CHAPTER 6- TUNNELING PHENOMENA

I. THE BARRIER POTENTIAL

Consider a single step barrier potential. A particle of mass m is incident from the left with unit amplitude. The height of the barrier is Vo. Since the potential is discontinuous at x=0, we will divide the problem into 2 regions (1) x<0 and (2) x>0 to solve. The wave function and its derivative must be continuous at x=0!

**Case I: E,> Vo**

\[
\psi_1(x) = I \exp(+ikx) + R \exp(-ikx) \quad k = \sqrt{\frac{2mE}{\hbar^2}}
\]

\[
\psi_2(x) = T \exp(+ik'x) \quad k' = \sqrt{\frac{2m(E-Vo)}{\hbar^2}} > 0
\]

We will assume \( I = 1 \!\)

In general \( |I|^2 = \text{number of particles per second}. \)

\[
\psi_1(0) = \psi_2(0) \rightarrow 1 + R = T \quad (1)
\]

\[
\psi_1'(0) = \psi_2'(0) \rightarrow ik - ikR = ik' T \quad (2)
\]

Solving (1) and (2) for R and T

\[
R = k - k'/k + k' \quad \text{reflection amplitude}
\]

\[
T = 2k/k + k' \quad \text{transmission amplitude}
\]

\[
|R|^2 = R^*R = (k-k'/k + k') (k-k'/k + k') = (k-k'/k + k')^2 \quad \text{reflection probability}
\]

\[
|T|^2 = T^*T = (2k/k + k') (2k/k + k') = (2k/k + k')^2 \quad \text{transmission probability}
\]

For \( E \gg Vo \) then \( T \rightarrow 1 \quad R \rightarrow 0 \)

For \( E \sim Vo \) then \( T \rightarrow 0 \quad R \rightarrow 1 \)

Also \( R + T = I \)
Case II (Absorption in region (2) and $k'$ complex wave number $k' = i \kappa$)

\[
\psi_1(x) = I \exp(+ix) + R \exp(-ix) \quad k = \sqrt{\frac{2mE}{\hbar^2}}
\]

\[
\psi_2(x) = T \exp(+i'x) = T \exp(-k'x) \quad k' = i\kappa = i \sqrt{\frac{2m(E-V_0)}{\hbar^2}}
\]

Complex wave # $k' = i\kappa$ implies absorption in region (2)

\[
\psi_1(0) = \psi_2(0) \Rightarrow 1 + R = T \quad (1)
\]

\[
\psi_1'(0) = \psi_2'(0) \Rightarrow ik - ikR = -kT \quad (2)
\]

Solving (1) and (2) for R and T

\[
R = \frac{(ik+\kappa)/(ik-\kappa)}{(ik+\kappa)/(ik-\kappa)} \quad \text{reflection amplitude}
\]

\[
T = \frac{2k}{(ik-\kappa)} \quad \text{transmission amplitude}
\]

\[
|R|^2 = R*R = \frac{(-ik+\kappa)/(-ik-\kappa)}{((ik+\kappa)/(ik-\kappa))} = 1 \quad \text{and} \quad T = 0
\]

\[
\text{But,} \quad T = |T|^2 = T*T = 4k^2/(k^2 + \kappa^2) \neq 0 \quad !!!
\]

The wave in the barrier is called an evanescent wave and has no physical interpretation, since the barrier is infinitely long $0 < x, \infty$. We can not see in the barrier or measure what comes out!

In the next section we define the general probability current density for a QM wave.

\[
J = \left(\frac{\hbar}{2im}\right) \left[ \psi^*\psi' - \psi\psi' \right]
\]

To find the mass flow in the barrier we have we would evaluate $<J>$ in region 2!

\[
J = \left(\frac{\hbar}{2im}\right) \left[ \psi^*\psi' - \psi\psi' \right]
= \left(\frac{\hbar}{2im}\right) \left[ T^*\exp(-\kappa x) \{-\kappa\} T\exp(-\kappa x) \right] - \left[ T^*\exp(-\kappa x) \{-\kappa\} T\exp(-\kappa x) \right] = 0 \quad !!!
\]

So even though $\psi_2^*\psi_2 = |\psi_2|^2 = |T|^2 \neq 0$ no particle current is flowing!!
BARRIER PENETRATION

Now consider a finite barrier of height $V_0$. Particles of energy $E < V_0$ are incident. The problem is solved exactly in most QM texts. We will only give a simplified argument for a general barrier solution here!

The transmission amplitude $T$ at $x = a$ must be about

$$T = I_0 - R = I_0 \exp(-\kappa a) = \exp(-\kappa a)$$

It is clear that the energy difference $\Delta = V_0 - E$ plays a major role in the probability of barrier penetration. For $E \sim V_0$ and $T = 0$.

We can calculate the transmission probability through a smoothly shaped barred $V(x)$

$$\kappa(x) = \sqrt{\frac{2m(V(x)-E)}{h^2}}$$

$$\mathcal{T}(E) = \exp\left(-2 \int_a^b \kappa(x) \, dx\right)$$
**ALPHA PARTICLE ESCAPE PROBABILITY** and Decay

Alpha particle groups have strong binding and p-p-n-n combinations occur readily in any nucleus. Some nuclei are not stable to alpha decay. The nuclear binding energy is about 8 MeV per nucleon so there is little chance the alpha particle can escape over the top. Scientist realize that the alphas can escape by approaching the barrier at a frequency of $f = 10^{21}$/s penetration! 

$$d = v T \quad \text{or} \quad f = 1/T = v/d$$

$$\kappa a = \sqrt{\frac{2m}{\hbar^2} (V_0 - E)}$$

$$a = \sqrt{\frac{2}{\hbar} \times 3800 \text{MeV} \times (92 \text{MeV})} \times 9 \times 10^{-13} \text{cm}$$

$$a = \frac{836 \text{MeV}}{197.35 \text{MeV} \cdot \text{fm}} \times 9 \text{ fm}$$

$$a = 4.28 \times 9 = 38.$$

$$\tau = \exp(-2 \kappa a) = \exp(-76) = 9.85 \times 10^{-34}$$

$$R = \text{Rate of Escape} = f \tau = (10^{21}/s) (9.85 \times 10^{-34}) = 9.85 \times 10^{-13}/s$$

$$\tau = 1/R = 1.0 \times 10^{12} \text{ s} = 88.5 \text{ yr average lifetime}$$