Problem 1. Show that, with $\lambda$ taken as an affine parameter, the Lagrangian

$$L = \frac{1}{2} g_{\alpha\beta}(x) \dot{x}^\alpha \dot{x}^\beta$$

is an equivalent Lagrangian to

$$L = \sqrt{-g_{\alpha\beta}(x) \dot{x}^\alpha \dot{x}^\beta},$$

where $\dot{x}^\alpha = dx^\alpha/d\lambda$.

Problem 2. For a test particle in a two-dimensional Euclidean spacetime: a) Write the Lagrangian (1) in polar coordinates $(r, \varphi)$; b) Show that the equations of motion are $\ddot{r} - r\dot{\varphi}^2 = 0$ and $r^2 \dot{\varphi} = \text{constant}$; c) Integrate the equations of motion and find the trajectory. What kind of trajectory the test particle does follow?

Problem 3. Starting from the ansatz:

$$ds^2 = -A(r) dt^2 + B(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

find the Schwarzschild solution by solving the Einstein equations in vacuo (without using Maple, Mathematica, etc.).

Problem 4 (Only for 629 level or for extra credit). Prove the Birkhoff theorem by showing that the most general spherically symmetric solution of Einstein equations in vacuo,

$$ds^2 = -A(r, t) dt^2 + B(r, t) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

is indeed static, i.e., $A(r, t) = A(r)$ and $B(r, t) = B(r)$. 