ANNOUNCEMENTS

- Homework #4, due Monday, September 17:

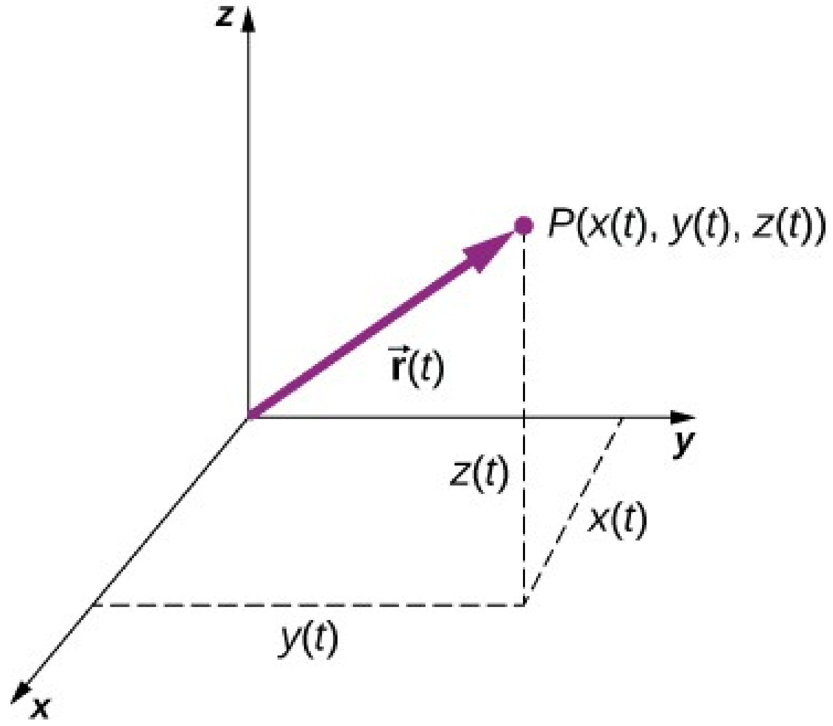
  Conceptual questions: Chapter 4, #6 and #10
  Problems: Chapter 4, #26, #42

- Read Sections 4.4-4.5 before next class (Friday)

- 5-minute quiz on Chapter 4:
  Wednesday, September 19 at beginning of class

- First in-class test will be Friday, September 21

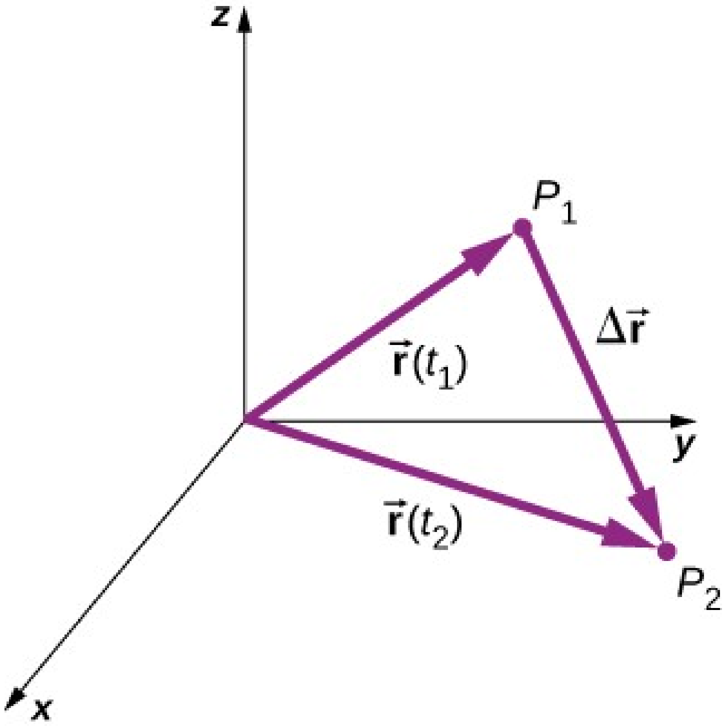
MOTION IN TWO AND THREE DIMENSIONS



A three-dimensional coordinate system with a particle at position *P*(*x*(*t*), *y*(*t*), *z*(*t*)).

$$\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}. \tag{4.2}$$

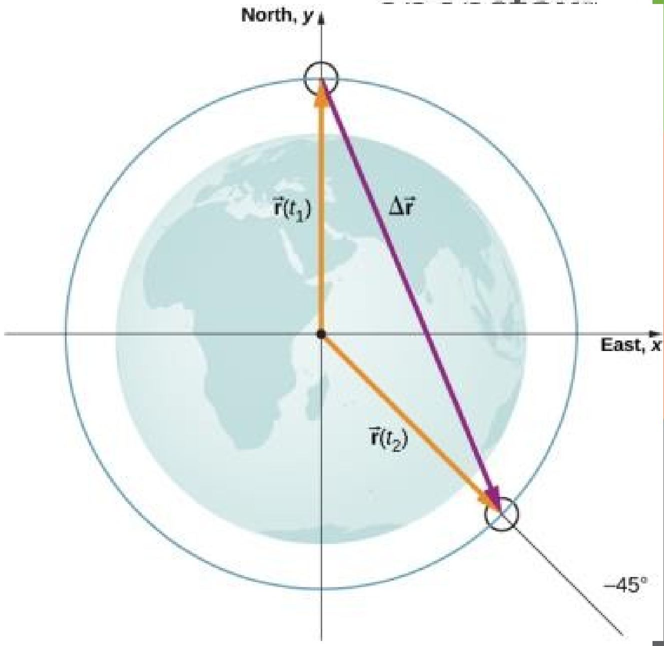
DISPLACEMENT



The displacement $\Delta\vec{r} = \vec{r}(t_2) - \vec{r}(t_1)$ is the vector from P_1 to P_2.

WORKING WITH VECTORS

Two position vectors are drawn from the center of Earth, which is the origin of the coordinate system, with the *y*-axis as north and the *x*-axis as east. The vector between them is the displacement of the satellite.



In unit vector notation, the position vectors are

$$\vec{\mathbf{r}}(t_1) = 6770.\ \text{km}\hat{\mathbf{j}}$$
$$\vec{\mathbf{r}}(t_2) = 6770.\ \text{km}\ (\cos 45°)\hat{\mathbf{i}} + 6770.\ \text{km}\ (\sin(-45°))\hat{\mathbf{j}}.$$

Evaluating the sine and cosine, we have

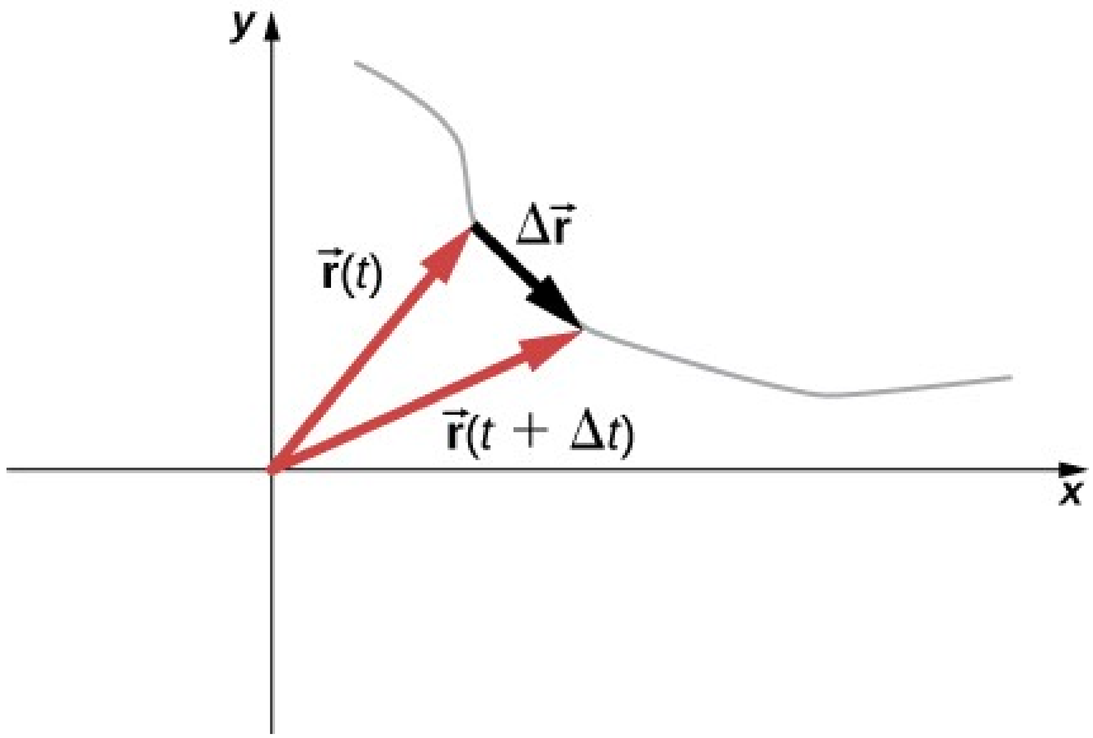
$$\begin{aligned} \vec{\mathbf{r}}(t_1) &= 6770.\hat{\mathbf{j}} \\ \vec{\mathbf{r}}(t_2) &= 4787\hat{\mathbf{i}} - 4787\hat{\mathbf{j}}. \end{aligned}$$

Now we can find $\Delta\vec{\mathbf{r}}$, the displacement of the satellite:

$$\Delta\vec{\mathbf{r}} = \vec{\mathbf{r}}(t_2) - \vec{\mathbf{r}}(t_1) = 4787\hat{\mathbf{i}} - 11{,}557\hat{\mathbf{j}}.$$

The magnitude of the displacement is $\left|\Delta\vec{\mathbf{r}}\right| = \sqrt{(4787)^2 + (-11{,}557)^2} = 12{,}509\ \text{km}.$ The angle the displacement makes with the *x*-axis is $\theta = \tan^{-1}\left(\frac{-11{,}557}{4787}\right) = -67.5°.$

VELOCITY



A particle moves along a path given by the gray line. In the limit as Δ*t* approaches zero, the velocity vector becomes tangent to the path of the particle.

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} = \frac{d\vec{r}}{dt}. \tag{4.4}$$

Equation 4.4 can also be written in terms of the components of $\vec{\mathbf{v}}(t)$. Since

$$\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}},$$

we can write

$$\vec{\mathbf{v}}(t) = v_x(t)\hat{\mathbf{i}} + v_y(t)\hat{\mathbf{j}} + v_z(t)\hat{\mathbf{k}} \tag{4.5}$$

where

$$v_x(t) = \frac{dx(t)}{dt}, \quad v_y(t) = \frac{dy(t)}{dt}, \quad v_z(t) = \frac{dz(t)}{dt}. \tag{4.6}$$

If only the average velocity is of concern, we have the vector equivalent of the one-dimensional average velocity for two and three dimensions:

$$\vec{\mathbf{v}}_{\text{avg}} = \frac{\vec{\mathbf{r}}(t_2) - \vec{\mathbf{r}}(t_1)}{t_2 - t_1}. \tag{4.7}$$

EXAMPLE 4.3

Calculating the Velocity Vector

The position function of a particle is $\vec{\mathbf{r}}(t) = 2.0t^2\hat{\mathbf{i}} + (2.0 + 3.0t)\hat{\mathbf{j}} + 5.0t\hat{\mathbf{k}}\text{m}$. (a) What is the instantaneous velocity and speed at t = 2.0 s? (b) What is the average velocity between 1.0 s and 3.0 s?

Solution

Using Equation 4.5 and Equation 4.6, and taking the derivative of the position function with respect to time, we find

(a) $v(t) = \frac{d\mathbf{r}(t)}{dt} = 4.0t\hat{\mathbf{i}} + 3.0\hat{\mathbf{j}} + 5.0\hat{\mathbf{k}}\text{m/s}$

$\vec{\mathbf{v}}(2.0s) = 8.0\hat{\mathbf{i}} + 3.0\hat{\mathbf{j}} + 5.0\hat{\mathbf{k}}\text{m/s}$

Speed $\left|\vec{\mathbf{v}}(2.0\text{ s})\right| = \sqrt{8^2 + 3^2 + 5^2} = 9.9\text{ m/s}.$

(b) From Equation 4.7,

$$\begin{aligned}\vec{\mathbf{v}}_{\text{avg}} &= \frac{\vec{\mathbf{r}}(t_2)-\vec{\mathbf{r}}(t_1)}{t_2-t_1} = \frac{\vec{\mathbf{r}}(3.0\text{ s})-\vec{\mathbf{r}}(1.0\text{ s})}{3.0\text{ s}-1.0\text{ s}} = \frac{(18\hat{\mathbf{i}}+11\hat{\mathbf{j}}+15\hat{\mathbf{k}})\text{ m}-(2\hat{\mathbf{i}}+5\hat{\mathbf{j}}+5\hat{\mathbf{k}})\text{ m}}{2.0\text{ s}} \\ &= \frac{(16\hat{\mathbf{i}}+6\hat{\mathbf{j}}+10\hat{\mathbf{k}})\text{ m}}{2.0\text{ s}} = 8.0\hat{\mathbf{i}} + 3.0\hat{\mathbf{j}} + 5.0\hat{\mathbf{k}}\text{m/s}.\end{aligned}$$

Significance

We see the average velocity is the same as the instantaneous velocity at t = 2.0 s, as a result of the velocity function being linear. This need not be the case in general. In fact, most of the time, instantaneous and average velocities are not the same.

$$\vec{\mathbf{a}}(t) = \lim_{t \to 0} \frac{\vec{\mathbf{v}}(t + \Delta t) - \vec{\mathbf{v}}(t)}{\Delta t} = \frac{d\vec{\mathbf{v}}(t)}{dt}. \tag{4.8}$$

The acceleration in terms of components is

$$\vec{\mathbf{a}}(t) = \frac{dv_x(t)}{dt}\hat{\mathbf{i}} + \frac{dv_y(t)}{dt}\hat{\mathbf{j}} + \frac{dv_z(t)}{dt}\hat{\mathbf{k}}. \tag{4.9}$$

Also, since the velocity is the derivative of the position function, we can write the acceleration in terms of the second derivative of the position function:

$$\vec{\mathbf{a}}(t) = \frac{d^2x(t)}{dt^2}\hat{\mathbf{i}} + \frac{d^2y(t)}{dt^2}\hat{\mathbf{j}} + \frac{d^2z(t)}{dt^2}\hat{\mathbf{k}}. \tag{4.10}$$

EXAMPLE 4.5

Finding a Particle Acceleration

A particle has a position function $\vec{\mathbf{r}}(t) = (10t - t^2)\hat{\mathbf{i}} + 5t\hat{\mathbf{j}} + 5t\hat{\mathbf{k}}\text{m}$. (a) What is the velocity? (b) What is the acceleration? (c) Describe the motion from $t = 0$ s.

Strategy

We can gain some insight into the problem by looking at the position function. It is linear in *y* and *z*, so we know the acceleration in these directions is zero when we take the second derivative. Also, note that the position in the *x* direction is zero for $t = 0$ s and $t = 10$ s.

Solution

(a) Taking the derivative with respect to time of the position function, we find

$$\vec{\mathbf{v}}(t) = (10 - 2t)\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 5\hat{\mathbf{k}}\ \text{m/s}.$$

The velocity function is linear in time in the *x* direction and is constant in the *y* and *z* directions.

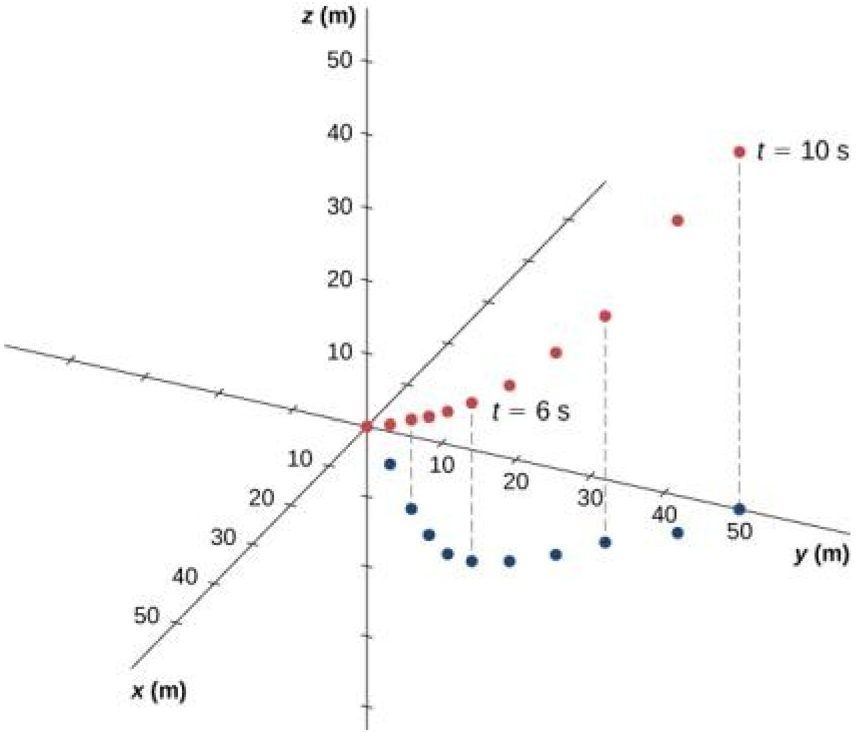
(b) Taking the derivative of the velocity function, we find

$$\vec{\mathbf{a}}(t) = -2\hat{\mathbf{i}}\ \text{m/s}^2.$$

The acceleration vector is a constant in the negative *x*-direction.

FIGURE 4.9

The particle starts at point $(x, y, z) = (0, 0, 0)$ with position vector $\vec{r} = 0$. The projection of the trajectory onto the *xy*-plane is shown. The values of *y* and *z* increase linearly as a function of time, whereas *x* has a turning point at $t = 5$ s and 25 m, when it reverses direction. At this point, the *x* component of the velocity becomes negative. At $t = 10$ s, the particle is back to 0 m in the *x* direction.



INDEPENDENCE OF MOTION

In the kinematic description of motion, we are able to treat the horizontal and vertical components of motion separately. In many cases, motion in the horizontal direction does not affect motion in the vertical direction, and vice versa.

SPECIAL CASE: CONSTANT ACCELERATION

$$x(t) = x_0 + (v_x)_{\text{avg}} t \qquad \boxed{(4.11)}$$

$$v_x(t) = v_{0x} + a_x t \qquad \boxed{(4.12)}$$

$$x(t) = x_0 + v_{0x} t + \frac{1}{2} a_x t^2 \qquad \boxed{(4.13)}$$

$$v_x^2(t) = v_{0x}^2 + 2a_x(x - x_0) \qquad \boxed{(4.14)}$$

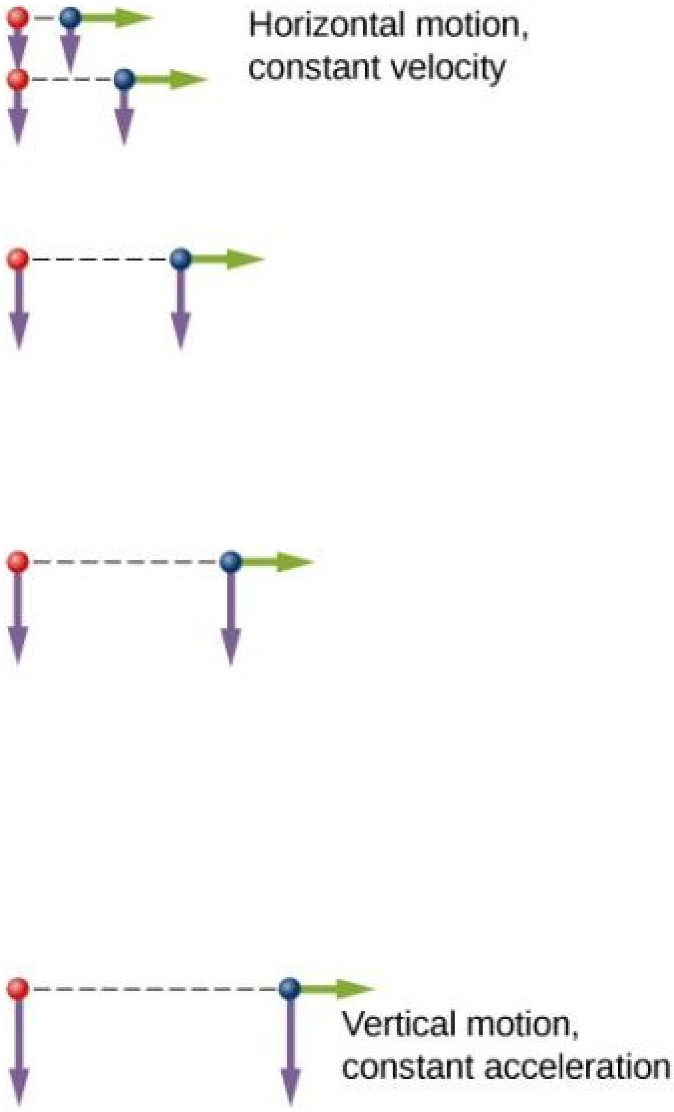
$$y(t) = y_0 + (v_y)_{\text{avg}} t \qquad \boxed{(4.15)}$$

$$v_y(t) = v_{0y} + a_y t \qquad \boxed{(4.16)}$$

$$y(t) = y_0 + v_{0y} t + \frac{1}{2} a_y t^2 \qquad \boxed{(4.17)}$$

$$v_y^2(t) = v_{0y}^2 + 2a_y(y - y_0). \qquad \boxed{(4.18)}$$

SPECIAL CASE: CONSTANT VELOCITY IN ONE DIRECTION



A diagram of the motions of two identical balls: one falls from rest and the other has an initial horizontal velocity. Each subsequent position is an equal time interval. Arrows represent the horizontal and vertical velocities at each position. The ball on the right has an initial horizontal velocity whereas the ball on the left has no horizontal velocity. Despite the difference in horizontal velocities, the vertical velocities and positions are identical for both balls, which shows the vertical and horizontal motions are independent.

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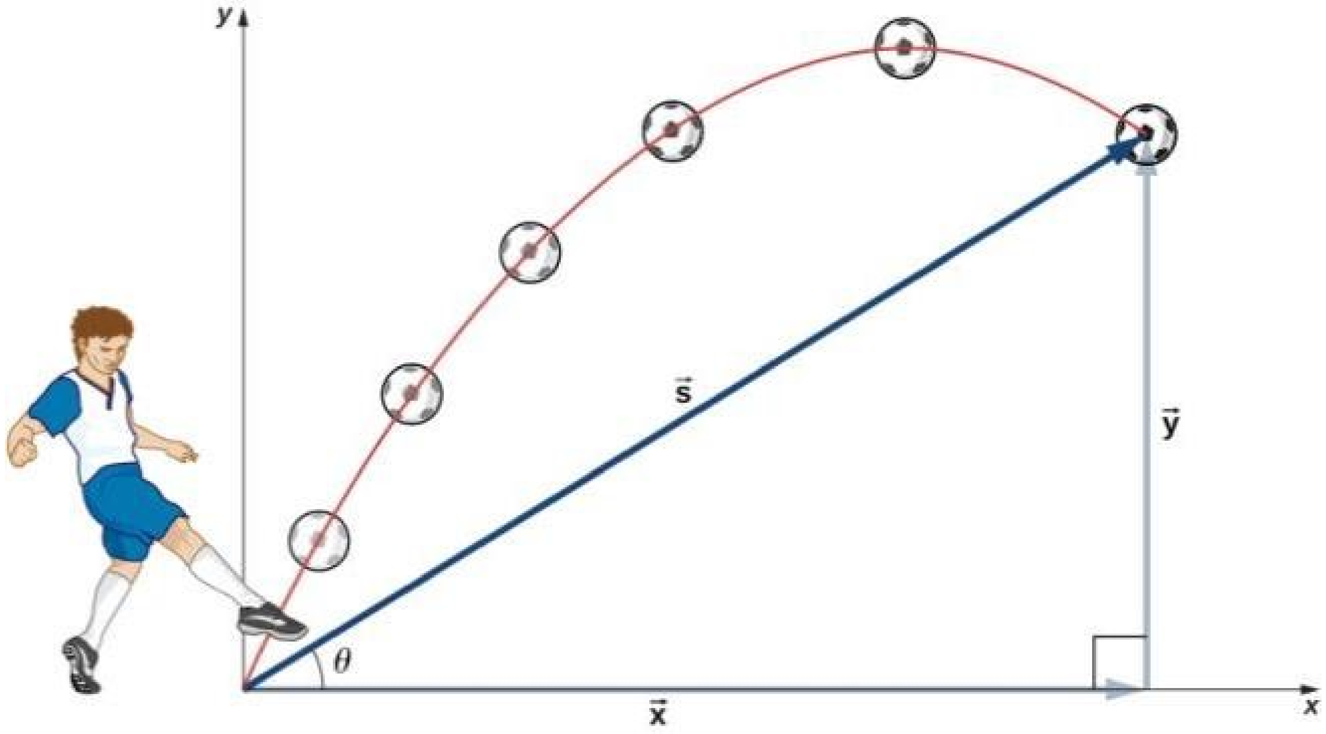
PARABOLIC (PROJECTILE) MOTION



The total displacement s of a soccer ball at a point along its path. The vector $\vec{s}$ has components $\vec{x}$ and $\vec{y}$ along the horizontal and vertical axes. Its magnitude is s and it makes an angle θ with the horizontal.

PARABOLIC (PROJECTILE) MOTION

Horizontal Motion

$$v_{0x} = v_x, \; x = x_0 + v_x t \tag{4.19}$$

Vertical Motion

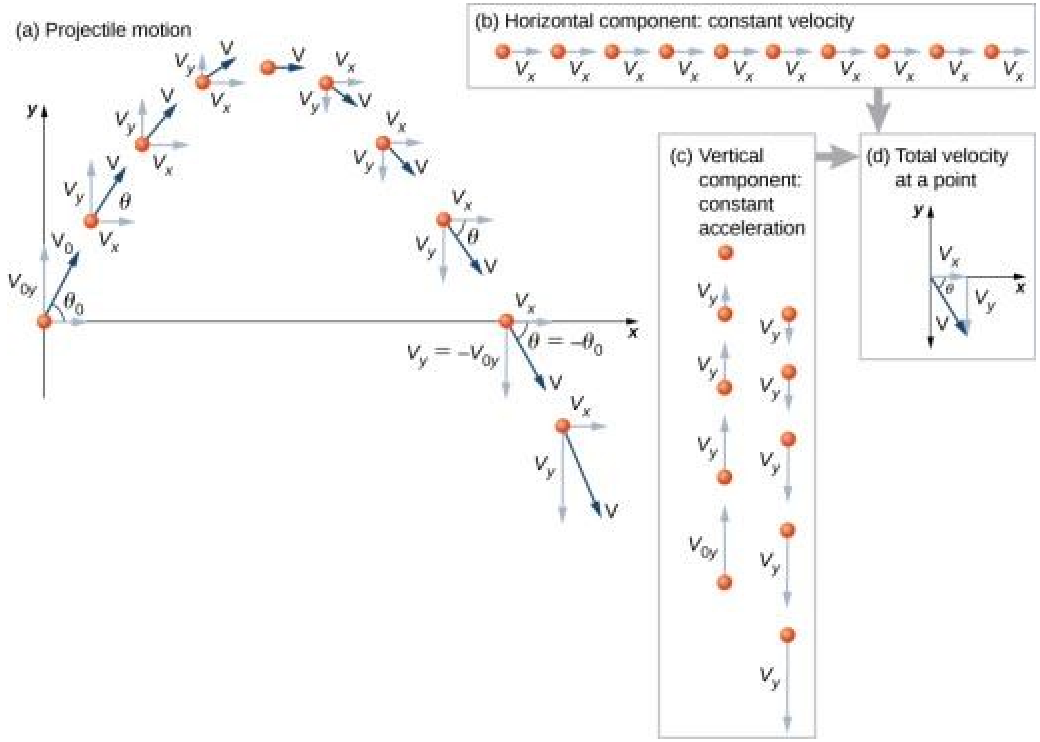
$$y = y_0 + \frac{1}{2}(v_{0y} + v_y)t \tag{4.20}$$

$$v_y = v_{0y} - gt \tag{4.21}$$

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2 \tag{4.22}$$

$$v_y^2 = v_{0y}^2 - 2g(y - y_0) \tag{4.23}$$

PARABOLIC (PROJECTILE) MOTION



(a) We analyze two-dimensional projectile motion by breaking it into two independent one-dimensional motions along the vertical and horizontal axes.

(b) The horizontal motion is simple, because $a_x = 0$ and v_x is a constant.

(c) The velocity in the vertical direction begins to decrease as the object rises. At its highest point, the vertical velocity is zero. As the object falls toward Earth again, the vertical velocity increases again in magnitude but points in the opposite direction to the initial vertical velocity.

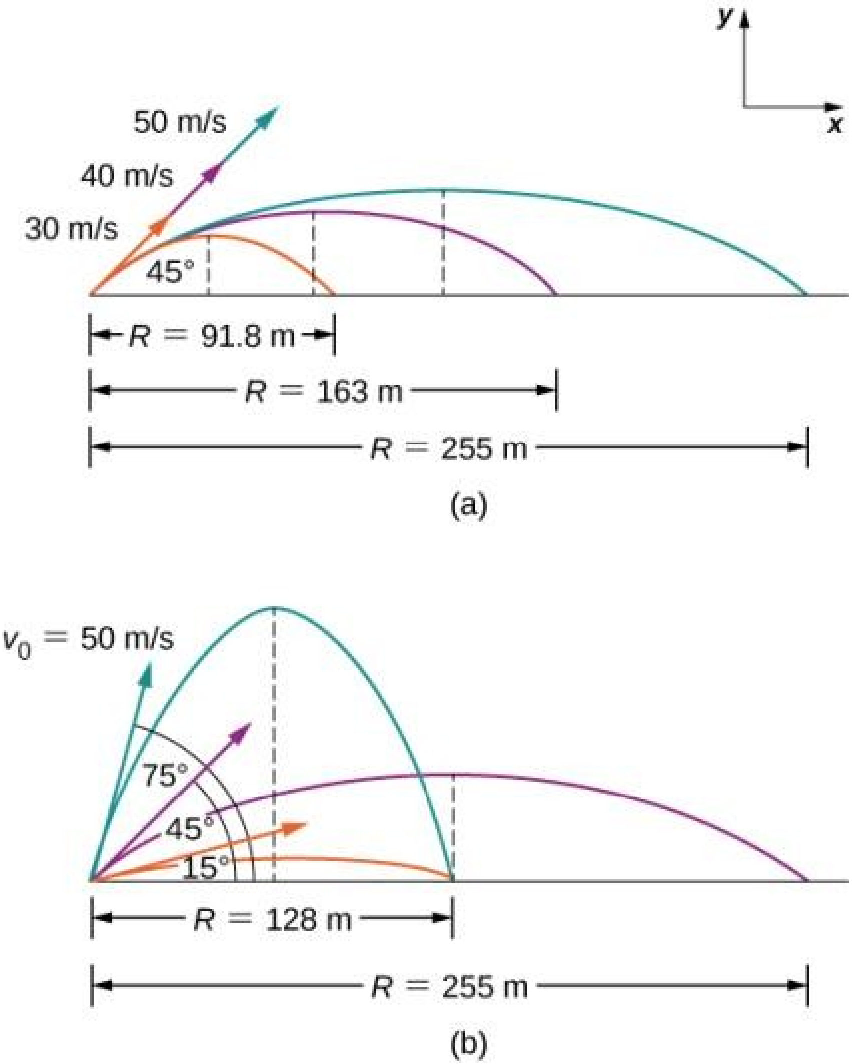
(d) The x and y motions are recombined to give the total velocity at any given point on the trajectory.

FIGURE 4.15

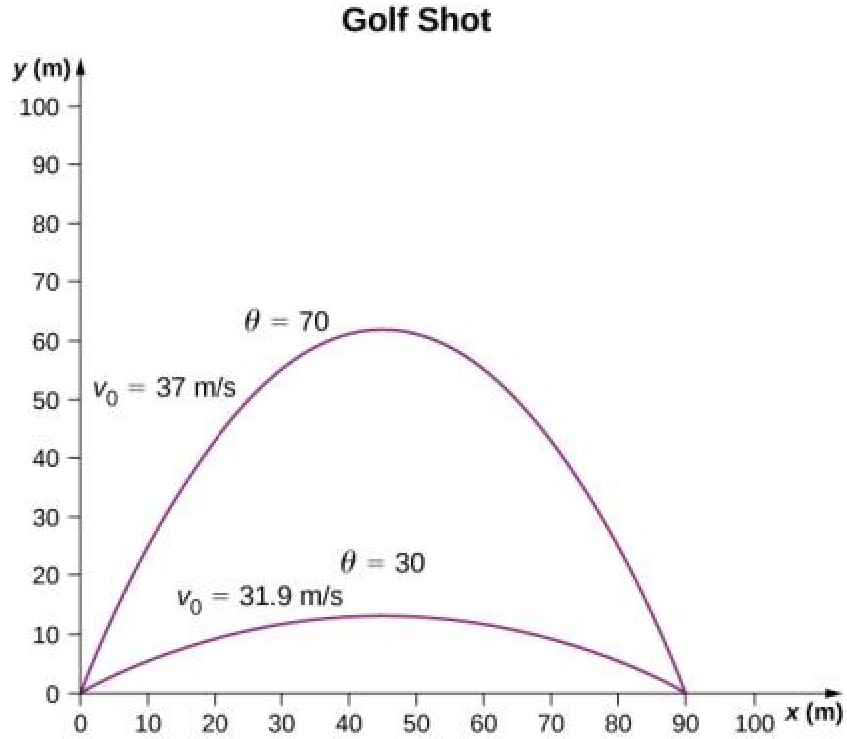
Trajectories of projectiles on level ground.

(a) The greater the initial speed v_0, the greater the range for a given initial angle.

(b) The effect of initial angle θ_0 on the range of a projectile with a given initial speed. Note that the range is the same for initial angles of 15° and 75°, although the maximum heights of those paths are different.



PARABOLIC (PROJECTILE) MOTION



Two trajectories of a golf ball with a range of 90 m. The impact points of both are at the same level as the launch point.

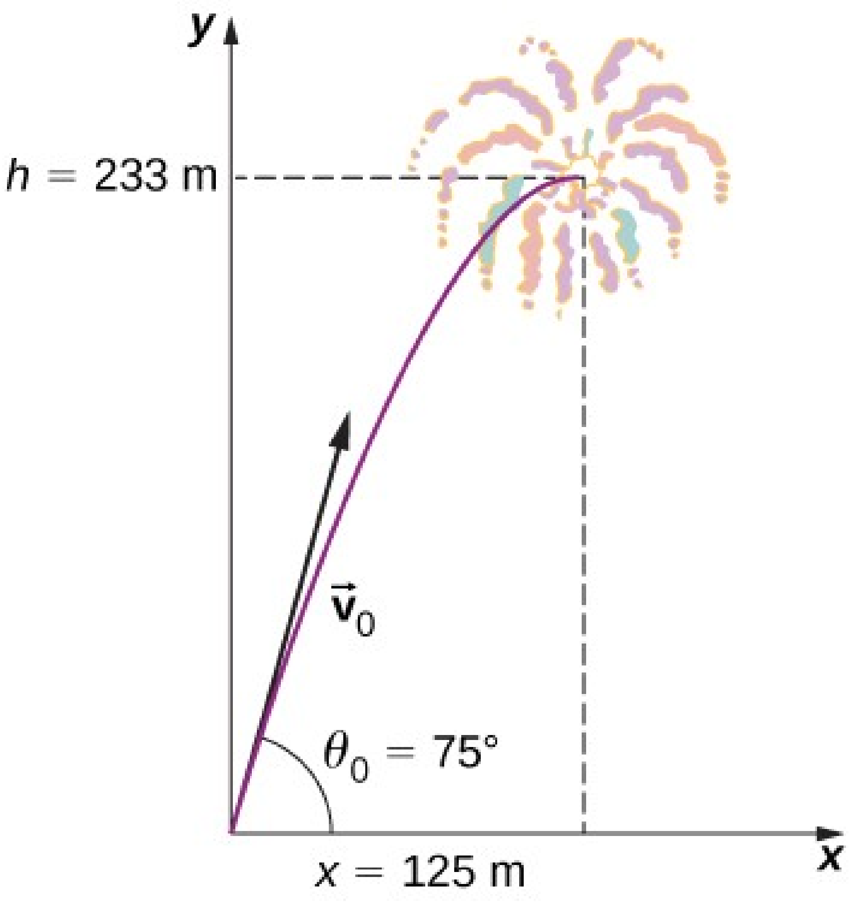
Problem-Solving Strategy: Projectile Motion

1. Resolve the motion into horizontal and vertical components along the *x*- and *y*-axes. The magnitudes of the components of displacement $\vec{\mathbf{s}}$ along these axes are *x* and *y*. The magnitudes of the components of velocity $\vec{\mathbf{v}}$ are $v_x = v\cos\theta$ and $v_y = v\sin\theta$, where *v* is the magnitude of the velocity and *θ* is its direction relative to the horizontal, as shown in Figure 4.12.
2. Treat the motion as two independent one-dimensional motions: one horizontal and the other vertical. Use the kinematic equations for horizontal and vertical motion presented earlier.
3. Solve for the unknowns in the two separate motions: one horizontal and one vertical. Note that the only common variable between the motions is time *t*. The problem-solving procedures here are the same as those for one-dimensional kinematics and are illustrated in the following solved examples.
4. Recombine quantities in the horizontal and vertical directions to find the total displacement $\vec{\mathbf{s}}$ and velocity $\vec{\mathbf{v}}$. Solve for the magnitude and direction of the displacement and velocity using

$$s = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}(y/x), \quad v = \sqrt{v_x^2 + v_y^2},$$

where *θ* is the direction of the displacement $\vec{\mathbf{s}}$.

EXAMPLE



The trajectory of a fireworks shell. The fuse is set to explode the shell at the highest point in its trajectory, which is found to be at a height of 233 m and 125 m away horizontally.

Solution

(a) By “height” we mean the altitude or vertical position *y* above the starting point. The highest point in any trajectory, called the *apex*, is reached when $v_y = 0$. Since we know the initial and final velocities, as well as the initial position, we use the following equation to find *y*:

$$v_y^2 = v_{0y}^2 - 2g(y - y_0).$$

Because y_0 and v_y are both zero, the equation simplifies to

$$0 = v_{0y}^2 - 2gy.$$

Solving for *y* gives

$$y = \frac{v_{0y}^2}{2g}.$$

Now we must find v_{0y}, the component of the initial velocity in the *y* direction. It is given by $v_{0y} = v_0 \sin\theta_0$, where v_0 is the initial velocity of 70.0 m/s and $\theta_0 = 75°$ is the initial angle. Thus,

$$v_{0y} = v_0 \sin\theta = (70.0 \text{ m/s}) \sin 75° = 67.6 \text{ m/s}$$

and *y* is

$$y = \frac{(67.6 \text{ m/s})^2}{2(9.80 \text{ m/s}^2)}.$$

Thus, we have

$$y = 233 \text{ m}.$$

(b) As in many physics problems, there is more than one way to solve for the time the projectile reaches its highest point. In this case, the easiest method is to use $v_y = v_{0y} - gt$. Because $v_y = 0$ at the apex, this equation reduces to simply

$$0 = v_{0y} - gt$$

or

$$t = \frac{v_{0y}}{g} = \frac{67.6 \text{ m/s}}{9.80 \text{ m/s}^2} = 6.90\text{s}.$$

This time is also reasonable for large fireworks. If you are able to see the launch of fireworks, notice that several seconds pass before the shell explodes. Another way of finding the time is by using$y = y_0 + \frac{1}{2}(v_{0y} + v_y)t$. This is left for you as an exercise to complete.

(c) Because air resistance is negligible, $a_x = 0$ and the horizontal velocity is constant, as discussed earlier. The horizontal displacement is the horizontal velocity multiplied by time as given by $x = x_0 + v_x t$, where x_0 is equal to zero. Thus,

$$x = v_x t,$$

where v_x is the *x*-component of the velocity, which is given by

$$v_x = v_0 \cos\theta = (70.0 \text{ m/s})\cos 75° = 18.1 \text{ m/s}.$$

Time *t* for both motions is the same, so *x* is

$$x = (18.1 \text{ m/s})6.90 \text{ s} = 125 \text{ m}.$$

Horizontal motion is a constant velocity in the absence of air resistance. The horizontal displacement found here could be useful in keeping the fireworks fragments from falling on spectators. When the shell explodes, air resistance has a major effect, and many fragments land directly below.

USEFUL EQUATIONS

TIME OF FLIGHT

$$T_{\text{tof}} = \frac{2(v_0 \sin\theta_0)}{g}. \tag{4.24}$$

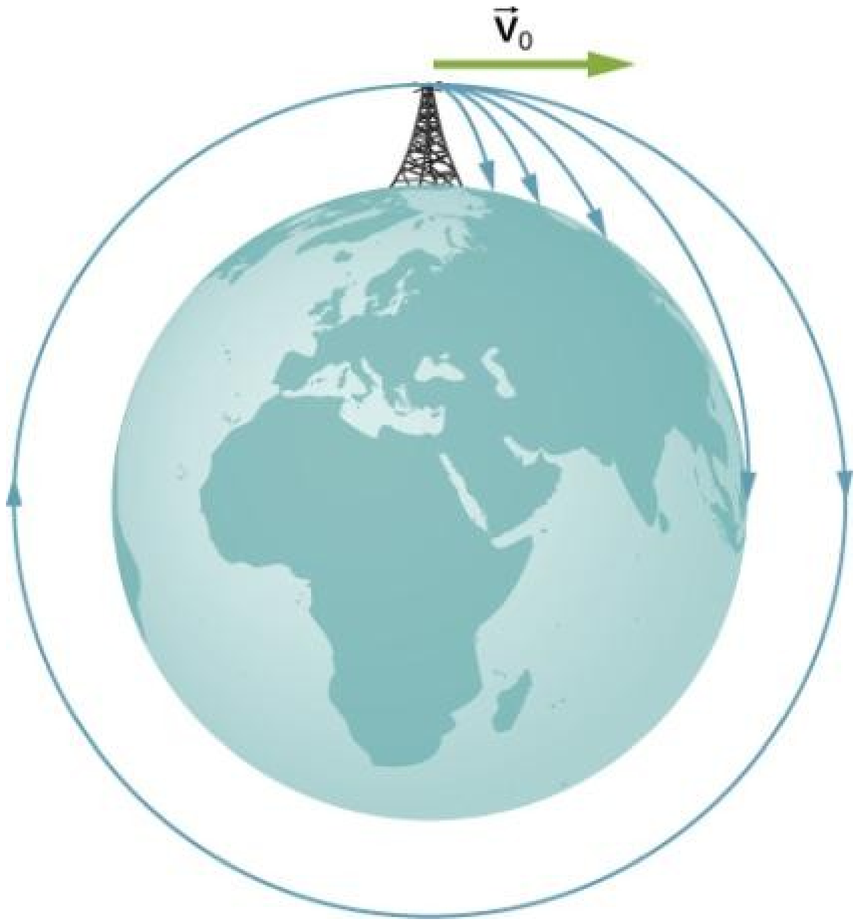
TRAJECTORY

$$y = (\tan\theta_0)x - \left[\frac{g}{2(v_0 \cos\theta_0)^2}\right]x^2. \tag{4.25}$$

RANGE

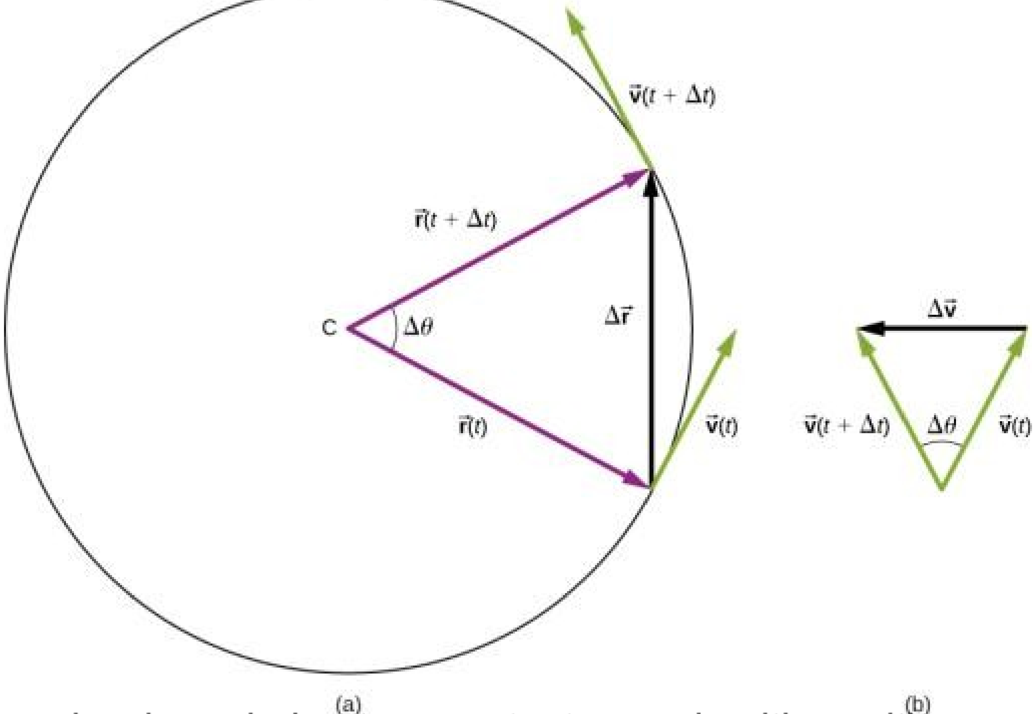
$$R = \frac{v_0^2 \sin 2\theta_0}{g}. \tag{4.26}$$

ORBITS



Projectile to satellite. In each case shown here, a projectile is launched from a very high tower to avoid air resistance. With increasing initial speed, the range increases and becomes longer than it would be on level ground because Earth curves away beneath its path. With a speed of 8000 m/s, orbit is achieved.

UNIFORM CIRCULAR MOTION



(a) A particle is moving in a circle at a constant speed, with position and velocity vectors at times t and $t + \Delta t$.

(b) Velocity vectors forming a triangle. The two triangles in the figure are similar. The vector $\Delta\vec{v}$ points toward the center of the circle in the limit $\Delta t \to 0$.

$$a_C = \frac{v^2}{r}. \qquad (4.27)$$

EXAMPLE 4.10

Creating an Acceleration of 1 *g*

A jet is flying at 134.1 m/s along a straight line and makes a turn along a circular path level with the ground. What does the radius of the circle have to be to produce a centripetal acceleration of 1 *g* on the pilot and jet toward the center of the circular trajectory?

Strategy

Given the speed of the jet, we can solve for the radius of the circle in the expression for the centripetal acceleration.

Solution

Set the centripetal acceleration equal to the acceleration of gravity: $9.8\ \text{m/s}^2 = v^2/r$.

Solving for the radius, we find

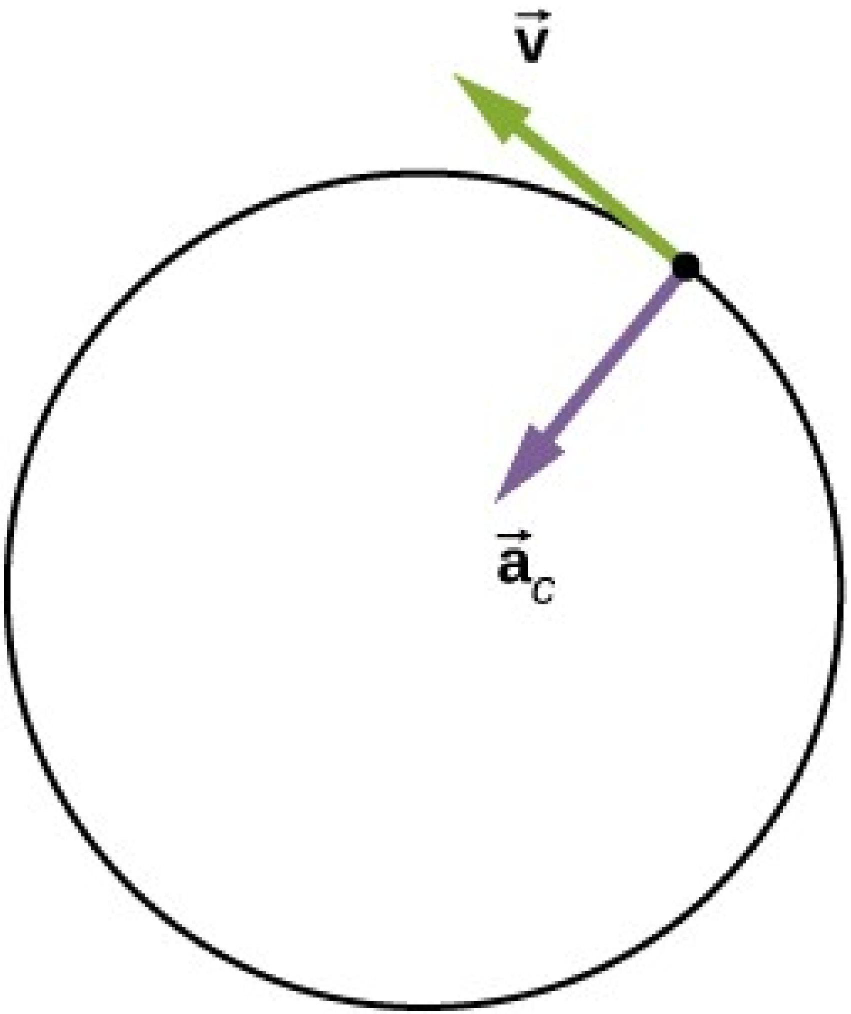
$$r = \frac{(134.1\ \text{m/s})^2}{9.8\ \text{m/s}^2} = 1835\ \text{m} = 1.835\ \text{km}.$$

Significance

To create a greater acceleration than *g* on the pilot, the jet would either have to decrease the radius of its circular trajectory or increase its speed on its existing trajectory or both.

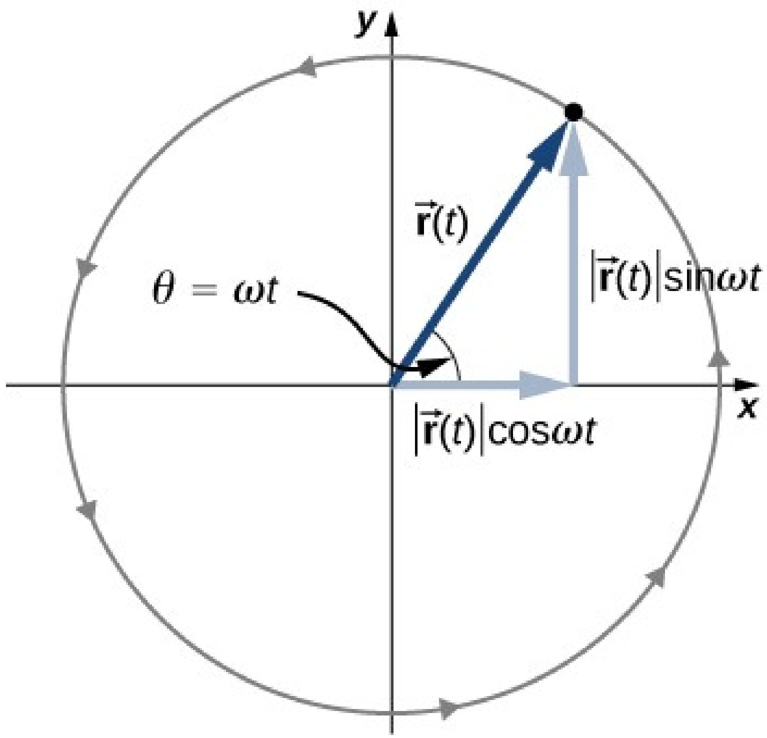
CENTRIPETAL ACCELERATION

The centripetal acceleration vector points toward the center of the circular path of motion and is an acceleration in the radial direction. The velocity vector is also shown and is tangent to the circle.



ANGULAR FREQUENCY

$$\omega = \frac{2\pi}{T}.$$



The position vector for a particle in circular motion with its components along the *x*- and *y*-axes. The particle moves counterclockwise. Angle θ is the angular frequency in radians per second multiplied by *t*.

$$\vec{\mathbf{r}}(t) = A\cos\omega t\hat{\mathbf{i}} + A\sin\omega t\hat{\mathbf{j}}. \qquad (4.28)$$

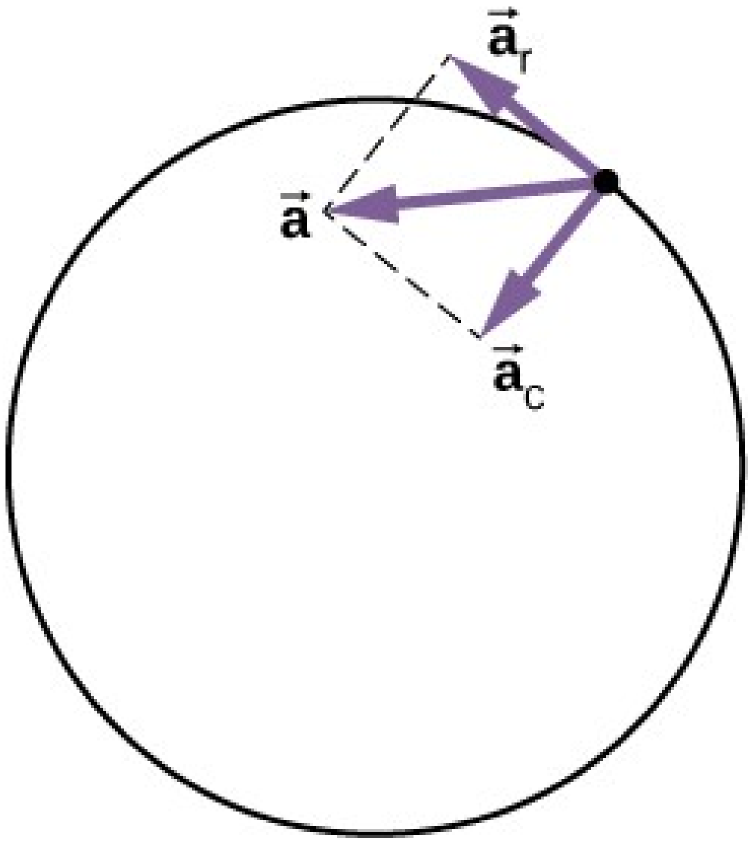
CIRCULAR MOTION: POSITION, VELOCITY AND ACCELERATION

$$\vec{\mathbf{r}}(t) = A \cos \omega t \hat{\mathbf{i}} + A \sin \omega t \hat{\mathbf{j}}. \tag{4.28}$$

$$\vec{\mathbf{v}}(t) = \frac{d\vec{\mathbf{r}}(t)}{dt} = -A\omega \sin \omega t \hat{\mathbf{i}} + A\omega \cos \omega t \hat{\mathbf{j}}. \tag{4.29}$$

$$\vec{\mathbf{a}}(t) = \frac{d\vec{\mathbf{v}}(t)}{dt} = -A\omega^2 \cos \omega t \hat{\mathbf{i}} - A\omega^2 \sin \omega t \hat{\mathbf{j}}. \tag{4.30}$$

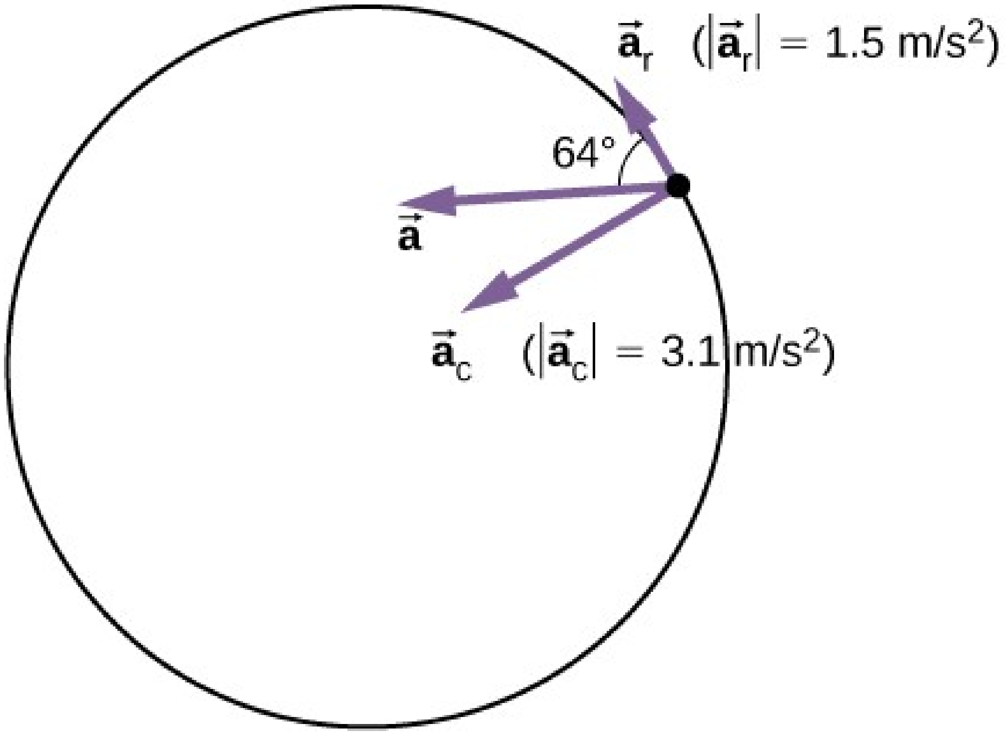
NON-UNIFORM CIRCULAR MOTION



The centripetal acceleration points toward the center of the circle. The tangential acceleration is tangential to the circle at the particle's position. The total acceleration is the vector sum of the tangential and centripetal accelerations, which are perpendicular.

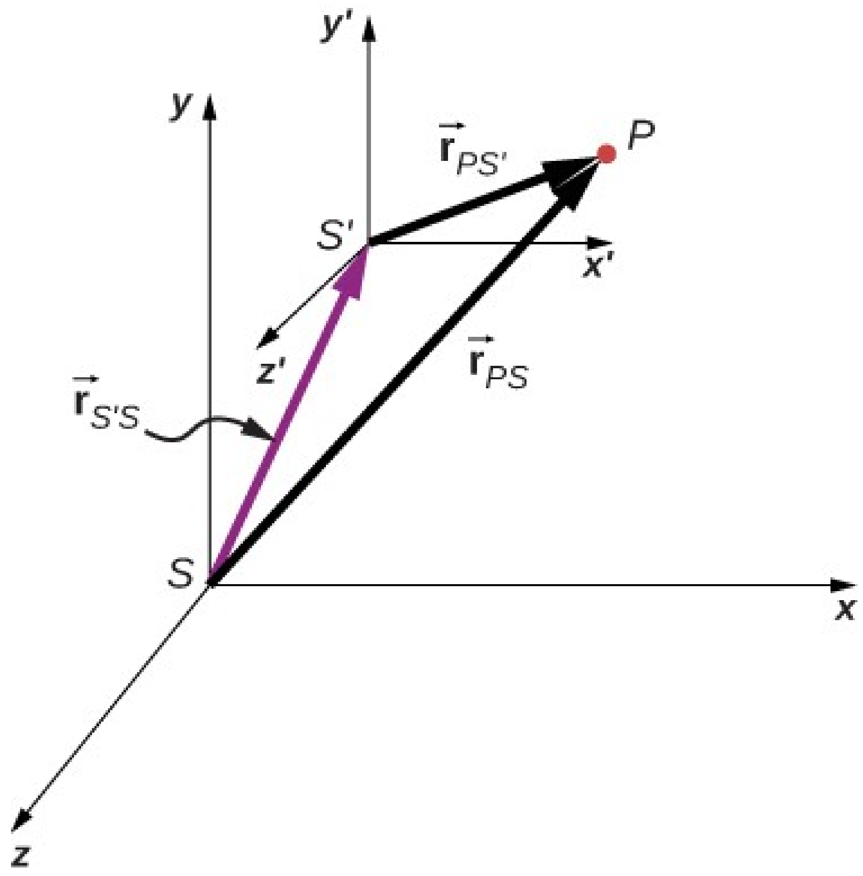
$$a_T = \frac{d\left|\vec{v}\right|}{dt}. \tag{4.31}$$

TANGENTIAL ACCELERATION



The tangential and centripetal acceleration vectors. The net acceleration is the vector sum of the two accelerations.

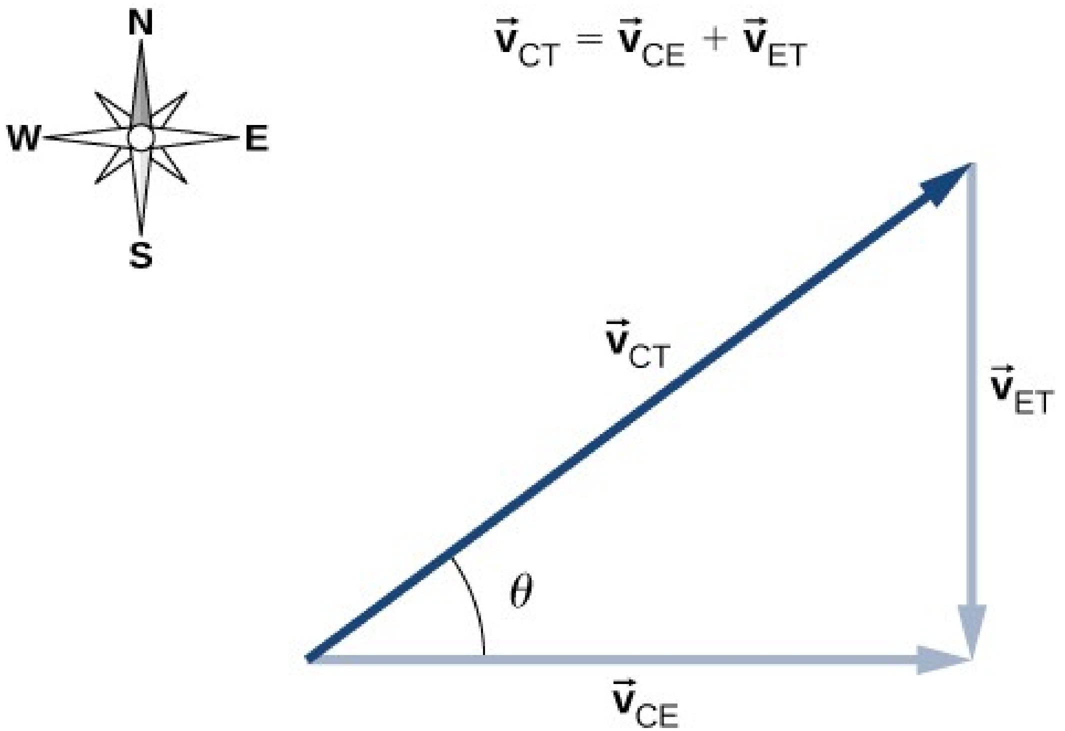
The positions of particle P relative to frames S and S' are $\vec{\mathbf{r}}_{PS}$ and $\vec{\mathbf{r}}_{PS'}$, respectively.



ADDITION OF VELOCITIES

Velocity vectors of the train with respect to Earth, person with respect to the train, and person with respect to Earth.

EXAMPLE



Vector diagram of the vector equation $\vec{v}_{CT} = \vec{v}_{CE} + \vec{v}_{ET}$.

EXAMPLE

Vector diagram for **Equation 4.34** showing the vectors $\vec{\mathbf{v}}_{PA}$, $\vec{\mathbf{v}}_{AG}$, $\vec{\mathbf{v}}_{PG}$.