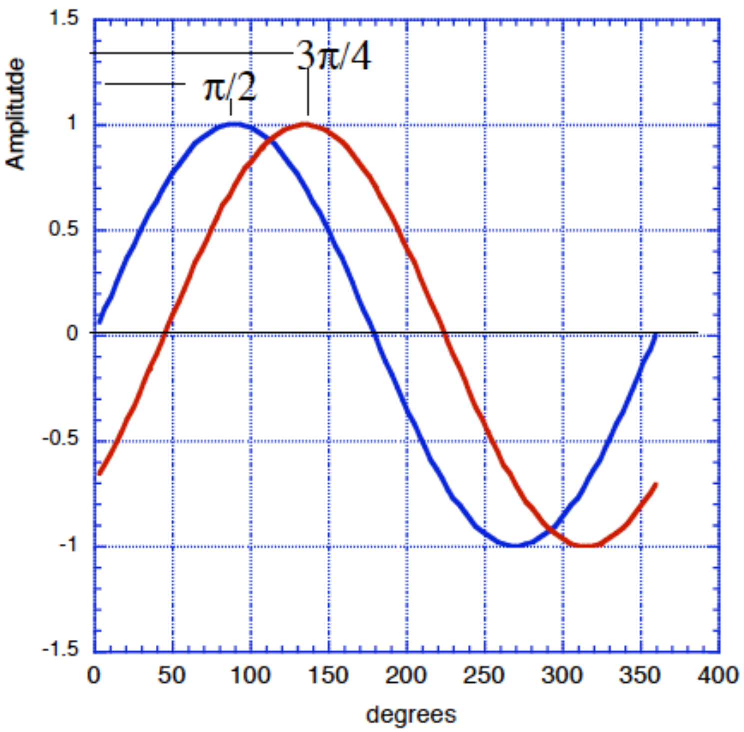
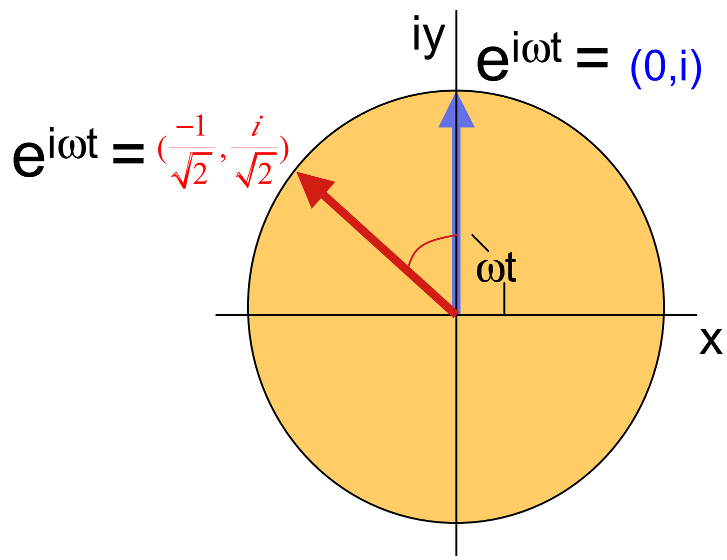
Chapter 3 Alternating Current Circuits I

- AC Voltage and Current - Phasors
- RMS Voltage and Current
- Reactance and Impedance
- High Pass and Low Pass filters
- RLC Resonance Circuits

Waves and Phasors



Phasor



$y(t) = A\cos(\omega t-\phi)$

$\omega = 2\pi f$ where $f=1/T$

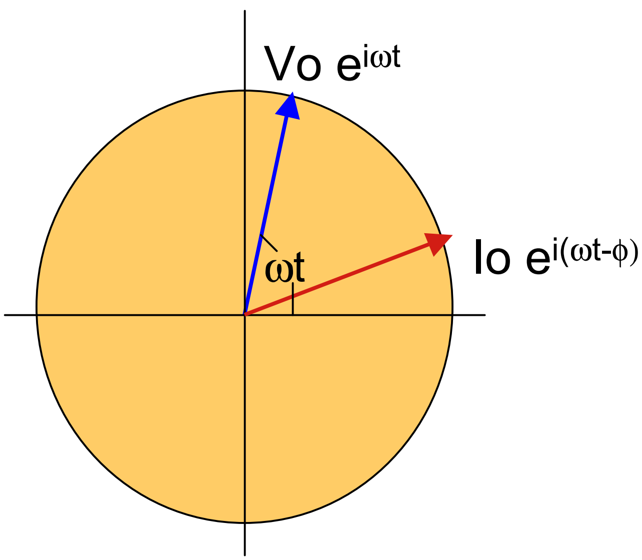
$z = (x, iy) = |z|e^{i\pi/2} = (\cos(\pi/2),\ i\sin(\pi/2)) = (0,i)$

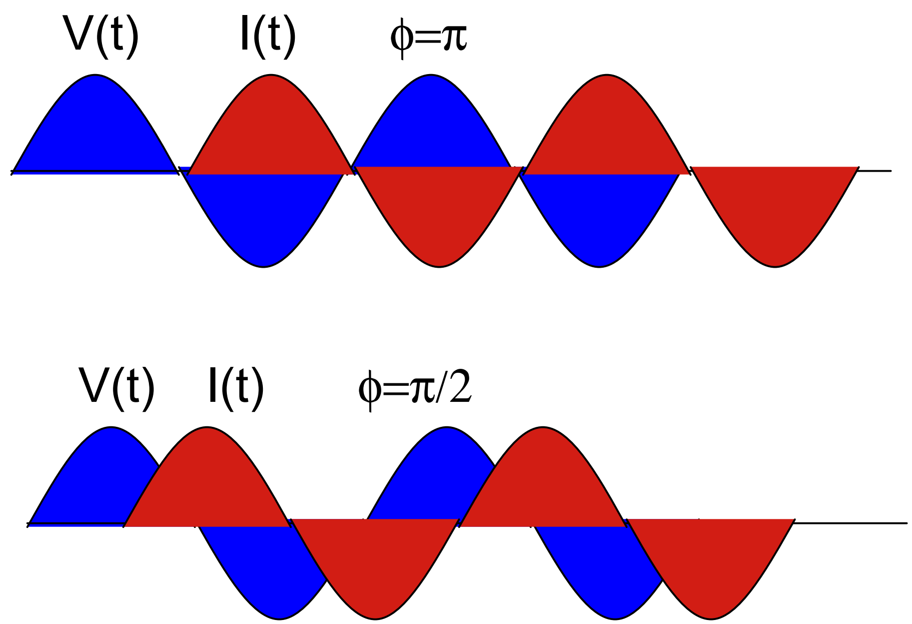
$z = (x, iy) = |z|e^{i\pi/4} = (\cos(3\pi/4),\ i\sin(3\pi/4)) = \left(\frac{-1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right)$

Phase Lag in an AC Circuit

In a general AC circuit (RLC) we have to consider that the voltage and current may be out of phase due to the circuit elements.

$$V(t) = V_o \sin(\omega t) \sim V_o e^{i\omega t}$$
$$I(t) = I_o \sin(\omega t - \phi) \sim I_o e^{i(\omega t - \phi)}$$





Average and RMS Voltage, Current, Power

$$V_{AVG} = \frac{1}{T}\int_0^T V(t)\ dt \qquad I_{AVG} = \frac{1}{T}\int_0^T I(t)\ dt \qquad P_{AVG} = \frac{1}{T}\int_0^T I(t)V(t)\ dt$$

$$\left(V_{RMS}\right)^2 = \frac{1}{T}\int_0^T |V(t)|^2\ dt = \frac{V_0^2}{T}\int_0^T \sin^2(\omega t)\ dt = \frac{Vo^2}{T}\int_0^T \frac{1+\cos(2\omega t)}{2}\ dt = \frac{Vo^2}{2} + \frac{Vo^2}{2T}\frac{1}{2\omega}\cos(2\omega t)\Big|_0^T$$

$$\left(V_{RMS}\right)^2 == \frac{Vo^2}{2} + \frac{Vo^2}{2T}\frac{1}{2\omega}\left(\cos(2\omega T)-1\right) = \frac{Vo^2}{2} + \frac{Vo^2}{2T}\frac{1}{2\omega}\underbrace{\left(\cos(2\omega\frac{2\pi}{\omega})-1\right)}_{=0} = \boxed{\mathbf{V_{RMS}} = \frac{\mathbf{Vo}}{\sqrt{\mathbf{2}}}}$$

$$\left(I_{RMS}\right)^2 = \frac{1}{T}\int_0^T |I(t)|^2\ dt = \frac{Io^2}{2} \qquad \boxed{\mathbf{I_{RMS}} = \frac{\mathbf{Io}}{\sqrt{\mathbf{2}}}}$$

$$\left(P_{RMS}\right)^2 = \frac{1}{T}\int_0^T |I(t)V(t)|^2\ dt = \frac{I_0V_0}{T}\int_0^T \sin^2(\omega t+\varphi)\sin^2(\omega t)dt$$

Reactance, Impedance, and Phasors

Consider the general RLC circuit with $V(t) = V_0 e^{j\omega t}$, $I(t) = I_0 e^{j(\omega t - \phi)}$:

$$V(t) = L\frac{dI}{dt} + IR + \frac{1}{C}\int I(t)dt$$

$$V(t) = I(t)\underbrace{\left(i\omega L + R + \frac{1}{i\omega C} \right)}_{\substack{\text{Reactance or} \\ \text{Complex Impedance}}} \quad Ohm's\ Law\ Complex\ (AC)\ Form$$

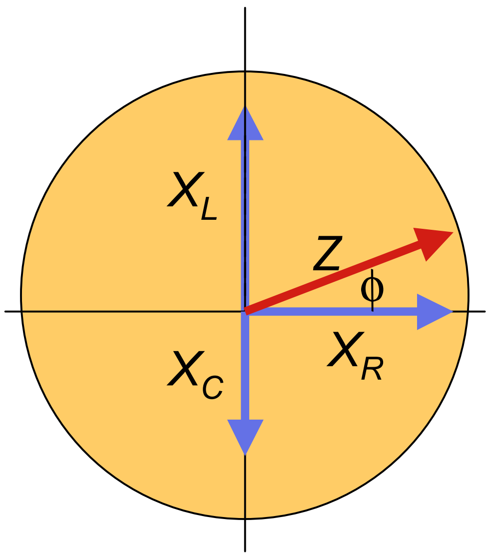
In any AC Circuit the Resistive, Capacitive, and Inductive elements can be replaces by their Complex Impedances!

$Z_R = R = X_L$	Resistive Reactance	$1 = e^{+i\ 0}$	$\phi = 0$
$Z_L = i(\omega L) = +i\ X_L$	Inductive Reactance	$+i = e^{+i\ \pi/2}$	$\phi = +\pi/2$
$Z_C = \frac{1}{i}\left(\frac{1}{\omega C}\right) = -i\ X_C$	Capacitive Reactance	$-i = e^{-i\ \pi/2}$	$\phi = -\pi/2$

Magnitude and Phase

The total Re*ac* tan *ce can be written as* $Z = R + i\left(\omega L - \frac{1}{\omega C}\right)$

corresponding to a complex number



The magnitude (length) of Z is

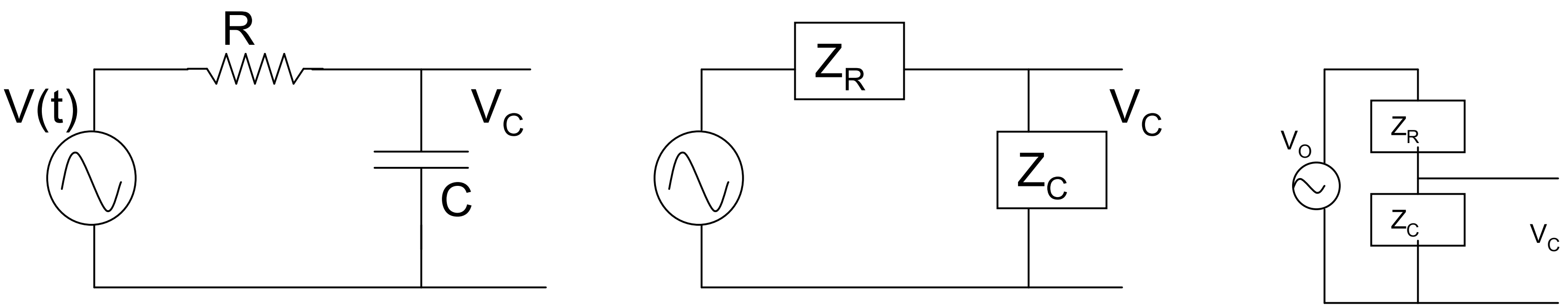
$$|Z| = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}$$

The phase angle between X_R *and Z*

$$\varphi_Z = \tan^{-1}(y/x) = \tan^{-1}\frac{\omega L - \frac{1}{\omega C}}{R}$$

$$cos(\varphi) = \frac{R}{|Z|}$$

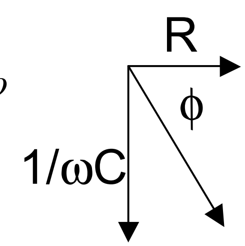
Simple RC Circuit - Low pass Filter (1)



What is the output voltage V and phase θ looking across the capacitor?

$$V_C = \frac{Z_C}{Z_R + Z_C} Vo\, e^{i\omega t} = \frac{-i/\omega C}{R - i/\omega C} Vo\, e^{i\omega t} \quad \leftarrow \text{voltage divider eq} \qquad gain = \left|V_C / V_0\right|$$

$$|V_C| = \sqrt{V_C V_C^*} = \sqrt{\frac{(-i/\omega C)(+i/\omega C)}{(R - i/\omega C)(R + i/\omega C)}} Vo = \frac{1/\omega C}{\sqrt{R^2 + (1/\omega C)^2}} Vo = \frac{1/\omega RC}{\sqrt{1 + (1/\omega RC)^2}} Vo$$



$$Phase: V_C = (\overbrace{\frac{-i/\omega C}{R - i/\omega C}}^{=Z})(\overbrace{\frac{R + i/\omega C}{R + i/\omega C}}^{=1}) = \overbrace{\left(\frac{1/\omega^2 C^2}{R^2 + (1/\omega C)^2}\right)}^{X} - i \overbrace{\left(\frac{R/\omega C}{R^2 + (1/\omega C)^2}\right)}^{Y}$$

$$\phi_C = \tan^{-1}\left(\frac{Y}{X}\right) = \tan^{-1}\left(\frac{-R/\omega C}{1/\omega^2 C^2}\right) = \tan^{-1}(-\omega RC) = \cot^{-1}\left(\frac{-1}{\omega RC}\right) = \tan^{-1}\left(\frac{1}{\omega RC}\right) - \pi/2$$

$$At \quad \omega = 1/RC \quad \phi_C = \tan^{-1}(1) - \pi/2 = \pi/4 - \pi/2 = -\pi/4$$

Simple RC Circuit - Low pass Filter (2)

$$Break\ \ Frequency\ \ occurs\ \ when\quad \omega_{break} = \frac{1}{RC} \Rightarrow |V_C| = \frac{1}{\sqrt{2}} Vo = 0.707\, Vo$$

$$|V_C| = \frac{1}{\sqrt{1+(\omega RC)^2}} Vo \qquad \underbrace{f_{break} = \frac{1}{2\pi RC}}_{break\ frequency}$$

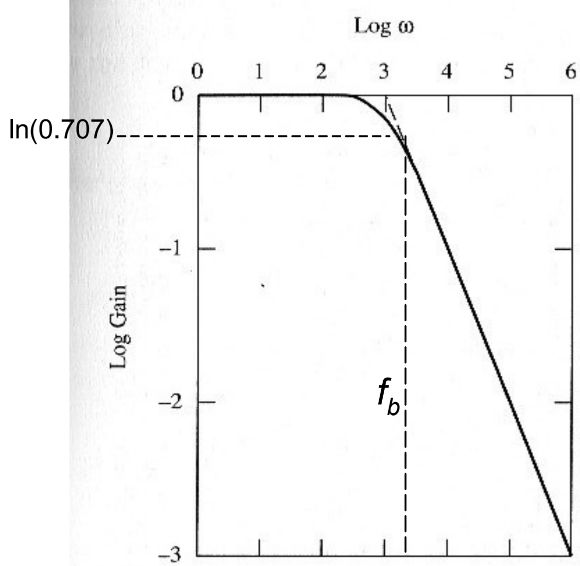


FIGURE 3.9A **Gain *versus* frequency for the circuit of Figure 3.8 (low pass).**

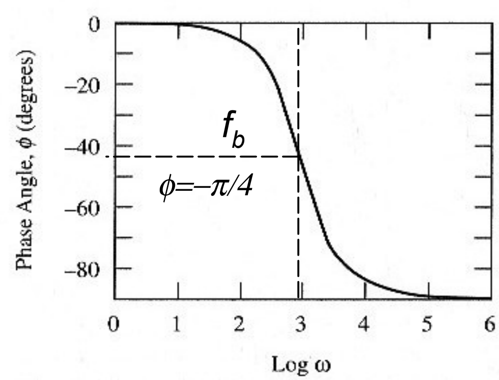
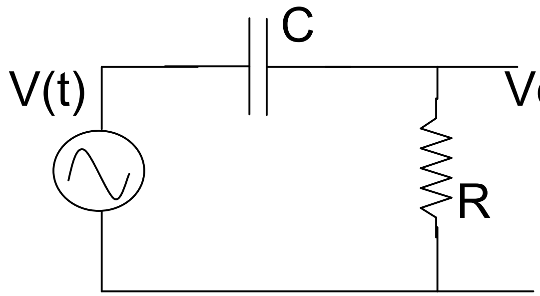
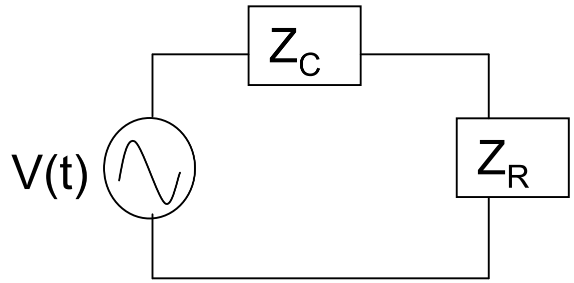


FIGURE 3.9B **Phase angle *versus* frequency for the circuit of Figure 3.8 (low pass).**

Simple RC Circuit - High Pass Filter

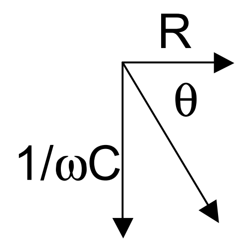




What is the output voltage V_R *and phase* ϕ_R *looking across the resistor?*

$$V_R = \left(\frac{Z_R}{Z_R + Z_C}\right) Vo\ e^{j\omega t} = \frac{R}{(R - j/\omega C)}\ Vo\ e^{j\omega t}$$

$$|V_R| = \sqrt{V_R V_R^*} = \sqrt{\frac{R^2}{(R - j/\omega C)(R + j/\omega C)}} Vo = \frac{R}{\sqrt{R^2 + (1/\omega C)^2}} Vo = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}} Vo$$



$$gain = V_R / V_O = \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}}$$

$$Phase: V_R = \left(\frac{R^2}{R^2 + (1/\omega C)^2}\right) + j\left(\frac{R/\omega C}{R^2 + (1/\omega C)^2}\right) \quad \Rightarrow\ \phi_R = \tan^{-1}\left(\frac{Y}{X}\right) = \tan^{-1}\left(\frac{1}{\omega RC}\right)$$

$$At\quad \omega = 1/RC \quad \phi_R = \tan^{-1}(1) = \pi/4$$

Simple RC Circuit - High Pass Filter (2)

$$|V_R| = \frac{\omega RC}{\sqrt{1+(\omega RC)^2}} Vo \qquad \underbrace{f_{break} = \frac{1}{2\pi RC}}_{break\ frequency} \qquad |V_R| = \frac{1}{\sqrt{2}} Vo = 0.707\, Vo$$

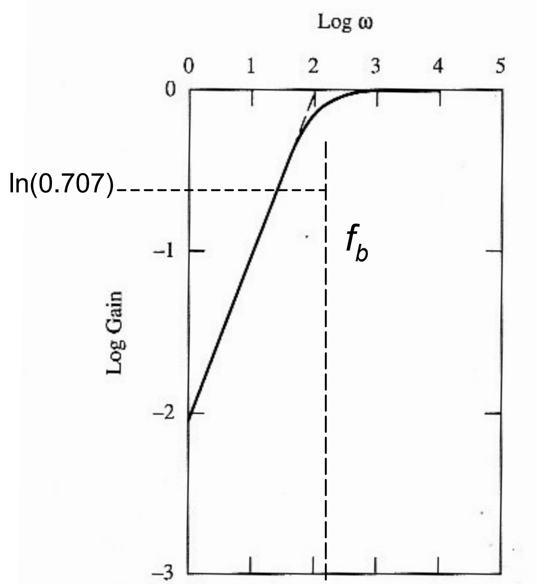


FIGURE 3.11A **Gain *versus* frequency for the circuit of Figure 3.10 (high pass).**

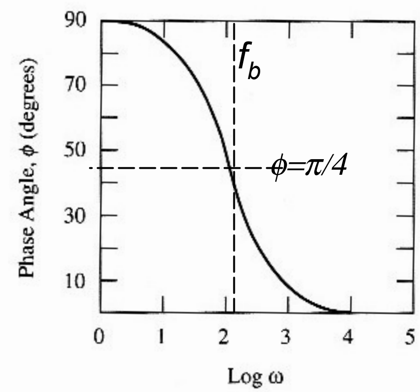
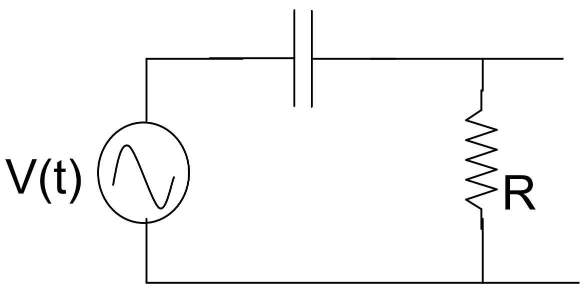
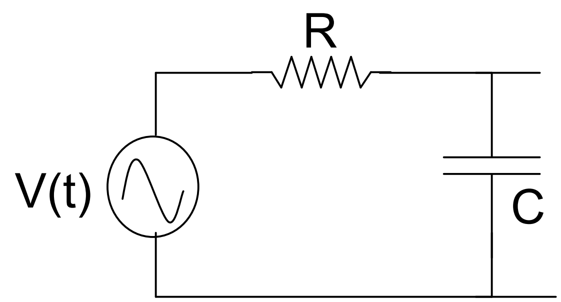


FIGURE 3.11B **Phase angle *versus* frequency for the circuit of Figure 3.10 (high pass).**

Differentiator and Integrator Connection





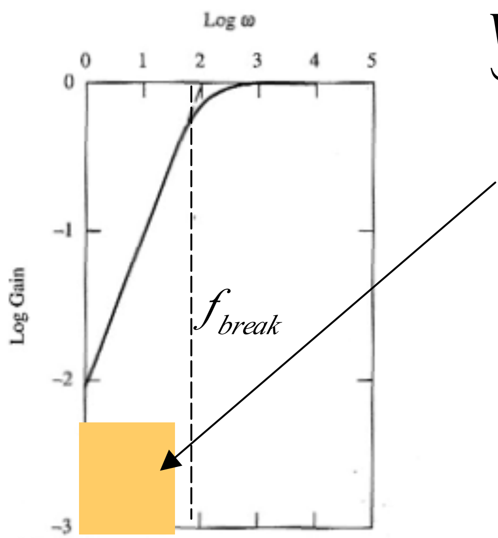


FIGURE 3.11A Gain *versus* frequency for the circuit of Figure 3.10 (high pass).

$$V_R = R\,dq/dt$$

differentiator
$RC << T$
$1/T << 1/RC$
$f << 1/RC$
$f < f_{break}$

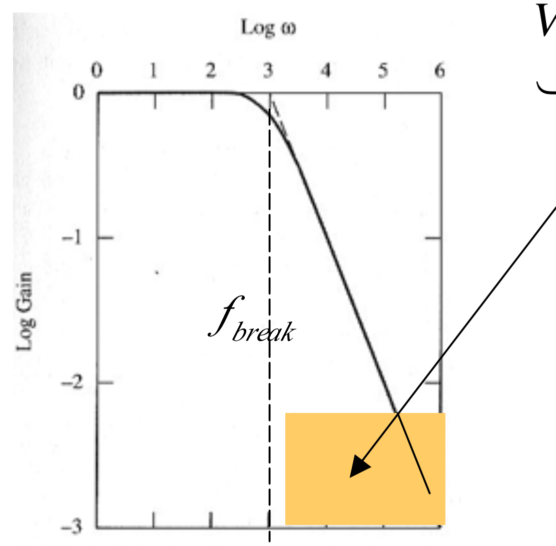


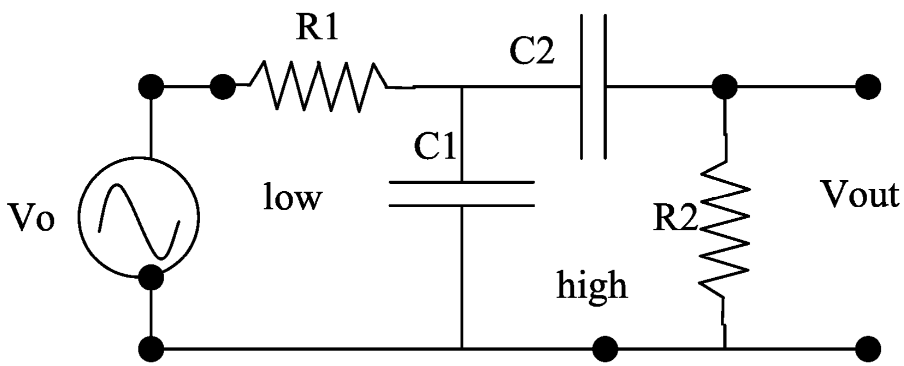
FIGURE 3.9A Gain *versus* frequency for the circuit of Figure 3.8 (low pass).

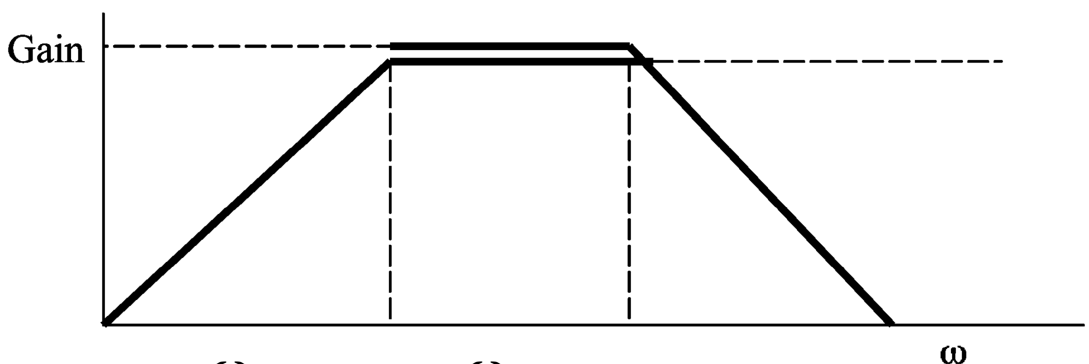
$$V_C = \frac{1}{C}\int i\,dt$$

int *egrator*
$RC >> T$
$1/T >> 1/RC$
$f >> 1/RC$
$f > f_{break}$

Band Pass or Notch Filter

Band Pass Filter - Combination of Low and High Pass

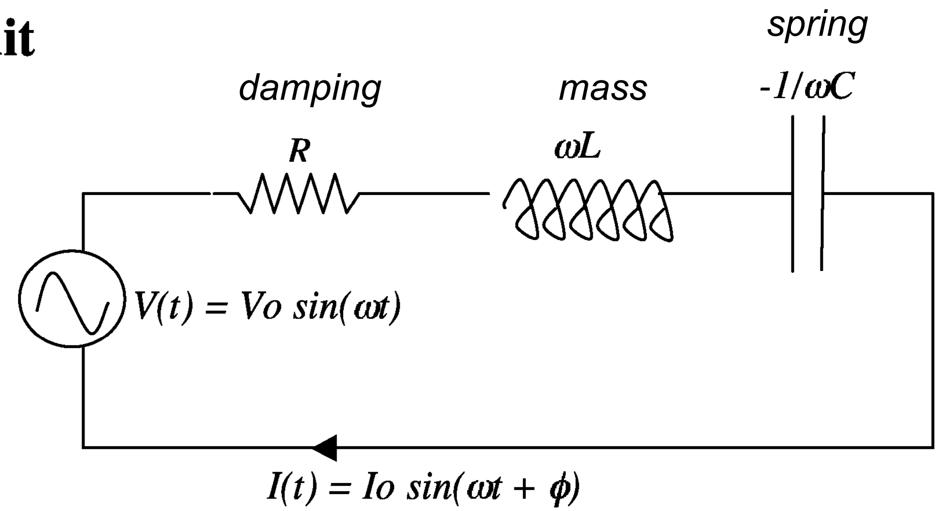




$\omega_{HI} = 1/R_2C_2$ $\omega_{LO} = 1/R_1C_1$

RLC Circuit

RLC Circuit



$Z = R + i(\omega L - 1/\omega C)$

$|Z| = \sqrt{R^2 + (\omega L - 1/\omega C)^2}$

$I(t) = V(t)/|Z|$

$|I(t)| = V(t) / \sqrt{R^2 + (\omega L - 1/\omega C)^2}$

$V_R = I(t)\, R = V(t)\, R / |Z|$

$\phi = \tan^{-1}\{(\omega L - 1/\omega C)/R\}$

$V_L = I(t)\, e^{+i\pi/2} \{(\omega L) / |Z|\}$

$V_C = I(t)\, e^{-i\pi/2} \{(1/\omega C) / |Z|\}$

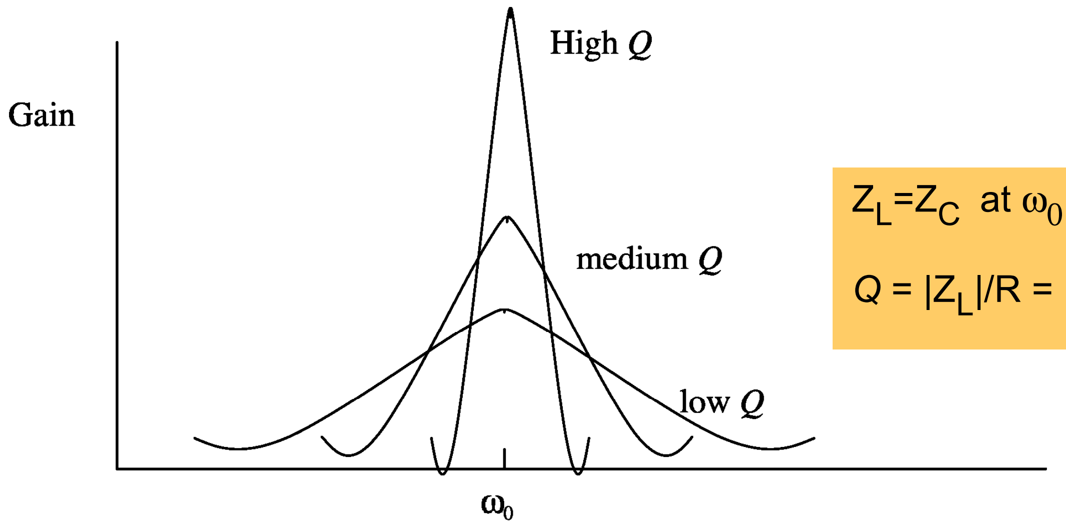
Resonance Condition and Q-factor

$$|I(t)| = V(t) \,/\, \sqrt{R^2 + (\omega L - 1/\omega C)^2}$$

When $\omega_0 L = 1/\omega_0 C$ the current flowing in the circuit and Gain reach a maximum. → $\omega_0^{\,2} = 1/LC$

$$f_0 = 1/2\pi\sqrt{LC}$$

The Q-factor of the circuit is defined as the $Q \sim G_{peak}/\Delta G$ ~ *(maxAmplitde)/(Width at half max)*



$Z_L = Z_C$ at ω_0

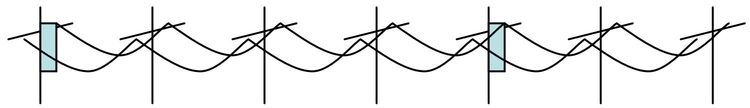
$Q = |Z_L|/R = |Z_C|/R$

$$Q = |Z_L| / R = \omega_0 L / R$$

ω_o

Power in AC Circuits

Power in AC Circuits



$P(t) = I(t)\ V(t)$ *Instantaneous Power*

$P_{AVG} = 1/T \int_0^T I(t)V(t)dt$ *Average Power*

$= Io\ Vo/\ T \int_0^T \sin(\omega t)\ \sin(\ \omega t - \phi)\ dt$

$= 1/2\ Io\ Vo\ cos(\phi) = Io/\sqrt{2}\ \ Vo/\sqrt{2}\ \ cos(\phi) \ \ = I_{RMS}\, V_{RMS}\, cos(\phi)$

$$P_{AVG} = Io/\sqrt{2}\ \ Vo/\sqrt{2}\ \ cos(\phi)$$

$cos(\phi) = R/\sqrt{R^2 + (\omega L - 1/\omega C)^2}$ *RLC Circiut*

Maximum power transfer to the circuit occurs when $cos(\phi) = 1$ or at resonance→ $\omega_0 L = 1/\ \omega_0 C$

If the load is strongly inductive (motors, compressors, etc) then capacitors are added to the circuit the by power companies to create a more favorable power transfer.

Decibel Scale

DECIBEL SCALE

When measuring power gain and voltage gain in an amplifier or circuit we often use the dcibel scale.

Power

P db = 10 log(Pout/Pin)

Voltage

$V\,db = 20 \log_{10}(Vout/Vin) = 20 \log_{10}(gain)$

Sound Power
- Near total silence - 0 dB
- A whisper - 15 dB
- Normal conversation - 60 dB
- A lawnmower - 90 dB
- A car horn - 110 dB
- A rock concert or a jet engine - 120 dB
- A gunshot or firecracker - 140 dB

Vdb at the Break Frequency

At the breaking frequency the gain Vout/Vin drops by a factor of $1/\sqrt{2}$. This is called the -3 db point. Can you jusify this rema rk?