## PHYS 721 – Legendre Polynomials $P_l(x)$ – Useful formulas

Definition range: $(a,b)=(-1,1)$.

Standard normalization: $P_l(1)=1$.

Rodriguez formula:

$$P_l(x)=\frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l\,.$$

Differential equation:

$$(1-x^2)P_l''(x)-2xP_l'(x)+l(l+1)P_l(x)=0\,,$$

$$[(1-x^2)P_l'(x)]'+l(l+1)P_l(x)=0\,.$$

Recurrence relation:

$$(l+1)P_{l+1}(x)=(2l+1)xP_l(x)-lP_{l-1}(x)\,.$$

Derivative:

$$(1-x^2)P_l'(x)=l[P_{l-1}(x)-xP_l(x)]=(l+1)[xP_l(x)-P_{l+1}(x)]\,.$$

Other relations with derivatives:

$$xP_l'(x)-P_{l-1}'(x)=lP_l(x)\,,$$

$$P_{l+1}'(x)-xP_l'(x)=(l+1)P_l(x)\,,$$

$$(2l+1)P_l(x)=P_{l+1}'(x)-P_{l-1}'(x)\,.$$

Generating function:

$$\sum_{l=0}^{\infty}P_l(x)z^l=\frac{1}{\sqrt{1-2xz+z^2}}\,,\qquad -1<x<1\,,\quad |z|<1\,.$$

Series representation:

$$P_l(x)=2^{-l}\sum_{k=0}^{[l/2]}(-1)^k\binom{l}{k}\binom{2l-2k}{l}x^{l-2k}\,,$$

$$P_l(\cos\theta)=\sum_{k=0}^{l}g_k\,g_{l-k}\cos[(l-2k)\theta]\,,$$

where $[l/2]$ = maximum integer smaller than $l/2$, and

$$g_k = \frac{(2k-1)!!}{2^k k!}\,.$$

Symmetry property:

$$P_l(-x) = (-1)^l P_l(x)\,.$$

Special values:

$$\begin{aligned}
&P_l(\pm 1) = (\pm 1)^l\,,\\
&P_{2l}(0) = (-1)^l g_l\,,\\
&P_{2l+1}(0) = 0\,,\\
&P'_{2l}(0) = 0\,,\\
&P'_{2l+1}(0) = (-1)^l (2l+1) g_l\,.
\end{aligned}$$

First polynomials:

$$\begin{aligned}
&P_0(x) = 1\,,\\
&P_1(x) = x\,,\\
&P_2(x) = \frac{1}{2}(3x^2 - 1)\,,\\
&P_3(x) = \frac{1}{2}(5x^3 - 3x)\,,\\
&P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)\,.
\end{aligned}$$

Orthonormality:

$$\int_{-1}^{1} dx\, P_l(x) P_{l'}(x) = \frac{2}{2l+1}\delta_{ll'}\,.$$

Other useful relations:

$$\int_{-1}^{1} dx\, x\, P_l(x) P_{l'}(x) = \begin{cases} \dfrac{2(l+1)}{(2l+1)(2l+3)}\delta_{l,l'-1} \\ \dfrac{2l}{(2l-1)(2l+1)}\delta_{l,l'+1} \end{cases}\,,$$

$$\int_{-1}^{1} dx\, x^2\, P_l(x) P_{l'}(x) = \begin{cases} \dfrac{2(l+1)(l+2)}{(2l+1)(2l+3)(2l+5)}\delta_{l,l'-2} \\ \dfrac{2(2l^2+2l-1)}{(2l-1)(2l+1)(2l+3)}\delta_{ll'} \end{cases}\,.$$