_2(z)$ is a rather singular perturbation of $L_1$. In particular, the continuous spectra of $L$ and $L_1$ are drastically different. As discussed in the previous section the continuous spectrum of $L$ is the entire real line, $(-\infty, \infty)$, regardless of how slowly $v_2$ increases with height. Further, in the self-adjoint case, $B$
$=0$, it was seen that $L_1$ has point spectrum while $L$ does not. This self-adjoint case has been studied in detail in a classic paper by Titchmarsh. What remains of the point spectrum in this case are poles, called resonances, in the analytic continuation in $\varepsilon$ of the continuum eigenfunctions. It is clear from (13) that the resonances are solutions of the eigenvalue equation (21) for which the corresponding solution of (6) is not decreasing with height; it follows that the imaginary part of the resonance must be negative. Titchmarsh shows that the real part of a resonance is well approximated by formal perturbation theory while the imaginary part is smaller than any power of $v_2(z)$ (which is to say that it is exponentially small in the size of the perturbation).

An approach similar to that of Titchmarsh can be taken here in this non-self-adjoint context. Perturbation theory can be adapted in a straightforward manner to the bi-orthogonality relevant to the current situation. One finds that, to leading order,

$$\eta_j = \eta_j^{(1)} + \int_0^\infty \phi_j^{(1)}(z)^2 v_2(z) \, dz + \cdots.$$  \hspace{1cm} (23)

In practice the approximation (23) has proven to be sufficiently accurate to provide an initial point from which standard numerical routines converge rapidly to the solution of the eigenvalue equation (21).

As discussed in the previous section, the mode $\phi_j$ corresponding to $\eta_j$ extends much higher into the atmosphere than $\phi_j^{(1)}$, although in most cases with very small amplitude. In particular $\phi_j$ has slightly less contact with the ground than $\phi_j^{(1)}$ and thus is attenuated slightly less by the ground. One thus expects that

$$\operatorname{Im} \eta_j < \operatorname{Im} \eta_j^{(1)},$$

just as in the self-adjoint case. This expectation is borne out by the numerical results presented below. Indeed, it is possible that the perturbation causes the imaginary part of $\eta_j$ to be negative; as pointed out above, in the self-adjoint case all of the eigenvalues acquire a negative imaginary part. If the imaginary part becomes negative, then the corresponding solution of (6) and (7) grows exponentially as $z \to \infty$ and the eigenvalue becomes a resonance and ceases to be in the point spectrum. It follows that $N \leq M$: the number of eigenvalues in the point spectrum of $L$ is less than or equal to the number of eigenvalues in the point spectrum of $L_0$, the operator relevant to a strictly downward refracting atmosphere with no attenuation at the ground.

Note that for a given $v_2$ the magnitude of the perturbation is determined by the vertical extent of the unperturbed mode. If the mode is concentrated near the ground and has insignificant magnitude for $z>H$, then the perturbation will be insignificant as well. If the magnitude of the mode decreases only slowly with height and is still appreciable for $z>H$, then the perturbation can be significant, perhaps sufficiently so that $\eta_j$ is not in the point spectrum. Since one has

$$|\phi_j^{(1)}(z)| \sim \text{const} \, e^{-\sqrt{\eta_j^{(1)}-k_0^2} z}$$

for large $z$ the perturbation is significant only if $\eta_j^{(0)}$, the corresponding eigenvalue of $L_0$, is sufficiently close to $k_0^2$.

B. Numerical methods

The form (21) for the eigenvalue equation is not the most convenient for numerical analysis since it has poles at the values of $\eta$ for which the Dirichlet problem $\psi_+ (\eta,0) = 0$ has a solution. An eigenvalue equation must be found which has no poles in the region of interest. A convenient choice is

$$\psi_+ (\eta,0) + C \psi_+ (\eta,0) - i \psi_+ (\eta,0) = 0.$$  \hspace{1cm} (24)

This form still has poles, at the zeros of $\psi_+ (\eta,0)$, but they are in the lower half-plane sufficiently distant from the real line. To compute $\psi_+ (\eta,0)$ fix some value of $z_0$ which is large enough for the asymptotic form (20) to be accurate and then solve (6), using any convenient numerical solver, subject to (20) imposed as a boundary condition at $z_0$.

In practice the estimate (23) is found to be quite accurate, except for those occasions in which $\eta_j^{(1)}$ is too close to $k_0^2$. Even in these cases, however, it has been found that (23) is sufficiently accurate to provide a starting point from which the Newton–Raphson method applied to (24) converges rapidly, even when $\operatorname{Im} \eta_j$ turns out to be negative. In the examples worked in this paper the bisection method was employed to find the point spectrum of $L_0$. The perturbative estimate of Ref. 10 was then used as a starting point for a Newton–Raphson solver applied to the eigenvalue equation for $L_1$.

The dispersion obtained from applying these methods to sound speed profile 1 from Fig. 1 is plotted in Fig. 3. Plotted are the real and imaginary parts of $k_{ij} = \sqrt{k_0^2 + \eta_j}$ as a function of frequency. The real part is normalized by $k_0$. The real
part is referred to as the horizontal wave number and the imaginary part as the attenuation constant. In Fig. 4 the dispersion of mode 5 for the asymptotically constant approximation and the full asymptotically upward refracting profile are compared. In the upper plot $\sqrt{k_0^2 + \eta_5} - k_0$ and $\sqrt{k_0^2 + \eta_5^{(1)}} - k_0$ are tracked in the complex plane. In the lower plot the attenuation coefficients $\text{Im} \sqrt{k_0^2 + \eta_5}$ and $\text{Im} \sqrt{k_0^2 + \eta_5^{(1)}}$ are plotted as functions of frequency. Note that the asymptotically constant approximation can have more modes than in the asymptotically upward refracting case. In both cases the mode vanishes at the frequency at which the attenuation constant becomes zero; however, in the asymptotically constant approximation the horizontal wave number approaches $k_0$ while in the asymptotically upward refracting case the horizontal wave number remains slightly larger than $k_0$.

Once the eigenvalues are found, the corresponding modes are obtained by computing $\psi_\eta(\eta, z)$ and then normalizing according to (11). Computing the normalization from (10) can be difficult, however, since $\psi_\eta(\eta, z)$ can decrease very slowly as $z \to \infty$. Instead (A8) is used.

In Fig. 5 the modes at 50 Hz for sound speed profile 1 are plotted. They are labeled by their corresponding horizontal wave number and attenuation coefficient. In Fig. 6 the same plots are found, but for 41.6 Hz. This is a transition frequency at which the number of modes increases from three to four (see Fig. 3). Accordingly, the fourth mode has horizontal wave number very close to $k_0$.

Note that the fourth mode at 41.6 Hz extends high up into the atmosphere oscillating before it eventually slowly decays in accordance with (9). In fact, all of the modes have this behavior, although except in the exceptional case of a transition frequency at which the number of modes changes, the oscillations at high altitude have negligibly small amplitude. This is demonstrated in Fig. 7 in which the fourth mode at 50 Hz is plotted at high altitudes. Physically this long tail at high amplitudes represents the leaking of acoustic energy from the duct up into the atmosphere. The eventual decay of the mode with height reflects the fact that the rate at which energy is lost into the ground is much greater than the rate at which it can leak into the upper atmosphere.

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**FIG. 4.** The modal dispersion relations for the fifth mode in the asymptotically constant case and the asymptotically upward refracting case are compared near a transition frequency. Plotted are the attenuation constants, $\text{Im} \sqrt{k_0^2 + \eta_5}$, versus the horizontal wave numbers, $\sqrt{k_0^2 + \eta_5} - k_0$, and then the attenuation constants versus frequency.

**FIG. 5.** The modes at 50 Hz for sound speed profile 1.

**FIG. 6.** The modes at the transition frequency 41.6 Hz for sound speed profile 1.
IV. THE MODES OF THE CONTINUOUS SPECTRUM

Consider the modes of the continuous spectrum \( \psi_u(z) \) given by (14). The dependence of \( \psi_u(z) \) on \( \epsilon \) may be divided into three general types according to whether \( \epsilon \) is somewhat less than \( k_0^2 \), comparable to \( k_0^2 \), or somewhat greater than \( k_0^2 \). In Fig. 8 continuum modes at 50 Hz for sound speed profile 1 displaying the three types of behavior are plotted.

When \( \epsilon \) is somewhat less than \( k_0^2 \) the behavior of the modes is oscillatory over the entire domain. These are the modes which contain that part of the sound field which propagates from low altitudes up into the atmosphere or the reverse. Note that in this case the magnitude of a mode does not change dramatically with increasing height.

When \( \epsilon \) is somewhat greater than \( k_0^2 \) there are one or more turning points at height \( z \) for which \( k(z)^2 = \epsilon \). For sound speed profile 1 there is always one turning point, to be denoted \( \tau_1(\epsilon) \) for purposes of discussion, above the sound speed inversion. If \( \epsilon < k(0)^2 \), there is a second, to be denoted \( \tau_0(\epsilon) \), below the sound speed inversion. If \( \epsilon = k(0)^2 \), let \( \tau_0(\epsilon) = 0 \). Above the turning points, for \( z > \tau_1(\epsilon) \), the modes are oscillatory with behavior given, for sufficiently large \( z \), by (8). Between the two turning points, for \( \tau_0(\epsilon) < z < \tau_1(\epsilon) \), the modes grow exponentially with height.\(^{14}\) It follows that in this case the modes have negligible amplitude near the ground. Their amplitude increases rapidly with height up to the vicinity of the turning point \( \tau_1(\epsilon) \), above which they oscillate with gradually decreasing amplitude. These modes contain that part of the sound field radiated from a source above the sound speed inversion which is refracted upwards away from the ground, not penetrating below the inversion.

When \( \epsilon \) is comparable to \( k_0^2 \) two factors can complicate the behavior of the continuum modes: there is the coalescence of the two turning points discussed in the last paragraph and there is the proximity of the point spectrum. At high altitudes the continuum modes will be oscillatory. At low altitudes the coalescence of the turning points means that no simple and general behavior can be expected. The behavior of the modes reflects the details of the low altitude form of the sound speed profile.

The proximity of the point spectrum produces sharp peaks in the low altitude behavior of those modes for which \( \epsilon \) is in the vicinity of the real part of an eigenvalue in the point spectrum. This is seen clearly in (14): the eigenvalue condition (21) is precisely the condition that \( F_u(C, \eta) = 0 \) so that \( \psi_u \) has poles where the eigenvalue condition is satisfied. The effect of these poles is tempered by the fact that if \( \epsilon \) is sufficiently greater than \( k_0^2 \), then the corresponding mode is still exponentially decreasing with decreasing \( z \) below the inversion.

This behavior is demonstrated in Figs. 9(a) and 9(b) in which the continuum mode amplitude on the ground, \( |\psi_u(0)| \), for sound speed profile 1 is plotted as a function of \( \epsilon \) for 50, 52, 54, 56, and 58 Hz. Both \( |\psi_u(0)| \) and \( \log |\psi_u(0)| \) are plotted in Figs. 9(a) and 9(b), respectively. Note that the peaks in \( |\psi_u(0)| \) are sharpest for transition frequencies near which the number of modes changes. Indeed these are the frequencies for which one of the eigenvalues in the point spectrum can be arbitrarily close to the continuous spectrum. The plot of \( \log |\psi_u(0)| \) shows that all of the eigenvalues in the point spectrum show up as peaks in the continuous spectrum; however, unless their real parts are sufficiently close to \( k_0^2 \), the exponential behavior of the mode overwhelms the peak.
To extract the contribution to integrals such as those that appear in (15) from these peaks it is useful to know the residues of the poles that cause them. Consider

$$
\psi_e(z_1)\psi_e(z_2) = -\frac{1}{\pi F_+^1(\epsilon)F_-^1(\epsilon)} \psi_-(C,\epsilon,z_1)\psi_-(C,\epsilon,z_2).
$$

(25)

Let $\eta_1, \ldots, \eta_M$ be the solutions of the eigenvalue condition (7) which were obtained in Sec. III by perturbing off of the asymptotically constant case. Recall that, depending on the signs of their imaginary parts, not all of the $\eta_j$ are necessarily in the point spectrum and thus not all represent true ducted modes. $F_\pm(\epsilon)$ are defined for $\pm \text{Im } \epsilon \neq 0$, but have straightforward analytic continuations into regions with $\pm \text{Im } \epsilon < 0$. One then has $F_+(\eta_j) = 0$, regardless of the sign of $\text{Im } \eta_j$. Thus, $\psi_e(z_1)\psi_e(z_2)$ has poles, as a function of $\epsilon$, at the $\eta_j$. When $\text{Im } \eta_j > 0$, so that $\eta_j$ is in fact in the point spectrum, then $-2\pi i$ times the residues at these poles gives the modes, $\phi_j$, corresponding to $\eta_j$:

$$
\phi_j(z_1)\phi_j(z_2) = \frac{2i}{F_-(\eta_j)F_+^1(\epsilon)} \psi_-(C,\eta_j,z_1)\psi_-(C,\eta_j,z_2)
$$

(26)

[This follows from the fact that $\psi_+(\epsilon+i0^+,0)\psi_-(C,\eta_j,z) = \psi_+(\epsilon+i0^+,z)$ and from (A4) and (A5); see the Appendix]. When $\text{Im } \eta_j < 0$, introduce the notion of the quasi-mode $\phi_j$ defined by (26).

V. LONG RANGE PROPAGATION FROM A POINT SOURCE

Consider the sound field produced by a point source at $x_H = 0$, $z = z_0$. The solution to (4) for a point source at $x_H = 0$, $z = z_0$ is proportional to the Green’s function

$$
G(\omega,x_H,z_0) = \sum_{j=1}^{N} G_H(\eta_j,x_H)\phi_j(z_0)\phi_j(z)
$$

$$
+ \int_{-\infty}^{\infty} G_H(\epsilon,x_H)\psi_\epsilon(z_0)\psi_\epsilon(z)\, d\epsilon,
$$

(27)

where $G_H(\epsilon,x_H)$ is the free space Green’s function satisfying $(\nabla_x^2 + \epsilon)G_H(\epsilon,x_H) = \delta(x_H)$. In polar coordinates $r, \theta$, $G_H$ is independent of $\theta$. One finds

$$
G_H(\epsilon,x_H) = -\frac{i}{4} H_0^{(1)}(\sqrt{r}),
$$

(28)

where $H_0$ is the Hankel function.

Once the modes $\phi_j$ have been computed, computing the contribution to (27) from the modal sum is straightforward. The contribution from the integral over the continuum modes must be computed numerically. This computation can be intensive. However, if one is interested in the sound field near the ground and far from the source, then it can be expected that the integral over the continuum spectrum is insignificant so that one need only compute the modal sum.

For the sound speed profiles considered here it will be shown that it is indeed the case for most frequencies that the continuous spectrum is insignificant for low altitudes at long ranges. The exceptions are those narrow bands of transition frequencies in which the number of modes changes. As discussed in the previous section, for these frequencies the emerging mode manifests itself as a sharp peak in the continuum mode amplitudes.

The long range approximation to the Green’s function $G$ is obtained by replacing the Hankel function in (28) by its large argument asymptotic form,

$$
G_H(\epsilon,r) \approx \sqrt{-\frac{i}{8\pi \sqrt{r}}} e^{i\sqrt{\epsilon r}},
$$

and dropping the part of the integral in (27) which does not propagate horizontally, that is, that part of the integral with $\epsilon \ll 0$. One obtains

$$
G(\omega,r,z_0,z) \approx -\sqrt{-\frac{i}{8\pi r}} \sum_{j=1}^{N} \eta_j^{-1/4} e^{i\sqrt{\eta_j r}} \phi_j(z_0)\phi_j(z)
$$

$$
+ \int_{0}^{\infty} e^{-1/4} e^{i\sqrt{\epsilon r}} \psi_\epsilon(z_0)\psi_\epsilon(z)\, d\epsilon.
$$

(29)

Note that for fixed $z$ the continuum modes $\psi_\epsilon(z)$ go to zero exponentially in $\epsilon$ as $\epsilon \rightarrow -\infty$ so that the improper integral in (29) converges rapidly. As mentioned above, performing these integrals is computationally intensive. However, it is generally expected to be unnecessary to compute the continuum integral since at long ranges the term $e^{i\sqrt{\epsilon r}}$ oscillates rapidly compared to the variations of $e^{-1/4}\psi_\epsilon(z_0)\psi_\epsilon(z)$, causing the integral to be negligible.

An exception to this rule is the case in which $\psi_\epsilon(z_0)\psi_\epsilon(z)$ is sharply peaked. As discussed in the previous section this occurs near those frequencies at which the num-
number of modes changes. The contribution from these peaks can be extracted by deforming the integration contour in (29) below the real line to the contour \( \Gamma \) depicted in Fig. 10. The term \( e^{i\tau} \) grows exponentially with range for \( \Im \eta > 0 \), however, the contour \( \Gamma \) need only descend below the real line a distance large compared to the \( |\Im \eta| \) and thus may be chosen so that this exponential growth is insignificant for ranges \( r \ll 1/|\Im \eta| \) (here \( \eta_N \) is used here since the imaginary parts of the \( \eta_j \) typically decrease with increasing \( j \)). In the examples considered here \( |\Im \eta| \leq 10^{-4} \) per meter so that for \( r \leq 10 \) km this deformation can be made. In addition, the analytic continuations of the continuum modes to complex \( \epsilon \) will grow exponentially as \( z \to \infty \). However, for low altitudes, for ranges up to 10 km, and for \( \Gamma \) sufficiently far from the poles the contribution from the resulting integral is negligible.

If any poles are encountered in making this deformation, their residues (26) must be subtracted from the integral. The long range contribution from the continuum integral is contained entirely in these residues. Ignoring the continuum integral over the deformed contour \( \Gamma \) one obtains as an approximation to (29)

\[
G(\omega, r, z_0, z) \approx -\sqrt{\frac{i}{8\pi r}} \sum_{j=1}^{N} \eta_j e^{i\sqrt{\eta_j} \phi_j(z_0) \phi_j(z)} + \sum_{j=N+1}^{M} \eta_j e^{i\sqrt{\eta_j} \phi_j(z_0) \phi_j(z)}
\]

where, for \( j > N \), the quasi-modes defined by (26) are used.

To check the accuracy of the approximation (30) the continuum integral has been computed using a trapezoidal rule approximation with variable step size (a finer step size is used in the region near \( \Re \epsilon - k_0^2 \) where the integrand is more rapidly varying). The integral is performed along the deformed contour \( \Gamma \). In Fig. 11, ground to ground propagation, \( z_0 = z = 0 \), is studied. Ten times the logarithm of the magnitude of the Green’s function times range, \( |x_H|G(\omega, x_H, 0, z) \), is displayed. This quantity is the magnitude in dB of the sound field produced by a point source relative to the field that would be produced by a point source of equal strength at a distance \( |x_H| \) from the source in an empty homogeneous space. The full field is compared to the individual contributions from the modal sum and the continuum integral. Frequencies 50 Hz, which is not a transition frequency, and 54 Hz, which is a transition frequency, are shown in Figs. 11(a) and 11(b), respectively. The small fluctuations in the computation of the continuum integral at short ranges is the result of numerical noise associated with the vertical discretization of the continuum modes as \( \epsilon \to 0 \). The sum of modes and quasi-modes alone, (30), is seen to be an excellent approximation to the full field.

VI. CONCLUSIONS

A vertical modal expansion has been produced for the model commonly used to describe outdoor sound propagation on calm nights over flat ground. In this model the sound field is refracted downwards near the ground, upwards at high altitudes and is attenuated by its interaction with the ground. The full sound field is given by a sum over ducted modes plus an integral over continuum modes as given in (15). The ducted modes decrease exponentially at high altitudes while the continuum modes oscillate with very slowly decreasing amplitude at high altitudes.

Except for a discrete set of narrow frequency bands in which the number of ducted modes changes, at long ranges the sound field near the ground is well approximated by the modal sum alone. For these exceptional frequencies the continuum integral has a long range contribution to the sound field. At low altitudes, however, the contribution from the continuum integral is well approximated by a sum over quasi-modes. The quasi-modal sum has the same form as the modal sum except that the quasi-modes increase exponentially at high altitudes.

The modes and quasi-modes, and the associated wave numbers and attenuation constants, can be obtained by perturbing about a sound speed profile which is constant at high altitudes, reducing the problem to the type solved in Ref. 10. Given the modes and quasi-modes, the result, presented in (30) for propagation from a point source, is an easily evalu-
APPENDIX: NORMALIZING THE MODES

In this appendix it is shown how, under rather general conditions, the ducted and continuum modes can be normalized. Normalizing the ducted modes by computing the normalization integral (10) can be difficult since the modes can extend very high up into the atmosphere. Instead, formulas are developed which do not depend on the specific large $z$ asymptotic form of the modes. The approach used is a generalization of a technique developed to normalize continuum modes in atomic physics. 21

Let $\eta \in \mathbb{R}$ and consider the Green’s function,

$$
\left( \frac{d^2}{dz^2} + v(z) - \eta \right) g(C, \eta, z_1, z_2) = \delta(z_1 - z_2),
$$

g(C, \eta, z_1, z_2) = g(C, \eta, z_2, z_1) \quad \text{and} \\
\left. \frac{\partial}{\partial z_1} g(C, \eta, z_1, z_2) \right|_{z_1 = 0} = -C g(C, \eta, 0, z_2).
$$

Here $v(z) = k(z)^2$ is positive. One has

$$
g(C, \eta, z_1, z_2) = \frac{1}{W(\psi_-(C, \eta, \cdot), \psi_+(\eta, \cdot))} \times \psi_-(C, \eta, z_1) \psi_+(\eta, z_2), \tag{A1}
$$

where $z_{<}(z_{>)}$ is the smaller (larger) of $z_1$ and $z_2$. $\psi_-(C, \eta, z)$ and $\psi_+(\eta, z)$ are solutions of the differential equation

$$
\left( \frac{d^2}{dz^2} + v(z) - \eta \right) \psi(z) = 0,
$$

$\psi_-(C, \eta, z)$ satisfies the initial conditions

$$
\psi'_-(C, \eta, 0) = -C \psi_-(C, \eta, 0), \tag{A2}
$$

$\psi_+(\eta, z)$, defined only for $\eta \in \mathbb{R}$, is the exponentially decreasing solution, and $W$ denotes the Wronskian. Note that it follows from (A2) and (A3) that

$$
W(\psi_-(C, \eta, \cdot), \psi_+(\eta, \cdot)) = - (\psi'_+(\eta, 0) + C \psi_+(\eta, 0)). \tag{A4}
$$

Further, using the asymptotic form (9), one finds that

$$
W(\psi_+(e + i0^+, \cdot), \psi_+(e - i0^+, \cdot)) = -2i. \tag{A5}
$$

In addition to (A1) one has the eigenfunction expansion,

$$
g(C, \eta, z_1, z_2) = \sum_{j=1}^{\infty} \frac{1}{\eta_j - \eta} \phi_j(z_1) \phi_j(z_2)
$$

$$
+ \int_{-\infty}^{\infty} \frac{1}{\eta - \eta} \psi_+(\eta) \psi_+(\eta) \ d \epsilon,
$$

which follows directly from the definitions of $\phi_j$ and $\psi_+$ and from the completeness relation (12). Using the fact that when $\eta = \eta_j$

$$
\psi_+(\eta_j, 0) \psi_-(C, \eta_j, z) = \psi_+(\eta_j, z) \tag{A7}
$$

and comparing the residues of $g(C, \eta, z_1, z_2)$ at the $\eta_j$ computed from (A1) and (A6), one finds that

$$
\phi_j(z_1) \phi_j(z_2) = \left. \frac{\psi_+(\eta, 0)}{(d/d\eta) W(\psi_-(C, \eta, \cdot), \psi_+(\eta, \cdot))} \right|_{\eta = \eta_j} \times \psi_-(C, \eta_j, z_1) \psi_-(C, \eta_j, z_2).
$$

Using (A7) the equality

$$
\int_{0}^{\infty} \psi_+(\eta_j, z)^2 \ d z = -\psi_+(\eta_j, 0)
$$

$$
\times \left. (\frac{d}{d\eta}) \left( \psi'_+(\eta, 0) + C \psi_+(\eta, 0) \right) \right|_{\eta = \eta_j} \tag{A8}
$$

for the normalization of the eigenfunctions in the point spectrum follows.

As regards the normalization of the eigenfunctions in the continuous spectrum, note that

$$
g(C, \epsilon_0 + i0^+, z_1, z_2) - g(C, \epsilon_0 - i0^+, z_1, z_2)
$$

$$
= \lim_{\delta \downarrow 0} \int_{-\infty}^{\infty} \frac{2i \delta}{(\epsilon - \epsilon_0)^2 + \delta^2} \psi_+(\epsilon_0) \psi_+(\epsilon) \ d \epsilon
$$

$$
= 2 \pi i \psi_+ \left( \frac{1}{\epsilon} \psi_+(\epsilon_0) \right) \psi_+(\epsilon_0). \tag{A9}
$$

Let $\epsilon \in \mathbb{R}$ and note that
Here the fact that $\psi_\pm(C, \eta, z)$ is continuous in $\eta$ as $\eta$ crosses the real line while $\psi_\pm(\eta, z)$ is not has been used. Equations (14) and (13) now follow from (A4) and (A5).