PHYS 721 – LEGENDRE FUNCTIONS & SPHERICAL HARMONICS

LEGENDRE FUNCTIONS

Differential equation

$$(1-z^2)u''(z) - 2zu'(z) + \left(\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right)u(z) = 0\,,$$

Solution in the region $|z| < 1$:

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\mu/2} F(-\nu, \nu+1; 1+\mu; (1-z)/2)\,,$$

is called Legendre function of the first kind. It is also represented as

$$P_\nu^\mu(z) = \frac{2^\mu}{\Gamma(1-\mu)}(z^2-1)^{-\mu/2} F(1-\mu+\nu, -\mu-\nu; 1-\mu; (1-z)/2)\,.$$

Note the property:

$$P_\nu^\mu(z) = P_{-\nu-1}^\mu(z)\,.$$

For $\mu = 0$:

$$P_\nu(z) \equiv P_\nu^0(z) = F(-\nu, \nu+1; 1; (1-z)/2)\,.$$

are the Legendre functions, which reduce to the Legendre polynomials for $\nu = l$ (integer). They satisfy the differential equation

$$(1-z^2)P_\nu''(z) - 2zP_\nu'(z) + \nu(\nu+1)P_\nu(z) = 0\,,$$

If $\mu \neq m$ (integer), $P_\nu^\mu(z)$ and $P_\nu^{-\mu}(z)$ are independent. Instead of $P_\nu^{-\mu}(z)$ it is customary to introduce:

$$Q_\nu^\mu(z) = \frac{\pi e^{i\pi\mu}}{2\sin\pi\mu}\left(P_\nu^\mu(z) - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}P_\nu^{-\mu}(z)\right)\,.$$

(Legendre function of the second kind.)

If $\mu = m$ (integer) $P_\nu^m(z)$ and $P_\nu^{-m}(z)$ are proportional:

$$P_\nu^{-m}(z) = (-1)^m \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)} P_\nu^m(z)\,.$$

and

$$P_\nu^{-m}(z) = \frac{1}{\Gamma(1+m)}\left(\frac{z+1}{z-1}\right)^{-m/2} F(-\nu, \nu+1; 1+m; (1-z)/2)\,,$$

Relation between $P_\nu^m(z)$ and the Legendre functions $P_\nu(z)$:

$$P_\nu^m(z) = (z^2-1)^{m/2}\frac{d^m}{dz^m}P_\nu(z)\,, \qquad m > 0\,.$$

ASSOCIATED LEGENDRE FUNCTIONS

If $\nu = l$ (integer) the associated Legendre functions are regular in $-1 \leq x \leq 1$. They are defined as:

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x)\,, \qquad m > 0$$

Note the extra $(-1)^{3m/2}$ factor (conventions). Sometimes a normalization without $(-1)^m$ is also used.

Rodrigues' formula for the associated Legendre Polynomials:

$$P_l^m(x) = \frac{(-1)^m}{2^l l!} (1-x^2)^{m/2} \frac{d^{m+l}}{dx^{m+l}} (x^2-1)^l\,, \qquad m > 0$$

Relation to $m < 0$:

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x)\,.$$

Normalization:

$$\int_{-1}^{1} dx\, P_l^m(x) P_{l'}^m(x) = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{ll'}\,.$$

This implies:

$$|m| < l\,, \qquad \text{or} \qquad -l \leq m \leq l\,.$$

SPHERICAL HARMONICS

Differential equation:

$$\left[\frac{\partial^2}{\partial\theta^2} + \operatorname{cotan}\theta\,\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + l(l+1)\right]Y(\theta,\varphi) = 0\,,$$

The regular solution in $-1 \le \cos\theta \le 1$ with boundary condition

$$Y(\theta,\varphi) = Y(\theta,\varphi+2\pi)$$

is

$$Y_{lm}(\Omega) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\,P_l^m(\cos\theta)\,e^{im\varphi}\,.$$

or

$$Y_{lm}(\Omega) = \frac{1}{2^l l!}\left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2}(-\sin\theta)^m\frac{d^{m+l}}{d(\cos\theta)^{m+l}}(\cos^2\theta-1)^l e^{im\varphi}\,.$$

Normalization:

$$\int_{S^2} d\Omega\,Y^\star_{lm}Y_{l'm'} = \delta_{ll'}\delta_{mm'}\,.$$

Symmetry property:

$$Y_{l,-m}(\Omega) = (-1)^m Y^\star_{lm}(\Omega)\,.$$

Completeness:

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^\star_{lm}(\Omega')Y_{lm}(\Omega) = \delta(\cos\theta-\cos\theta')\delta(\varphi-\varphi')\,.$$

Addition theorem:

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^\star_{lm}(\Omega')Y_{lm}(\Omega)\,,$$

where

$$\cos\gamma = \cos\theta'\cos\theta + \sin\theta'\sin\theta\cos(\varphi-\varphi')\,.$$