

BESSEL FUNCTIONS

Differential equation:

$$\left[z^2 \frac{d^2}{dz^2} + z \frac{d}{dz} + (z^2 - \nu^2) \right] u_\nu(z) = 0,$$

Independent solutions:

$$J_\nu(z) \quad \text{and} \quad J_{-\nu}(z).$$

Usually, instead of $J_{-\nu}(z)$ the following function is defined:

$$Y_\nu(z) = \frac{1}{\sin(\nu\pi)} [J_\nu(z) \cos(\nu\pi) - J_{-\nu}(z)].$$

J_ν and Y_ν are called Bessel functions of first and second kind.

Series representation:

$$J_\nu(z) = \left(\frac{z}{2} \right)^\nu \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{k! \Gamma(\nu + k + 1)}$$

Hankel functions of first and second kind:

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z), \quad H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z).$$

Properties:

$$H_{-\nu}^{(1)}(z) = e^{i\nu\pi} H_\nu^{(1)}(z), \quad H_{-\nu}^{(2)}(z) = e^{-i\nu\pi} H_\nu^{(2)}(z).$$

If $\nu = n$ is an integer:

$$J_{-n}(z) = (-1)^n J_n(z), \quad Y_{-n}(z) = (-1)^n Y_n(z).$$

$$Y_n(z) = -\frac{(z/2)^{-n}}{\pi} \sum_{k=0}^{\infty} \frac{(n-k-1)!}{k!} (z^2/4)^k + (2/\pi) \ln(z/2) J_n(z) + \\ -\frac{(z/2)^n}{\pi} \sum_{k=0}^{\infty} [\psi(k+1) + \psi(n+k+1)] \frac{(-z^2/4)^k}{k!(n+k)!},$$

where $\psi(n) = -\gamma + \sum_{k=1}^{n-1} k^{-1}$.

Integral representation:

$$J_n(z) = \frac{1}{\pi} \int_0^\pi d\theta \cos(z \sin \theta - n\theta),$$

Small arguments:

$$J_\nu(z) \sim \frac{(z/2)^\nu}{\Gamma(\nu+1)}, \quad \nu \neq -1, -2, -3 \dots$$

$$Y_\nu(z) \sim -iH_\nu^{(1)}(z) \sim iH_\nu^{(2)}(z) \sim -(1/\pi)\Gamma(\nu)(z/2)^{-\nu}, \quad \text{Re}(\nu) > 0,$$

$$Y_0(z) \sim -iH_0^{(1)}(z) \sim iH_0^{(2)}(z) \sim (2/\pi) \ln z, \quad \nu = 0.$$

Asymptotic expansion ($|z| \rightarrow \infty$):

$$J_\nu(z) = \sqrt{2/(\pi z)} \cos(z - \pi\nu/2 - \pi/4) + \dots,$$

$$Y_\nu(z) = \sqrt{2/(\pi z)} \sin(z - \pi\nu/2 - \pi/4) + \dots,$$

$$H_\nu^{(1)}(z) = \sqrt{2/(\pi z)} e^{i(z - \pi\nu/2 - \pi/4)} + \dots,$$

$$H_\nu^{(2)}(z) = \sqrt{2/(\pi z)} e^{-i(z - \pi\nu/2 - \pi/4)} + \dots.$$

Expansion for large orders ($\nu \rightarrow \infty$):

$$J_\nu(z) \sim \frac{1}{\sqrt{2\pi\nu}} \left(\frac{ez}{2\nu} \right)^\nu,$$

$$Y_\nu(z) \sim -\sqrt{\frac{2}{\pi\nu}} \left(\frac{ez}{2\nu} \right)^{-\nu}.$$

Recurrence relations ($\Omega_\nu =$ all Bessel/Hankel):

$$\Omega_{\nu-1}(z) + \Omega_{\nu+1}(z) = \frac{2\nu}{z} \Omega_\nu(z),$$

$$\Omega_{\nu-1}(z) - \Omega_{\nu+1}(z) = 2\Omega'_\nu(z),$$

$$\Omega'_\nu(z) = \Omega_{\nu-1}(z) - \frac{\nu}{z} \Omega_\nu(z),$$

$$\Omega'_\nu(z) = -\Omega_{\nu+1}(z) + \frac{\nu}{z} \Omega_\nu(z)$$

MODIFIED BESSEL FUNCTIONS

Differential equation:

$$\left[z^2 \frac{d^2}{dz^2} + z \frac{d}{dz} - (z^2 + \nu^2) \right] u_\nu(z) = 0,$$

Independent solutions:

$$I_\nu(z), \quad \text{and} \quad I_{-\nu}(z).$$

Usually, instead of $I_{-\nu}(z)$ the following function is defined:

$$K_\nu(z) = \frac{\pi}{2 \sin(\nu\pi)} [I_{-\nu}(z) - I_\nu(z)].$$

J_ν and Y_ν are called Bessel functions of first and second kind.

Series representation:

$$I_\nu(z) = \left(\frac{z}{2} \right)^\nu \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k! \Gamma(\nu + k + 1)}$$

If $\nu = n$ is an integer:

$$I_{-n}(z) = I_n(z), \quad K_{-n}(z) = K_n(z).$$

$$\begin{aligned} K_n(z) = & \frac{1}{2} \left(\frac{z}{2} \right)^{-n} \sum_{k=0}^{\infty} \frac{(n-k-1)!}{k!} (-z^2/4)^k - (-1)^n \ln(z/2) I_n(z) + \\ & + (-1)^n \frac{1}{2} \left(\frac{z}{2} \right)^n \sum_{k=0}^{\infty} [\psi(k+1) + \psi(n+k+1)] \frac{(z^2/4)^k}{k!(n+k)!}. \end{aligned}$$

Integral representation:

$$I_n(z) = \frac{1}{\pi} \int_0^\pi d\theta e^{z \cos \theta} \cos(n\theta).$$

Small arguments:

$$I_\nu(z) \sim \frac{(z/2)^\nu}{\Gamma(\nu+1)}, \quad \nu \neq -1, -2, -3 \dots$$

$$K_\nu(z) \sim (1/2) \Gamma(\nu) (z/2)^{-\nu}, \quad \text{Re}(\nu) > 0,$$

$$K_0(z) \sim -\ln z, \quad \nu = 0.$$

Asymptotic expansion ($|z| \rightarrow \infty$):

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}} [1 - (4\nu^2 - 1)/(8z) + \dots],$$

$$K_\nu(z) = \sqrt{\pi/(2z)} e^{-z} [1 + (4\nu^2 - 1)/(8z) + \dots].$$

Recurrence relations ($\Omega_\nu =$ all Modified Bessel):

$$\Omega_{\nu-1}(z) - \Omega_{\nu+1}(z) = \frac{2\nu}{z} \Omega_\nu(z),$$

$$\Omega_{\nu-1}(z) + \Omega_{\nu+1}(z) = 2\Omega'_\nu(z),$$

$$\Omega'_\nu(z) = \Omega_{\nu-1}(z) - \frac{\nu}{z} \Omega_\nu(z),$$

$$\Omega'_\nu(z) = \Omega_{\nu+1}(z) + \frac{\nu}{z} \Omega_\nu(z)$$