EXPERIMENTAL ERRORS

Experimental errors are generally divided in to two classes: statistical errors and systematic errors. **Statistical errors** are caused by random processes during your measurement, like electronic noise and other backgrounds. The size of the measurement sample will limit the measurement and will be part of your statistical error. There are also random reading errors, and other noise sources. Statistical errors will fluctuate up and down about the mean $A \pm \Delta A_{stat}$. **Sytematic errors** on a measurement are generally attributed to a some measurement bias. The electronic offset in a measuring device may be the cause. A one-sided bias in reading an instrument, or an assumption one makes about performing a measurement e.g. (temperature doesn't matter) or other reasons may cause a systematic error. The systematic error generally moves the measurement up or down by an amount, $A + \Delta A_{sys}$ *or* $A - \Delta A_{sys}$.

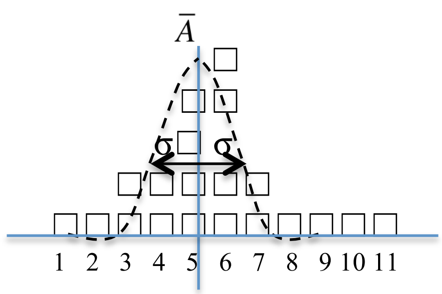
The Gaussian Distribution and it's relation to error

We generally assume that measurement of a quantity *A* will fluctuate around its **mean** or **average** value $\bar{A}$ with **standard deviation** σ_A, . Assuming we have taken a number of measurements *N* then we define $\bar{A}$ and σ_A as:

$$\bar{A} = \frac{1}{N}\sum_{i=1}^{N} A_i \text{ (mean)} \qquad \sigma_A = \sqrt{\frac{\sum_{i=1}^{N}(A_i - \bar{A})^2}{N-1}} \text{ (standard deviation)}$$

Because $\bar{A}$ is an estimate and not the true average we must divide by *N-1* instead of *N*. We have lost one degree of freedom. It is important to note that the error on the average $\bar{A}$ is the quantity $\sigma_{\bar{A}} = \sigma_A / \sqrt{N}$. σ_A reflects the width of the distribution of data points, but $\sigma_{\bar{A}}$ reflects the uncertainty in the mean. Consider the *N*=21 datapoints.
A_i = 1,2,3,3,4,4,5,5,5,5,6,6,6,6,6,7,7,8,9,10,11

$\bar{A}$=5.66 σ_A=2.50 $\sigma_{\bar{A}}$=0.54



In the limit as $N \to \infty$ the distribution of data around its mean will approach a Gaussian distribution (dotted line). 68% of the data lies within +/-1σ of the mean.

Gaussian Distribution

$$G(x, \sigma_x, \bar{x}) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-(x-\bar{x})^2/2\sigma_x^2}$$

Propagation of Error

Once the statistical and systematic errors on a measurement are known we must have a way of combining and propagating errors.

Consider the measurement of a quantity *F(A,B,C,...)* which depends on quantities A,B,C.
Assume we know $\bar{A}\pm\Delta A,\ \bar{B}\pm\Delta B,\ \bar{C}\pm\Delta C$ where $\Delta A=\sigma_{\bar{A}},\Delta B=\sigma_{\bar{B}},\ \Delta C=\sigma_{\bar{C}}$,
and the values A,B,C are fluctuating in a Gaussian manner. F will also fluctuate about its mean value $\bar{F}$. We can use the power of calculus to see the effect. Expand F in a Taylor series about $\bar{A},\bar{B},\bar{C}$:

$$\Delta F^2=(F(A,B,C)-\bar{F}(\bar{A},\bar{B},\bar{C}))^2= \qquad \textit{change in F about its mean} - \textit{squared}$$

$$\left(\left(\frac{\partial F}{\partial A}\right)\Delta A\pm\left(\frac{\partial F}{\partial B}\right)\Delta B\pm\left(\frac{\partial F}{\partial C}\right)\Delta C\right)^2$$

$$=\left(\frac{\partial F}{\partial A}\right)^2\Delta A^2+\left(\frac{\partial F}{\partial B}\right)^2\Delta B^2+\left(\frac{\partial F}{\partial C}\right)^2\Delta C^2 \quad \textit{quadratic term}$$

$$+2\left(\frac{\partial F}{\partial A}\cdot\frac{\partial F}{\partial B}\right)\Delta A\cdot\Delta B+2\left(\frac{\partial F}{\partial A}\cdot\frac{\partial F}{\partial C}\right)\Delta A\cdot\Delta C+2\left(\frac{\partial F}{\partial B}\cdot\frac{\partial F}{\partial C}\right)\Delta B\cdot\Delta C \quad \textit{covariance overlap term}$$

Ignoring the *covariance term* we have (we are assuming, A, B, C are independent.)

$$\Delta F^2=\left(\frac{\partial F}{\partial A}\right)^2\Delta A^2+\left(\frac{\partial F}{\partial B}\right)^2\Delta B^2+\left(\frac{\partial F}{\partial C}\right)^2\Delta C^2$$

Some Simple Rules

1) $F=A\pm B\pm C,$	$\Delta F=\sqrt{\Delta A^2+\Delta B^2+\Delta C^2}$	*addition*
2) $F=A\cdot B\cdot C$	$\frac{\Delta F}{F}=\sqrt{\left(\frac{\Delta A}{A}\right)^2+\left(\frac{\Delta B}{B}\right)^2+\left(\frac{\Delta C}{C}\right)^2}$	*products*
3) $F=A/BC$	$\frac{\Delta F}{F}=\sqrt{\left(\frac{\Delta A}{A}\right)^2+\left(\frac{\Delta B}{B}\right)^2+\left(\frac{\Delta C}{C}\right)^2}$	*division*
4) $F=nA$	$\Delta F=n\Delta A$	$n=\textit{constant}$
5) $F=\ln(A)$	$\Delta F=\frac{\Delta A}{A}$	*logorythm*
5) $F=e^{nA}$	$\frac{\Delta F}{F}=n\Delta A$	*exponentiaton*

Example: Find the error ΔF in the function $F=(A+B)\cdot C$

$$F=A\cdot C+B\cdot C \ \textit{and}\ \partial F/\partial A=C,\ \partial F/\partial B=C,\ \partial F/\partial C=A+B=F$$

$$\Delta F^2=(\partial F/\partial A)^2\Delta A^2+(\partial F/\partial B)^2\Delta B^2+(\partial F/\partial C)^2\Delta C^2$$

$$\Delta F=\sqrt{C^2(\Delta A^2+\Delta B^2)+F^2\Delta C^2}$$

Covariance Term

If variable *A,B,C* are not independent (e.g. jiggling *A* affects *B*) then the covariance term may be important to keep in the error. If we plot $A_{i's}$ vs $B_{i's}$ for a set of measurements we may be able to see a dependence.

Random dependence(low correlation, r=0)

```
  x  x x   x x
 x   x   x  x  x  x
 x  x   x  x x  x x
x  x   xx  x   x x
   x     x   x   x
x x   x x    x   x
       x   x        x
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Linear dependence(strong correlation, r=1)

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                x    x x
             x   x   x
        x x   x x
     x   x x
  x   x
    x
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The sample covariance S_{AB} is defined $\sigma_{AB}^2 = \dfrac{\sum_{i=1}^{N}(A_i-\bar{A})(B_i-\bar{B})}{N-1}$ $(\mathrm{cov}\,ariance)$

The sample variances are $\sigma_A^2 = \dfrac{\sum_{i=1}^{N}(A_i-\bar{A})^2}{N-1}$ $\sigma_B^2 = \dfrac{\sum_{i=1}^{N}(B_i-\bar{B})^2}{N-1}$ $(variances)$

The correlation coefficient (degree of correlation) is defined as $r = \sigma_{AB}/\sigma_A\sigma_B$, with r=0 indicating low correlation and r=1 indicating strong correlation.

Weighted Average

Consider averaging measurements that are not equally weighted or have different errors. A weighted average may be used.

$$\bar{x}_W = \frac{\sum_{i=1}^{N} x_i \cdot w_i}{\sum_{i=1}^{N} w_i} \quad \textit{weighted average with } w_i \textit{ the weightes of each measurent}$$

If we assume the weights are given by the inverse square of the measurement error $\boxed{w_i = 1/\sigma^2{}_i}$

$$\boxed{\bar{x}_W = \frac{\sum_{i=1}^{N} x_i \cdot \frac{1}{\sigma_i^2}}{\sum_{i=1}^{N} \frac{1}{\sigma_i^2}}} \quad \textit{and} \quad \mathrm{var}\,iance \quad \boxed{\sigma_{\bar{x}_W}^2 = \frac{1}{\sum_{i=1}^{N} \frac{1}{\sigma_i^2}}}$$

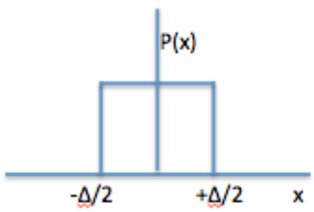
Error on a Single Measurement

Often we take a single measurement of a quantity. Our error formulas are not useful. What is the error? We must do our best to estimate the error by close inspection of the measuring apparatus for example. A digital voltmeter may fluctuate at the +/-1mV level or we may read a scale to +/-1mm as examples. For a binary type measurement with step or least count Δ (between two values of x) we can consider the error to be σ=+-1/2 Δ. Or even smaller! A careful treatment of probablity distributions and r.m.s error would predict:

$$\sigma^2_{rms} = <(x-\bar{x})^2> = \frac{1}{\Delta}\int_{-\Delta/2}^{+\Delta/2}(x-\bar{x})^2 P(x)dx \qquad P(x) = \begin{matrix} 1 & |x| \le \Delta/2 \\ 0 & |x| > \Delta/2 \end{matrix}$$

$$\sigma^2_{rms} = <(x-\bar{x})^2> = \frac{1}{\Delta}\int_{-\Delta/2}^{+\Delta/2} x^2\, dx \qquad with\ \bar{x} = 0$$

$$\boxed{\sigma^2_{rms} = \frac{1}{\Delta}\int_{-\Delta/2}^{+\Delta/2} x^2\, dx} \quad Can\ you\ show\ that \quad \boxed{\sigma = \frac{\Delta}{\sqrt{12}}}\ ?$$



Measuement occurs with equal probablility between +/-Δ/2

Phys 417 Homework - 0

1) A student makes five measurements of the electric charge e =15,17,18,14,16 x 10^{-20} C.
(a) Find the best estimate (mean) for e and the r.m.s. uncertainty σ_e. (b) Find the error on the mean.

2) 20 measurements are taken of a quantity x giving <x> = 25 +/- 5. What is the probability that the 21st measurement is above 30 (x≥30)? Assume a gaussian distribution of measurements.

3) Two independent variables are measured, x=6.0+/1.0 and y=3.0+/-1.0 . Use these values to calculate q = (x+y)/(x+1) and the uncertainty σ_q by propagation of error.

4) Prove expressions 4), 5) ,6) above.

5) Find the correlation coefficient r between student homework ($A_{i's}$) and exams scores ($B_{i's}$)

N=	1	2	3	4	5	6	7	8	9	10
A =	90	60	45	100	15	23	52	30	71	88
B =	90	71	65	100	45	60	75	85	100	80

6) Two measurements are taken of the speed of light. c = 3.01|/-0.02 x10^8 m/s and c = 2.96+/-0.05 x10^8 m/s. Find the weighted average of the two measurements and the error on the weighted average.

7) Show that $\sigma = \frac{\Delta}{\sqrt{12}}$ for a single measurement with least count Δ.