PHYS 503/629 – Homework No. 1 – DUE THURSDAY, 03/31/2005

Problem 1. Show that, with λ taken as an affine parameter, the Lagrangian

$$L = \frac{1}{2} g_{\alpha\beta}(x)\dot{x}^\alpha \dot{x}^\beta \tag{1}$$

is an equivalent Lagrangian to

$$L = \sqrt{-g_{\alpha\beta}(x)\dot{x}^\alpha \dot{x}^\beta}\,, \tag{2}$$

where $\dot{x}^\alpha = dx^\alpha/d\lambda$.

Problem 2. For a test particle in a two-dimensional Euclidean spacetime: **a)** Write the Lagrangian (1) in polar coordinates (r, φ); **b)** Show that the equations of motion are $\ddot{r} - r\dot{\varphi}^2 = 0$ and $r^2\dot{\varphi} =$ constant; **c)** Integrate the equations of motion and find the trajectory. What kind of trajectory the test particle does follow?

Problem 3. Starting from the ansatz:

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,,$$

find the Schwarzschild solution by solving the Einstein equations in vacuo (without using Maple, Mathematica, etc.!).

Problem 4 (Only for 629 level or for extra credit). Prove the Birkhoff theorem by showing that the most general spherically symmetric solution of Einstein equations in vacuo,

$$ds^2 = -A(r,t)dt^2 + B(r,t)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,,$$

is indeed static, i.e., $A(r,t) = A(r)$ and $B(r,t) = B(r)$.