PHYS 721 – HOMEWORK # 1 – DUE TUESDAY, SEPTEMBER 4, 2018

Problem 1. Using the definition of the Dirac δ function, prove the following properties:

1) $x^n \delta^{(n)}(x) = (-1)^n\, n!\, \delta(x)$, where $\delta^{(n)}(x)$ is the n-th derivative of the Dirac δ distribution.
2) $\delta[y(x)] = \sum_i \delta(x - x_i)|dy/dx|^{-1}_{x=x_i}$, where x_i are the simple zeros of the function $y(x)$.

Problem 2. Show that the one-dimensional integral representation of the Dirac δ function

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\ e^{ikx}$$

can be written as the distribution $\lim\limits_{n\to\infty} nG(nx)$, where (1) $G(x) = e^{-\pi x^2}$ and (2) $G(x) = \sin^2(x)/(\pi x^2)$.

Problem 3. Using the Green's function for the Laplacian in three-dimensional space $I\!R^3$:

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|},$$

show that any regular vector field $\mathbf{A}(\mathbf{x})$ defined over the domain $I\!R^3$ can be written as $\mathbf{A}(\mathbf{x}) = -\nabla\phi + \nabla \times \mathbf{C}$, where ϕ and $\mathbf{C}$ are a scalar and vector field defined as:

$$\phi(\mathbf{x}) = \frac{1}{4\pi}\int_{I\!R^3} d^3x' \frac{\nabla' \cdot \mathbf{A}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$
$$\mathbf{C}(\mathbf{x}) = \frac{1}{4\pi}\int_{I\!R^3} d^3x' \frac{\nabla' \times \mathbf{A}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

respectively. Specify what conditions $\mathbf{A}$, ϕ and $\mathbf{C}$ must satisfy for your derivation to be valid (regularity, behavior at infinity, etc.)

Problem 4. Using the Dirac delta and (if needed) Heaviside functions in the appropriate coordinates, express the following charge distributions as three-dimensional charge densities $\rho(\mathbf{x})$.

a) In spherical coordinates, a charge Q uniformly distributed over a spherical shell of radius R.
b) In cylindrical coordinates, a charge λ per unit length uniformly distributed over a cylindrical surface of radius R.
c) In cylindrical coordinates, a charge Q uniformly distributed over a flat annulus of negligible thickness and radii R_1 and $R_2 > R_1$.
d) In Cartesian coordinates, a charge per unit length λ uniformly distributed on a square loop of wire with negligible radius and side L.