Noncommutative Extra Dimensions: Cosmological Implications

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The Physical Theories

1. **Noncommutative Physics**

2. **Kaluza-Klein Theory**

3. **Casimir Energy**

4. **Cosmology: Dark Energy and Inflation**

Noncommutative quantum field theory

- ordinary spacetime variables commute

$$[x^\mu, x^\nu] = 0$$

- in order to be noncommutative they become operators which satisfy

$$[\hat{x}^\mu, \hat{x}^\nu] = i\Theta^{\mu\nu}$$

- spacetime uncertainty relations

$$\Delta x^\mu \Delta x^\nu \geq \frac{1}{2} |\Theta^{\mu\nu}|$$

  – space-time points $\mapsto$ Planck size area cells.

  – world appears to be fuzzy at small distances.

Noncommutative Sphere

- S_F^2: fuzzy version of the 2 dimensional sphere S^2

$$\left[\hat{x}^i, \hat{x}^j\right] = i\Theta^{ij} = -i\,\gamma\,\epsilon_{ijk}\hat{x}^k \qquad i,j = 1\ldots 3$$

$$\hat{x}^i\hat{x}^j\delta_{ij} = a^2\mathbf{1}$$

  – γ: noncummutative parameter, a: the radius of S^2.

- relationship between γ, a and the size of the representation $M+1$ of the $SU(2)$ algebra

$$\frac{a}{\gamma} = \sqrt{\frac{M}{2}\Big(\frac{M}{2}+1\Big)}$$

- S^2 is recovered when

$$\gamma \to 0 \qquad or \qquad M \to \infty \qquad \text{with } a \text{ fixed}$$

Kaluza-Klein theory

- topology of fifth dimension is a circle

$$\phi(x,y) = \phi(x, y + 2\pi r) = \sum_n \phi_n(x) e^{ip_n y} \qquad p_n = \frac{n}{r}$$

- cylinder condition on y for the $n = 0$ mode

$$\partial_y \phi(x,y) = 0 \qquad \partial_y g_{\mu\nu}(x,y) = 0 \qquad \partial_y A_\mu(x,y) = 0$$

- five dimensional metric

$$g_{ab} = \begin{pmatrix} g_{\mu\nu} + \kappa^2 A_{\mu\nu} & \kappa A_\nu \\ \kappa A_\mu & \phi \end{pmatrix}$$

$$a, b = 0 \ldots 4 \qquad \mu, \nu = 0 \ldots 3$$

- five dimensional Einstein equations

$$R_{ab} - \frac{1}{2} g_{ab} R = 0$$

- four dimensional field equations

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= \frac{\kappa^2 \phi^2}{2} T_{\mu\nu}^{EM} \\ \nabla^\mu F_{\mu\nu} &= 0 \end{aligned}$$

– $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $\phi(x) = \text{constant}$

- relevance of extra dimensions to include other interactions
five dimensions for $U(1) \longrightarrow$ eleven dimensions for $U(1) \times SU(2) \times SU(3)$

Casimir Energy

- zero point energy density

$$\frac{<0|H|0>}{V} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \hbar w_k$$

- Casimir energy between two plates

$$2 \sum_n \int \frac{dk_y dk_z}{(2\pi)^2} \frac{\hbar}{2} \sqrt{(\frac{\pi n}{d})^2 + k_y^2 + k_z^2} \simeq \frac{-\hbar c \pi^2}{720 d^3} \qquad k_x \rightarrow k_n = \frac{\pi n}{d}$$

– cutoff of higher frequencies

– negative value $\rightarrow$ attractive force between the plates

- Boundary conditions for S^N

compact space without boundaries $\longrightarrow$ periodic boundary conditions

Cosmology: Dark Energy

- Robertson-Walker metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a(t)^2 \Big[\frac{1}{1 - \kappa r^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \Big]$$

- energy-momentum tensor of a perfect fluid in the comoving frame

$$T_{00} = \varrho \qquad T_{ij} = p$$

- Einstein equation

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

- Friedman equations

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \Big(\varrho + 3p \Big) \qquad \Big(\frac{\dot{a}}{a} \Big)^2 = \frac{8\pi G}{3} \varrho - \frac{\kappa}{a^2}$$

- acceleration condition
  - if $\varrho + 3p < 0 \longrightarrow \ddot{a} > 0 \longrightarrow$ acceleration of the Universe
  - $p = \omega\varrho \longrightarrow \omega < -\frac{1}{3}$ set $\omega = -1 \longrightarrow p = -\rho$
- energy components

$$\Omega = \frac{\rho}{\rho_c} = \Omega_B + \Omega_R + \Omega_{DM} + \Omega_{DE} \simeq 1 \qquad \rho_c = \frac{3H^2}{8\pi G}$$
$$= .05 + 10^{-5} + .30 + .65$$

- dark energy problem: what is dark energy in term of physical theory?
  - cosmological constant Λ
  - quintessence
  - ...
- current research
  Dark energy from Casimir energy on noncommutative extra dimensions
  - S. Fabi, B. Harms, G. Karatheodoris, Phys.Rev.D **74**, 083506 (2006)

The energy density

- energy-momentum tensor for $M^4 \times S^2$

$$t_{ab} = \partial_a\varphi\partial_b\varphi - \frac{1}{2}g_{ab}\partial_c\varphi\partial^c\varphi \qquad (a,b=0\dots5)$$

- vacuum expectation value in term of the Green's function

$$< 0|t_{00}|0 >= \lim_{(x,y)\to(x',y')} \partial_0\partial_0'\mathrm{ImG}(x,y,x',y') \qquad x\in M^4 \quad y\in S^2$$

- Fourier expansion of the Green's function

$$G(x,y,x',y') = \int \frac{d^4k}{(2\pi)^4} e^{k_\mu(x^\mu - x'^\mu)} g(y,y',k^\lambda k_\lambda)$$

- vacuum energy density on M^4

$$u(a) = V_{S^2} < 0|t_{00}|0 >= -\frac{iV_{S^2}}{2(2\pi)^4}\int d^3k \int d\omega\omega^2 g(y,y,\kappa^\lambda\kappa_\lambda)$$

- reduced Green function equation

$$(\nabla^2 + k^\lambda k_\lambda) g(y, y', k^\lambda k_\lambda) = -\delta(y - y')$$

- reduced Green's funcion on S^2

$$g(y, y', \mu^2) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y^l_m(y) \bar{Y}^l_m(y')}{l(l+1) + \mu^2} \qquad \mu^2 = a^2(k^2 - \omega^2)$$

- vacuum energy density of M^4 in the commutative case

$$u(a) = -\frac{i}{(2\pi)^4} \int d^3k \int d\omega\, \omega^2 \sum_{l=0}^{\infty} \frac{2l+1}{\left(\frac{l(l+1)}{a^2} + k^2 - w^2\right)}$$

- energy density result

$$u(a) = \frac{\alpha_{KM}}{a^4} \ln \frac{a}{b} \qquad b \sim L_{Planck}$$

- reduced Green's function in the noncommutative case

$$g_M(y,y',\mu^2) = \sum_{l=0}^{M} \sum_{m=-l}^{l} \frac{Y_m^l(y)\bar{Y}_m^l(y')}{l(l+1)+\mu^2} \qquad \mu^2 = a^2(k^2-\omega^2)$$

- vacuum energy density of M^4 in the noncommutative case

$$u(a)_{NC} = \frac{\alpha_{TOT}}{a^4} \ln \frac{a}{b} \qquad \alpha_{TOT} = \alpha_{KM} + \alpha_{NC}(M) + \alpha_C(M)$$

- numerical values for $M^4 \times S_F^2$ and $M+1=2$

$$\alpha_{KM} = -\frac{1}{1260\pi} \quad \alpha_{NC} = -\frac{0.0060}{\pi^2} \quad \alpha_C = \frac{1}{12\pi^2} \quad \rightarrow \qquad \alpha_{TOT} = 0.0077 \quad > 0$$

- dark energy and numerical value of a

$$u(a)_{NC} = u_{\text{dark energy}} = \frac{3H_0^2}{8\pi G} = 1.05 \times 10^{-5} h_0^2 \text{GeVcm}^{-3}$$

$$\rightarrow a = \alpha_{TOT}(M)\Big[\ln(\tfrac{a}{L_{Pl}})\Big]^{\frac{1}{4}} 80\mu m = 71.92\mu m \quad \text{below experimental limits} \quad !$$

- commutative limit: $\lim_{M\to\infty} u(a)_{NC} = u(a)$

$$u(a) = -\frac{i}{(2\pi)^4}\int d^3k \int d\omega\, \omega^2 \sum_{l=0}^{\infty} \frac{2l+1}{\left(\frac{l(l+1)}{a^2} + k^2 - w^2\right)}$$

- the sum in the commutative case:

$$\frac{1}{2}\Big[-\psi(\frac{1}{2}+\beta) - \psi(\frac{1}{2}-\beta) - 2\gamma + 2\zeta(1)\Big]$$

- the sum in the non commutative case:

$$\frac{1}{2}\Big[-\psi(\frac{1}{2}+\beta) - \psi(\frac{1}{2}-\beta) + \psi(M+\frac{1}{2}+\beta) + \psi(M+\frac{1}{2}-\beta)\Big]$$

- Limit:

$$\lim_{M\to\infty} \psi(M+\frac{1}{2}+\beta) + \psi(M+\frac{1}{2}-\beta) = -2\gamma + 2\zeta(1)$$

Future prospects: dark energy

- coefficients dependence on M and N

$$\begin{array}{rcll} \alpha_{KM} \text{ as N increases} & \longrightarrow & \text{more negative} & S^2 \\ \alpha_{NC} \text{ as M increases} & \longrightarrow & \text{more negative} & S_F^2 \\ \alpha_{C} \text{ as M increases} & \longrightarrow & \text{more positive} & S_F^2 \end{array}$$

- others values of the size of the representation $M+1$
- higher dimensions
  - S_F^N, $S_F^2 \times S_F^2 \longrightarrow$ the Green function
- different manifolds $M^4 \times M'^N$
  - $M'^N = T_F^2$ the Torus $f(x^i) = f(x^i + 2\pi r^i)$

$$\hat{\alpha}_i = \frac{\hat{x}^i}{r^i} \longrightarrow \hat{U}_i = e^{i\Theta\epsilon_{ij}\hat{\alpha}_j}$$

$$\hat{U}_i\hat{U}_j = e^{2\pi\Theta\epsilon_{ij}}\hat{U}_j\hat{U}_i \qquad [\hat{\alpha}_i, \hat{\alpha}_j] = \frac{2\pi}{\Theta}\epsilon_{ij}$$

- additional fields: vectors, fermions and gravitons

S^N	Scalars	Vectors	Fermions	Gravitons
S^2	-8.04×10^{-4}	-8.04×10^{-5}	-7.94×10^{-4}	1.07×10^{-2}
S^3	7.57×10^{-5}	–	1.95×10^{-4}	–
S^4	-4.99×10^{-4}	1.21×10^{-2}	-6.64×10^{-3}	-0.489
S^5	4.28×10^{-4}	–	-1.14×10^{-4}	–
S^6	1.31×10^{-3}	4.90×10^{-2}	-3.02×10^{-2}	5.10
S^7	8.16×10^{-4}	–	5.96×10^{-5}	–

– positive values $\longrightarrow$ Dark Energy

– considering NC$\longrightarrow$ regularization(M) $\longrightarrow$ new possible solutions

Cosmology: Inflation

- inflaton field

$$t_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}[\partial_\lambda\varphi\partial^\lambda\varphi + V(\phi)] \rightarrow \begin{cases} \rho = t_{00} = \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ p = \frac{1}{3}t_{ii} = \frac{1}{2}\dot{\phi}^2 - V(\phi) \end{cases}$$

- equations of motion and chaotic potential $V(\phi)$:

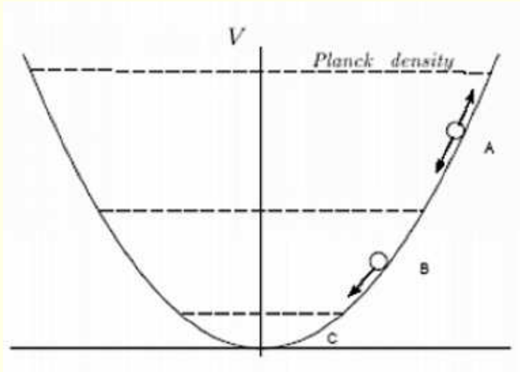
$$H^2 = \frac{1}{6}[\dot{\phi}^2 + m^2\phi^2] \qquad H = \frac{\dot{a}}{a}$$

$$\ddot{\phi} + 3H\dot{\phi} = -m^2\phi \qquad V(\phi) = \frac{1}{2}m^2\phi^2$$

- conditions for inflation to occur
  - if $\frac{1}{2}\dot{\phi}^2 \ll V(\phi) \rightarrow \rho = -p \rightarrow \ddot{a} > 0$
  - if $\phi \gg 1 \rightarrow H$ is large, $m \ll 1$, $V(\phi) \sim 1$ units $M_{Pl}^2 = 8\pi G = 1$

$$a(t) = a_0 e^{Ht} \qquad \phi(t) = \phi_0 - \frac{m}{2(3\pi)^{1/2}}t$$

- Initial condition for $V(\phi) = \frac{1}{2}m^2\phi^2, \quad \phi \gg 1$



–Slow rolling conditions of $\phi \rightarrow \rho \simeq -p \rightarrow$ Inflation occurs

Future prospects: inflation

- extra dimensions scenarios
  - ADD
  - RS
  - SUGRA (modular and brane inflation)
- ADD model

  $M_{Pl}^2 \sim M_{Pl(4+N)}^{2+N} R^N$ SM fields live on the *brane* (M^4)

  $N = 2: \; M_{Pl(6)}^4 \sim \frac{M_{Pl}^2}{R^2}$ Gravity lives on the *bulk* (the full $M^4 \times M'^N$)

  $$U(b) = \left(\frac{\alpha_{inf}}{b^4} + M_{ew}^4 V(b) \right)$$ C. Gardner (2001)

  - inflaton $\phi(t)$: radon $b(t)$, radius R of the extra dimensions
  - $\alpha_{inf} \ll M_{ew}^4 V(b)$ but if NC $\alpha_{inf} \to \alpha_{inf}(M) \sim M_{ew}^4 V(b)$
  - NC corrections to inflation

- Supergravity inflation

  – Modular (KKLT)

  $W = W_0 + Ae^{-a\rho} + B^{-b\rho}$

  – Brane

  Kähler potential $K = -3\log(\rho + \bar{\rho} - k(\phi\bar{\phi}))$

  ρ: radon and ϕ: inflaton, distance between the D3 and anti D3 branes

- noncommutative extensions ?

  – NC potential

  – Supergravity is local $\longleftrightarrow$ NC is not local

fine