05.12.22

Black-hole perturbation theory and quasinormal modes

General references:

- S. Chandrasekhar, "The mathematical theory of black holes".
- E. Berti, V. Cardoso, A. O. Starinets, "Quasinormal modes of black holes and black branes". 0905.2975
- K. D. Kokkotas & B. G. Schmidt, "Quasinormal modes of stars and black holes". gr-qc/9909058
- H.-P. Nollert, "Quasinormal modes: the characteristic 'sound' of black holes and neutron stars". CQG 16 (1999) R159 - R216.
- A. Pound & B. Wardell, "Black hole perturbation theory and gravitational self-force". 2101.04592.

LECTURE 1.

Start by studying a test field propagating on a black hole spacetime. For simplicity:

$$\Box \Phi = 0 \tag{1}$$

$$ds^2 = -f(r)\,dt^2 + f^{-1}(r)\,dr^2 + r^2\,(d\theta^2 + \sin^2\theta\,d\phi^2) \tag{2}$$

$$f = 1 - 2M/r \quad ; \quad r_H = 2M \quad , \quad c = G = 1. \tag{3}$$

In general,

$$\Box \Phi = \frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu \Phi\right) = 0 \qquad g := \det(g_{\mu\nu}) \tag{4}$$

Try to <u>separate variables</u>, assuming harmonic time dependence, as:

$$\Phi(t, r, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \frac{1}{r}\,\psi(r)\,Y_{\ell m}(\theta, \phi)\,e^{-i\omega t} \tag{5}$$

where

$Y_{\ell m}(\theta,\phi)$ = spherical harmonics, $P_\ell(\theta)\,e^{im\phi}$

$\ell = 0, 1, \dots$

$-|\ell| \leq m \leq |\ell|$.

Substitute (5) in (4). Use:

$$\sqrt{-g} = r^2 \sin\theta \quad ; \quad \frac{1}{\sin\theta}\,\partial_\theta(\sin\theta\,\partial_\theta P_\ell) - \frac{m^2}{\sin^2\theta}\,P_\ell = -\ell(\ell+1)\,P_\ell$$

to find:

$$\left(1 - \frac{2M}{r}\right)^2 \frac{d^2\psi}{dr^2} + \frac{2M}{r^2}\left(1 - \frac{2M}{r}\right)\frac{d\psi}{dr} + \left\{\omega^2 - \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3}\right]\right\}\psi = 0 \tag{6}$$

We can eliminate the $d\psi/dr$ by a smart choice of radial coordinate:

$$\frac{dr}{dx} := 1 - \frac{2M}{r} \quad ; \quad x = r + 2M \log\left(\frac{r}{2M} - 1\right) \tag{7}$$

(tortoise coordinate)

In terms of x:

$$\frac{d^2\psi}{dx^2} + \left[\omega^2 - V(r)\right]\psi = 0 \tag{8}$$

$$V := \left(1 - \frac{2M}{r}\right)\cdot\left[\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3}\right] \tag{9}$$

Note:

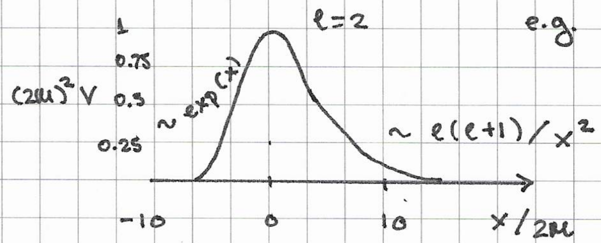
$r = 2M \Rightarrow x = -\infty$

$r = \infty \Rightarrow x = \infty$

That is we reduce the problem to a scattering problem, where a wave interacts with the potential V. [cf. Eq (9)].

|| Exercise: fill the steps leading to Eq. (8).

|| Exercise: produce plots of V as a function of x e.g.



Note: no m-dependence in Eq (8) and (9).

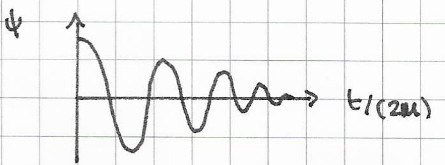
More generally, for bosonic massless fields we have:

$$V = \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} + (1-s^2)\frac{2M}{r^3}\right] \qquad (10)$$

$$s = \begin{cases} 0 & (\text{scalar}) \\ 1 & (\text{vector}) \\ 2 & (\text{tensor}) \end{cases}$$

1970: Vishveshwara scattered $s=2$ waves (time-evolution of $\psi_{s=2}$). Saw "characteristic" 'ringdown' at late times.

1971: Press identifies ringdown with free oscillation modes of the black hole.



What determines the free oscillations? Boundary conditions.

Note: as $x \to \pm\infty$ (or $r \to \frac{+\infty}{2M}$), Eq. (8) becomes:

$$\frac{d^2\psi}{dx^2} + \omega^2 \psi = 0$$

then

$$\psi \sim e^{\pm i\omega x} \qquad x \to \pm\infty$$

At $x \to -\infty$ we don't want waves coming out of the horizon:

$$\therefore \quad \psi \sim e^{-i\omega t}\, e^{-i x t} \qquad x \to -\infty$$

At $x \to +\infty$ we don't want waves coming from infinite (we want the "free" oscillation modes)

$$\therefore \quad \psi \sim e^{-i\omega t}\, e^{+i x t} \qquad x \to +\infty$$

Boundary value problem

$$\frac{d^2\psi}{dx^2} + [\omega^2 - V]\psi = 0 \qquad \begin{array}{ll} \psi \sim e^{-i\omega(t+x)} & x \to -\infty \\ \psi \sim e^{-i\omega(t-x)} & x \to +\infty \end{array}$$

fixes values of ω. These are the $\underbrace{\text{quasinormal}}_{\substack{\downarrow \\ \text{because we have a } \underline{\text{dissipative}} \text{ problem.}}}$ modes. (QNMs)

$$\underbrace{\omega \in \mathbb{R}}_{\text{normal mode}} \rightsquigarrow \underbrace{\omega \in \mathbb{C}}_{\text{quasinormal mode.}}$$

The QNMs are labelled as

$$\omega_{\ell m n}$$

ℓm angular indices

n overtone. $n=0$ is the <u>least damped</u> QNM of given ℓ and m.

Since $\omega \in \mathbb{C}$:

$$\omega = \omega_R + i\omega_I = \omega_R - i|\omega_I|$$

Some numbers? In units of $r_H/c = 1$ $(r_H = 2GM/c^2)$. We have for $\ell = 2$ and a Schwarzschild BH:

$$\omega \sim 0.747 - i\, 0.178 \tag{11}$$

Restoring units:

$$f = \frac{\omega_R}{2\pi} \sim 0.747 \frac{c}{r_H} \sim 12 \text{ kHz} \left(\frac{M_\odot}{M}\right)$$

$$\tau := 1/|\omega_I| \sim \frac{1}{0.178}\frac{r_H}{c} \sim 5.5 \times 10^{-5} \text{ s} \left(\frac{M}{M_\odot}\right) \tag{12}$$

- for $M = 10\, M_\odot$: $f \sim 1$ kHz $\tau \sim$ tens of ms
- for $M = 10^6\, M_\odot$: $f \sim 10$ mHz $\tau \sim 1$ minute.

EXERCISE: use the fitting formulas E5 - E6 in gr-qc/0512160 to check dependency of (12) with spin.

Let's explore V further. Define:

$$Q(x) := \omega^2 - V(r) \tag{13}$$

What are the extrema of Q? For brevity:

$$V = \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3}(1-s^2)\right]$$

$$:= \left(1 - \frac{2M}{r}\right)\left[\frac{\Lambda}{r^2} + \frac{2M\sigma}{r^3}\right] \qquad \begin{aligned}\sigma &= 1 - s^2\\ \Lambda &= \ell(\ell+1)\end{aligned} \tag{14}$$

Then:

$$\frac{dQ}{dx} = \frac{dr}{dx}\frac{dQ}{dr} = -\left(1 - \frac{2M}{r}\right)\frac{dV}{dr}$$

$$= \left(1 - \frac{2M}{r}\right)\left[\frac{2\Lambda}{r^3}\left(1 - \frac{3M}{r}\right) + \frac{2M\sigma}{r^4}\left(3 - \frac{8M}{r}\right)\right]$$

Set to zero. $r_0 = 2M$ is one. Another can be found at:

$$r_0 = \frac{3}{2}\frac{M}{\Lambda}\left\{\Lambda - \sigma + \left[\sigma^2 + \Lambda^2 + \frac{14}{9}\sigma\Lambda\right]\right\} \tag{15}$$

Notice that as $\ell \to \infty$ (i.e., $\Lambda \to \infty$) we have

$$r_0 = 3M \qquad \text{(at geometrical optics limit).} \tag{16}$$

What is value? Note that σ - dep. disappeared.

Relationship between QNMs and null geodesics as $\ell \to \infty$?

Goebel (1972).

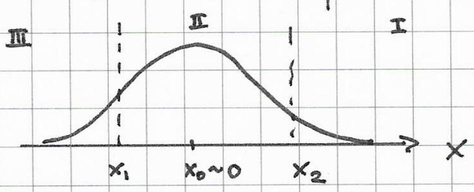
As $\ell \to \infty$, V becomes more peaked.

$\rightsquigarrow$ use WKB analysis. (Schutz and Will '85).

We will give an overview of the work. Consider:

$$(8) + (13) \rightsquigarrow \frac{d^2\psi}{dx^2} + Q(x)\psi = 0 \tag{17}$$

$Q \sim$ const. as $x \to \pm\infty$. Split:



Then:

$$\nu + \frac{1}{2} = -i \frac{Q_0}{(2Q_0'')^{1/2}}$$

$\uparrow$

now n (it was 'quantized' by BCs).

so

$$n + \frac{1}{2} = -i \frac{Q_0}{(2Q_0'')^{1/2}}$$

$$\frac{Q_0}{(2Q_0'')^{1/2}} = i\left(n + \frac{1}{2}\right) \qquad n = 0, 1, 2, \ldots \qquad (26)$$

We know that:

$Q_0 = \omega^2 - V(r_0)$; r_0 given by Eq. (15).

Q_0'' can be computed analytically (we know V and r_0)

We can solve for ω with this info:

QNM $\sim$ probe peak of effective potential

Take $\ell \to \infty$

$r_0 \approx 3M$

$$V \sim \left(1 - \frac{2M}{r}\right)\frac{\ell^2}{r^2} \quad \therefore \quad V_0'' \sim -\frac{2\ell^2}{81 M^4}$$

$$Q_0 \sim \omega^2 - \frac{\ell^2}{27M^2} \qquad Q_0'' \sim -\frac{2\ell^2}{81M^4}$$

From (26)

$$\omega^2 - \frac{\ell^2}{27M^2} = i\,(2Q_0'')^{1/2}\left(n + \frac{1}{2}\right)$$

$$\omega = \underbrace{\Omega_c}_{\frac{1}{3\sqrt{3}}\frac{1}{M}}\,\ell - i\left(n + \frac{1}{2}\right)\underbrace{|\lambda_0|}_{\frac{1}{3\sqrt{3}\,M}}$$

For $d \geq 4$

$$\frac{\omega}{\Omega_c} = \ell - i\sqrt{d-3}\left(n + \frac{1}{2}\right)$$

LECTURE II.

⇝ DO QUICK RECAP OF YESTERDAY

⇝ TODAY : REGGE - WHEELER - ZERILLI FORMALISM.

LEAVER'S METHOD

Yesterday, I just "postulated" the so-called REGGE - WHEELER equation

$$\frac{d^2\phi}{dx^2} + [\omega - V(r)]\phi = 0$$

$$V = \left(1 - \frac{2M}{r}\right) \cdot \left[\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right] . \qquad (1)$$

Today, we will derive it.

Start with coordinate split as:

$$x^\mu = (t, r, \theta, \phi) \rightsquigarrow x^\rho = (z^A, y^a)$$

$$z^A = \{t, r\} \qquad y^a = \{\theta, \phi\}$$

"slipt" angular direction from the rest.

Introduce two - sphere metric:

$$ds^2 = \gamma_{AB}\, dz^A\, dz^B = (d\theta^2 + \sin^2\theta\, d\phi^2)$$

i.e.,

$$\gamma_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \qquad ; \qquad \nabla_A := \text{covariant derivative on sphere.}$$

and def. the Levi - Civita symbol ε_{AB} :

$$\varepsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{or its tensor} \quad \varepsilon_{AB} = \sqrt{\gamma} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \sin\theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

What is this good for? Well, we can write the (scalar) spherical harmonics as:

$$\gamma^{AB}\, \nabla_A \nabla_B Y_{\ell m}(y^a) = -\ell(\ell+1)\, Y_{\ell m}(y^a)$$

<u>IDEA:</u> EXTEND this to vectors and tensors.

Regge-Wheeler showed that the 10 components of a metric perturbation $h_{\mu\nu}$:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$$

can be split in the following way when doing $(\theta, \phi) \to (\pi - \theta, \pi + \phi)$ [parity trans.] :

scalar, vector

$$h_{\mu\nu} = \begin{pmatrix} \boxed{h_{ab}} & \boxed{h_{a}{}^{B}} \\ \boxed{h^{A}{}_{b}} & \boxed{h_{AB}} \end{pmatrix}$$

vector, tensor

We can decompose each of this blocks according to <u>how</u> they transf. under parity. E.g.:

$$Y_{\ell m}(\pi - \theta, \pi + \phi) = (-1)^{\ell} \, Y_{\ell m}(\theta, \phi)$$

If we pick a $(-1)^{\ell}$ we say the quantity has <u>even</u> parity.
If we pick a $(-1)^{\ell+1}$ we say " " " <u>odd</u> parity.

Just a $Y_{\ell m}$ form a basis to expand scalar functions on the sphere, we can find vector and tensor basis to expand ~~ten~~ vector and tensor fields.

even ~ polar
odd ~ axial

VECTOR

Polar vector harmonics: $Y_A^{\ell m} := \nabla_A Y^{\ell m} = (\partial_\theta Y^{\ell m}, \partial_\phi Y^{\ell m})$

Axial " " : $S_A^{\ell m} := \epsilon_{AC} \, \gamma^{BC} \, \nabla_B Y^{\ell m}$

$$= \left(\frac{1}{\sin\theta} \partial_\phi Y^{\ell m}, - \sin\theta \, \partial_\theta Y^{\ell m} \right)$$

TENSOR

Polar rank-two tensor harmonics:

$$Z_{AB}^{\ell m} := \nabla_A \nabla_B Y^{\ell m} + \frac{1}{2} \ell(\ell+1) \gamma_{AB} Y^{\ell m}$$

$$= \frac{1}{2} \begin{pmatrix} W^{\ell m} & X^{\ell m} \\ X^{\ell m} & -\sin^2\theta \, W^{\ell m} \end{pmatrix}$$

Axial " " :

$$S_{AB}^{\ell m} := \frac{1}{2} \left(\nabla_B S_A^{\ell m} + \nabla_A S_B^{\ell m} \right) =$$

$$S_{AB}^{\ell m} = \frac{1}{2}\begin{pmatrix} \sin^{-1}\theta \; X^{\ell m} & -\sin\theta \; W^{\ell m} \\ -\sin\theta \; W^{\ell m} & -\sin\theta \; X^{\ell m} \end{pmatrix}$$

where

$$X^{\ell m} := 2\left(\partial_{\theta\phi} Y^{\ell m} - \cot\theta \, \partial_\phi Y^{\ell m} \right)$$

$$W^{\ell m} := \partial_{\theta\theta} Y^{\ell m} - \cot\theta \, \partial_\theta Y^{\ell m} - \sin^{-2}\theta \, \partial_{\phi\phi} Y^{\ell m}$$

Phew! But, now, we can write:

$$h_{ab}(y^a, z^A) \sim \bar{h}_{ab}^{\ell m}(y^a) \, Y^{\ell m}(z^A)$$

$$h_{aA}(y^a, z^A) \sim h^{pol}_{a,\ell m}(y^a) \, Y_A^{\ell m}(z^A) + h^{ax}_{a,\ell m}(y^a) \, S_A^{\ell m}(z^A)$$

$$h_{AB}(y^a, z^A) \sim r^2 \left\{ K_{\ell m}(y^a) \, \gamma_{AB} Y^{\ell m}(z^A) + G_{\ell m}(y^a) \, \nabla_A \nabla_B Y^{\ell m}(z^A) \right\} + 2 \, h_{\ell m}(y^a) \, S_{AB}(z^A).$$

These harmonics also are orthonormal after def. an inner product:

$$\langle f, g \rangle := \int d\Omega \; f^* g \quad ; \quad \text{e.g.,} \quad \langle Y^{\ell m}, Y^{\ell' m'} \rangle = \delta^{\ell\ell'} \delta^{mm'}$$

$$\langle f_A, g_A \rangle := \int d\Omega \; \gamma^{AB} f_A^* \, g_A$$

$$\langle f_{AB}, g_{AB} \rangle := \int d\Omega \; \gamma^{AC} \gamma^{BD} f_{AB}^* \, g_{CD}.$$

In general we could define

$$(R, T) = \int R_{\mu\nu} T_{\lambda\rho} \, \eta^{\mu\lambda} \eta^{\nu\rho} \, d\Omega$$

$$\eta^{\mu\nu} = \mathrm{diag}\left(-1, 1, r^{-2}, r^{-2}\sin^{-2}\theta \right)$$

Regge-Wheeler used this originally.

Now, we ~~can~~ have some gauge freedom associated to coordinate transf. The Regge-Wheeler gauge corresponds to:

$$h^{pol}_{a,\ell m} = G_{\ell m} = h_{\ell m} = 0$$

such that now:

$$h_{\mu\nu} = \begin{pmatrix} \bar{h}_{ab,\ell m} \, Y^{\ell m} & h^{ax}_{a,\ell m} \, S_A^{\ell m} \\ h^{ax}_{a,\ell m} \, S_A^{\ell m} & r^2 K_{\ell m} \, \gamma_{AB} \, Y^{\ell m} \end{pmatrix}$$

with

$h^{ax} = h$ from now on

$$\bar{h}_{ab,\ell m} = \begin{pmatrix} H_{0,\ell m} & H_{1,\ell m} \\ H_{1,\ell m} & H_{2,\ell m} \end{pmatrix}$$

3 scalar functions.

$$\bar{h}_{a,\ell m} = \begin{pmatrix} h_{0,\ell m} \\ h_{1,\ell m} \end{pmatrix}$$

drop ℓm now

In total:

2 axial vars. (h_0, h_1)

4 polar vars. (H_0, H_1, H_2, K)

$10 - 4 = 6$

2 "ax" 4 "pol"

Now we subs. this into the linearized Einstein's equation:

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$$

$$G_{\mu\nu}[\tilde{g}_{\mu\nu}] \sim \underbrace{G_{\mu\nu}[g_{\mu\nu}]}_{=0} + \underbrace{\delta G_{\mu\nu}[h_{\mu\nu}]}_{\text{want to compute this}} + \mathcal{O}(h^2)$$

For Ricci-flat backgrounds:

$$\delta G_{\mu\nu}[h_{\mu\nu}] = 0 \rightsquigarrow -\frac{1}{2} h_{\mu\nu;\alpha}{}^{;\alpha} + f_{(\mu;\nu)} - R_{\alpha\mu\beta\nu} h^{\alpha\beta}$$
$$- \frac{1}{2}(h^{\alpha\beta} g_{\alpha\beta})_{;\mu;\nu} - \frac{1}{2} g_{\mu\nu} \left(f_\lambda{}^{;\lambda} - h_{;\lambda}{}^{;\lambda}\right)$$
$$= 0 \qquad ; \quad f_\mu := h_{\mu\alpha}{}^{;\alpha}$$

Now, assume all functions are

$$f(x^a) = f(t,r) = f(r) \, e^{-i\omega t}$$

Let's focus on the axial case only. From the $t\phi$, $r\phi$, $\theta\phi$ components of Einstein's equations we find $r\phi$ and $\theta\phi$ yield

$$2ir^2\omega h_0 + [(r-2M)(2-\ell-\ell^2) + r^3\omega^2]\, h_1 - ir^3\omega h_0' = 0$$

$$ir^3\omega h_0 + (r-2M)[\,2Mh_1 + (r-2M)\, r\, h_1'\,] = 0$$

$' := d/dr$ — the $t\phi$ eq. is redundant. (it can be obtained from the two above).

Solve for h_0:

$$h_0 = \frac{i}{\omega r^2}\left(1-\frac{2M}{r}\right)[\,2Mh_1 + r(r-2M)\, h_1'\,]$$

then subs. in ( ) & and def:

$$\psi^{RW} := \frac{1}{r}\left(1-\frac{2M}{r}\right) h_1$$

by introducing tortoise coordinates [LECTURE I] $dr/dx = 1-\frac{2M}{r}$

we find:

EXERCISE FILL THE STEPS. ⇝

$$\frac{d^2\psi^{RW}}{dx^2} + \left\{\omega^2 - \left(1-\frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} - \frac{6M}{r^3}\right]\right\}\psi^{RW} = 0$$

Note: once know ψ^{RW} we can re-obtain h_0 and h_1, RW's!

h_1 is trivial, while $h_0 = \frac{i}{\omega}\frac{d}{dx}\left(r\psi^{RW}\right)$.

Note: the polar case is much more difficult. Heroic work by Zerilli 1970's. See SAGO - NAKANO - SASAKI PRD 67, 104 017 (2013)

For "executive summary + small misprints fixed.

Note: for the "Zerilli eq.":

$$\frac{d^2\psi^Z}{dx^2} + \left\{\omega^2 - \left(1-\frac{2M}{r}\right)\frac{1}{r^3}\left[\frac{2\lambda^2(\lambda+1)r^3 + 6\lambda^2 Mr + 18\lambda M^2 r + 18M^3}{(\lambda r+3M)^2}\right]\right\}\psi^Z = 0$$

$\lambda := (\ell-1)(\ell+2)/2$

Very diff. looking potentials, but quite similar in shape! Two master variables $\rightsquigarrow$ 2 DOF of grav. field

EXERCISE: plot V^{RW} and V^{Z}, and check this.

Remarkable property: QNMs of both eqs. are the same! i.e., isospectrality.

CHANDRASEKHAR '80

See also: GLAMPEDAKIS, JOHNSON, KENNEFICK PRD 96, 024036 (2017)

in general broken in non-GR theories.

Note: what about Kerr? We need new tools. ("CURVATURE perturbation"). Ultimately, we find the TEUKOLSKY EQ.

ApJ 185, 635 (1973)

$$\Delta^2 \frac{d}{dr}\left(\frac{1}{\Delta}\frac{dR}{dr}\right) - V_T R = 0 \qquad V_T := -\frac{K^2 + 4i(r-M)K}{\Delta} + 8i\omega r + \lambda$$

a "MIRACLE" it exists at all (~~red~~ read opening paragraph of sec. III B of GLAMPEDAKIS CQG 22, S605 (2005))

eigenvalue of ${}_{-2}S_{lm}$

WITH:

$$\lambda = E_{lm} + a^2\omega^2 - 2am\omega$$

$$K = (r^2 + a^2)\omega - ma$$

$$\Delta = r^2 - 2Mr + a^2$$

For Schwarzschild, we have the BARDEEN - PRESS EQ. JMP 14 7 (1973).

* LEAVER'S SOLUTION TO THE QNM PROBLEM

Our favorite eq:

$$\frac{d^2\psi^s}{dx^2} + \left[\omega^2 - V(r)\right]\psi^s = 0$$

$$V = \left(1 - \frac{2M}{r}\right)\cdot\left[\frac{\ell(\ell+1)}{r^2} + (1-s^2)\frac{2M}{r^3}\right]$$

Use $2M = 1$ from now on. Go from x to r, using

$$\frac{dr}{dx} = \left(1 - \frac{1}{r}\right) = \frac{r(r-1)}{r^2}.$$

then:

$$r(r-1)\frac{d^2\psi^s}{dr^2} + \frac{d\psi^s}{dr} - \left[\ell(\ell+1) - \frac{s^2-1}{r} - \omega^2\frac{r^3}{r-1}\right]\psi^s = 0 \qquad (*)$$

same eq as before, just written in a diff. way.

Make a series expansion:

$$(**) \qquad \psi = (r-1)^{-i\omega}\, r^{2i\omega}\, e^{i\omega(r-1)} \sum_{k=0}^{\infty} a_k \left(\frac{r-1}{r}\right)^k.$$

The prefactor has the QNM BCs by construction. will drop (s) now

* $r \to 2M$, i.e., $r \to 1$,

$$\psi \sim e^{-i\omega x}$$
$$\simeq e^{-i\omega[r + \log(r-1)]}$$
$$\sim e^{-i\omega \log(r-1)}$$

$x = r + \log(r-1)$

$$\psi \sim (r-1)^{-i\omega}$$

first term in $(*)$

* $r \to \infty$,

$$\psi \sim e^{+i\omega x}$$
$$\simeq e^{+i\omega[r + \log(r-1)]}$$
$$\sim e^{+i\omega[r + \log r]}$$
$$\psi \sim r^{i\omega} e^{i\omega r}$$

other prefactors in $(*)$.

Subs. $(**)$ in $(*)$ to find a three term recursion relation for a's:

$$\alpha_0 a_1 + \beta_0 a_0 = 0$$

$$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0 \qquad n = 1, 2, \ldots$$

where α_n, β_n, and γ_n depend on ω, ℓ, and s.

$$\alpha_n = n^2 + (2 - 2i\omega)n - 2i\omega + 1$$

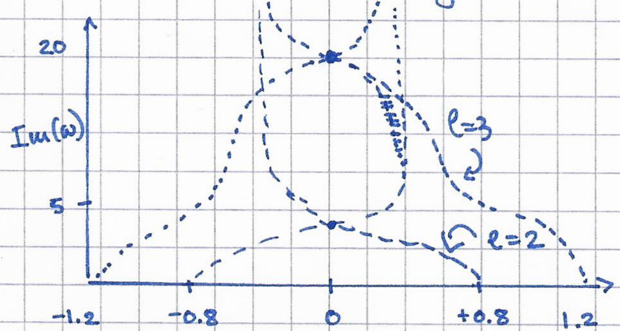
$$\beta_n = -[2n^2 + (2 - 8i\omega)n - 8\omega^2 - 4i\omega + \ell(\ell+1) + 1 - s^2]$$

$$\gamma_n = n^2 - 4in\omega - 4\omega^2 - s^2$$

A theorem by PINCHERLE says that when the continued fraction:

$$0 = \beta_0 - \cfrac{\alpha_0\gamma_1}{\beta_1 - \cfrac{\alpha_1\gamma_2}{\beta_2 - \cfrac{\alpha_2\gamma_3}{\beta_3 - \ddots}}} = \beta_0 - \frac{\alpha_0\gamma_1}{\beta_1 -}\,\frac{\alpha_1\gamma_2}{\beta_2 -}\cdots$$

is satisfied the series converges. This ~~serie~~ CF will be zero, given ℓ and s, only for certain ω's: the QNMs.



- algebraically special modes ω

$$\omega_R \sim 0 \qquad \omega_I \simeq \frac{1}{6}\frac{(\ell+2)!}{(\ell-2)!}$$

after LEAVER, PRSLA 402, 285 (1985)