PHYS 621 – HOMEWORK # 5 – DUE FRIDAY, 10/02/2009

Problem 1. A sphere of radius a has charge uniformly distributed over its surface with charge density $Q/(4\pi a^2)$, except for a spherical cap at the north pole defined by the cone $\theta = \alpha$, which is kept at zero potential. Show that the potential outside the sphere is:

$$\phi = \frac{Q}{8\pi\epsilon_0 a} \sum_{l=0}^{\infty} \frac{P_{l+1}(\cos\alpha) - P_{l-1}(\cos\alpha)}{2l+1} \left(\frac{a}{r}\right)^{l+1} P_l(\cos\theta)\,,$$

where $P_{-1}(\cos\alpha)$ is defined to be equal to -1. Discuss the limiting form of the potential as the spherical cap becomes very small or very large.

[Hint: You might find useful the following relation: $(2l+1)P_l(x) = P'_{l+1}(x) - P'_{l-1}(x)$.]

Problem 2. A thin flat conducting disc of radius a is maintained at constant potential V. If the surface charge density is proportional to $1/\sqrt{a^2 - d^2}$, where d is the distance from the center of the disc:

a) Show that the potential for $r > a$ is:

$$\phi = \frac{2Va}{\pi r} \sum_{l=0}^{\infty} \frac{(-1)^l}{2l+1} \left(\frac{a}{r}\right)^{2l} P_{2l}(\cos\theta)\,;$$

b) Find the potential for $r < a$;

c) Find the capacitance of the disc.

Problem 3. Consider two concentric spheres of radius a and b, held at constant potential V_a and V_b, respectively.

a) Using an expansion in Legendre polynomials, show that the potential between the two spheres is

$$\phi = A + \frac{B}{r}\,,$$

where

$$A = \frac{V_b/a - V_a/b}{1/a - 1/b} \qquad \text{and} \qquad B = \frac{V_a - V_b}{1/a - 1/b}\,.$$

b) Check the previous result using the Green's function of two concentric spheres

$$G(\mathbf{x}, \mathbf{x}') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y^*_{lm}(\theta', \varphi') Y_{lm}(\theta, \varphi)}{(2l+1)\left[1 - \left(\frac{a}{b}\right)^{2l+1}\right]} \left(r_-^l - \frac{a^{2l+1}}{r_-^{l+1}}\right) \left(\frac{1}{r_+^{l+1}} - \frac{r_+^l}{b^{2l+1}}\right).$$