Noncommutative Quantum Hall Effect and Aharonov-Bohm Effect

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Noncommutative quantum field theory

- inspired by quantum mechanics:

$$[p, q] = -i\hbar, \qquad \Delta q \Delta p \geq \hbar/2$$

- coordinates satisfy the "canonical" commutation relations:

$$[x^\mu, x^\nu] = i\Theta^{\mu\nu}$$

- spacetime uncertainty relations:

$$\Delta x^\mu \Delta x^\nu \geq \frac{1}{2}\Theta^{\mu\nu}$$

  – space-time points $\mapsto$ Planck size area cells.

  – world appears to be granular.

Quantum Hall effect

- Electron moving on the (x, y) plane in a uniform electric field $\vec{E} = -\vec{\nabla}\phi$ and the uniform magnetic field B:

$$H = \frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 - e\phi,$$

- symmetric gauge:

$$\vec{A} = \left(-\frac{B}{2}y, \frac{B}{2}x\right)$$

and the scalar potential is:

$$\phi = -Ex$$

- Hamiltonian becomes:

$$H(\vec{p}, \vec{r}) = \frac{1}{2m}\left[\left(p_x - \frac{eB}{2c}y\right)^2 + \left(p_y + \frac{eB}{2c}x\right)^2\right] + eEx,$$

- Eigenvalue problem

$$\hat{H}\Psi = E\Psi$$

- Change of variables:

$$\begin{aligned} \hat{z} &= \hat{x} + i\hat{y}, \\ \hat{p}_z &= \frac{1}{2}(\hat{p}_x - i\hat{p}_y). \end{aligned}$$

We define:

$$b^\dagger = -2i\hat{p}_{\bar{z}} + \frac{eB}{2c}\hat{z} + \lambda, \qquad b = 2i\hat{p}_z + \frac{eB}{2c}\hat{\bar{z}} + \lambda$$

and

$$d^\dagger = -2i\hat{p}_{\bar{z}} - \frac{eB}{2c}\hat{z}, \qquad d = 2i\hat{p}_z - \frac{eB}{2c}\hat{\bar{z}}$$

where $\lambda = mcE/B$ and:

$$\begin{aligned} \left[b, b^\dagger\right] &= 2m\hbar\omega \\ \left[d^\dagger, d\right] &= 2m\hbar\omega \end{aligned}$$

with $\omega = eB/mc$.

- Hamiltonian:

$$\hat{H} = \frac{1}{4m}(bb^\dagger + b^\dagger b) - \frac{\lambda}{2m}(d^\dagger + d) - \frac{\lambda^2}{2m}$$

$$\hat{H} = \hat{H}_{osc} - \hat{H}_1$$

- The harmonic oscillator part:

$$\hat{H}_{osc} = \frac{1}{4m}(bb^\dagger + b^\dagger b)$$

the eigenvalue equation $\hat{H}_{osc}\Phi_n = E_n^{osc}\Phi_n$ solution is:

$$\Phi_n = \frac{1}{\sqrt{(2m\hbar\omega)^n n!}}(b^\dagger)^n|0>,$$

$$E_n^{osc} = \frac{\hbar\omega}{2}(2n+1), \quad n = 0, 1, 2, ...$$

- The eigenvalue equation for $\hat{H}_1\phi_\gamma = E_\gamma\phi_\gamma$ in terms of eigenvalues of $\hat{r}_i$:

$$\phi_\gamma \;=\; exp\left(-i(\gamma y + \frac{m\omega}{2\hbar}xy)\right)$$

$$E_\gamma \;=\; \frac{\hbar\lambda}{m}\gamma + \frac{\lambda^2}{2m}, \quad \gamma \in R$$

- Energy spectrum of the Hamiltonian $\hat{H}$ is:

$$\Psi_{(n,\gamma)} = |n,\gamma\rangle \;=\; \frac{1}{\sqrt{(2m\hbar\omega)^n n!}}\, exp\left(-i(\gamma y + \frac{m\omega}{2\hbar}xy)\right)(b^\dagger)^n|0>,$$

$$E_{(n,\gamma)} \;=\; \frac{\hbar\omega}{2}(2n+1) - \frac{\hbar\lambda}{m}\gamma - \frac{\lambda^2}{2m}$$

- Current operator $\hat{\vec{J}}$ on the noncommutative plane as

$$\hat{\vec{J}} = \frac{ie\rho}{\hbar}\left[\hat{H}, \hat{\vec{r}}\right]$$

- The expectation value of $\langle\hat{\vec{J}}\rangle$ wrt eigenstates $|n, \gamma\rangle$ is:

$$\begin{aligned} \langle \hat{J}_x \rangle &= 0, \\ \langle \hat{J}_y \rangle &= -\frac{e\rho c}{B} E. \end{aligned}$$

- Hall conductivity is:

$$\begin{aligned} \sigma_H &= -\frac{e\rho c}{B} \\ &= -\frac{e^2}{h}\nu, \quad \nu = \frac{\Phi_0 \rho}{B} \end{aligned} \tag{1}$$

where $\Phi_0 = hc/e$

Noncommutative case:

- Commutation relations

$$
\begin{aligned}
[x_i, x_j] &= i\Theta_{ij}, \\
[p_i, p_j] &= i\hbar^2 \Xi_{ij}, \\
[x_i, p_j] &= i\hbar\,\delta_{ij}.
\end{aligned}
$$

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\end{aligned}
$$

- Commutative set of canonically conjugate coordinates (α_i, β_i):

$$
\begin{aligned}
[\alpha_i, \alpha_j] &= 0, \\
[\beta_i, \beta_j] &= 0, \\
[\alpha_i, \beta_j] &= i\hbar\, \delta_{ij}.
\end{aligned}
$$

- Commutation relations

$$[x_i, x_j] = i\Theta_{ij},$$
$$[p_i, p_j] = i\hbar^2\Xi_{ij},$$
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$$[\alpha_i, \beta_j] = i\hbar\,\delta_{ij}.$$

- Relation between coordinates:

$$x_i = a_{ij}\alpha_j + b_{ij}\beta_j$$
$$p_i = c_{ij}\beta_j + d_{ij}\alpha_j$$

- Transformation matrices relations:

$$\mathbf{a}\mathbf{b}^{\mathbf{T}} - \mathbf{b}\mathbf{a}^{\mathbf{T}} = \frac{\boldsymbol{\Theta}}{\hbar}$$
$$\mathbf{c}\mathbf{d}^{\mathbf{T}} - \mathbf{d}\mathbf{c}^{\mathbf{T}} = -\hbar\ \boldsymbol{\Xi}$$
$$\mathbf{c}\mathbf{a}^{\mathbf{T}} - \mathbf{b}\mathbf{d}^{\mathbf{T}} = \mathbf{I}$$

- We choose:

$$a_{ij} \equiv a\ \delta_{ij}, \qquad c_{ij} \equiv c\ \delta_{ij}$$
$$b_{ij} \equiv b\ \epsilon_{ij}, \qquad d_{ij} \equiv d\ \epsilon_{ij}$$

- Transformation matrices become:

$$a\ b = -\frac{\Theta}{2\hbar}$$
$$c\ d = \frac{\hbar\ \Xi}{2}$$
$$a\ c\ -b\ d = 1$$

- Then:

$$\begin{aligned} b &= -\frac{\Theta}{2a\hbar} \\ c &= \frac{1}{2a}\left(1 \pm \sqrt{\kappa}\right), \qquad \kappa \equiv 1 - \Theta\Xi \\ d &= \frac{\hbar a}{\Theta}\left(1 \mp \sqrt{\kappa}\right) \end{aligned}$$

- Substituting

$$\begin{aligned} x_i &= a_{ij}\alpha_j + b_{ij}\beta_j \\ p_i &= c_{ij}\beta_j + d_{ij}\alpha_j \end{aligned}$$

in the Hamiltonian we have:

$$H(\vec{\alpha},\vec{\beta}) = \frac{1}{2m}\left[h_1^2\left(\alpha_i\right)^2 + h_2^2\left(\beta_i\right)^2 - h_3\epsilon_{ij}\alpha_i\beta_j\right] + aeE\alpha_1 - \frac{\Theta}{2a}eE\beta_2$$

with

$$\begin{aligned} h_1^2 &= a^2\left[\frac{\hbar}{\Theta}\left(1 \mp \sqrt{\kappa}\right) - \left(\frac{eB}{2c}\right)\right]^2 \\ h_2^2 &= \frac{\Theta^2}{4\hbar^2 a^2}\left[\frac{\hbar}{\Theta}\left(1 \pm \sqrt{\kappa}\right) - \left(\frac{eB}{2c}\right)\right]^2 \\ h_3 &= \left[\left(\frac{eB}{2c}\right)^2\Theta + \hbar^2\,\Xi - \frac{\hbar eB}{c}\right] \end{aligned}$$

- Coordinate transformation:

$$\beta_1 \rightarrow \beta_1$$
$$\beta_2 \rightarrow \beta_2 - \frac{\frac{\Theta}{2a} eE}{2h_2^2}$$

and the Hamiltonian becomes:

$$H(\vec{\alpha}, \vec{\beta}) = \frac{1}{2m}\left[h_1^2 \left(\alpha_i\right)^2 + h_2^2 \left(\beta_i\right)^2 + h_3 \epsilon_{ij} \alpha_i \beta_j\right] + h_4 \alpha_1 - h_5$$

where,

$$h_4 = \pm 2eEa \frac{\sqrt{\kappa}}{\Theta\left[\frac{\hbar}{\Theta}\left(1 \pm \sqrt{\kappa}\right) - \left(\frac{eB}{2c}\right)\right]}$$
$$h_5 = \frac{1}{2m} \frac{e^2 E^2}{4\left[\frac{\hbar}{\Theta}\left(1 \pm \sqrt{\kappa}\right) - \left(\frac{eB}{2c}\right)\right]^2}$$

- Eigenvalue problem

$$\hat{H}\Psi = E\Psi$$

- Change of variables:

$$\begin{aligned} \hat{z} &= \alpha_1 + i\alpha_2, \\ \hat{p_z} &= \frac{1}{2}(\beta_1 - i\beta_2). \end{aligned}$$

We define:

$$b^\dagger = -2ih_2\hat{p}_{\bar{z}} + h_1\hat{z} + \lambda, \qquad b = 2ih_2\hat{p}_z + h_1\hat{\bar{z}} + \lambda$$

and

$$d^\dagger = -2ih_2\hat{p}_{\bar{z}} - h_1\hat{z}, \qquad d = 2ih_2\hat{p}_z - h_1\hat{\bar{z}}$$

where $\lambda = \mp meE\sqrt{\kappa}/h_3$ and:

$$\left[b, b^\dagger\right] = 2m\hbar\omega$$
$$\left[d^\dagger, d\right] = 2m\hbar\omega$$

with $\omega = h_3/m$.

- Hamiltonian:

$$\hat{H} = \frac{1}{4m}(bb^\dagger + b^\dagger b) - \frac{\lambda}{2m}(d^\dagger + d) - \frac{\lambda^2}{2m} - h_5$$
$$\hat{H} = \hat{H}_{osc} - \hat{H}_1$$

- The harmonic oscillator part:

$$\hat{H}_{osc} = \frac{1}{4m}(bb^\dagger + b^\dagger b)$$

the eigenvalue value equation $\hat{H}_{osc}\Phi_n = E_n^{osc}\Phi_n$ solution is:

$$\Phi_n = \frac{1}{\sqrt{(2m\hbar\omega)^n n!}}(b^\dagger)^n|0>,$$
$$E_n^{osc} = \frac{\hbar\omega}{2}(2n+1), \quad n = 0, 1, 2, ...$$

- The eigenvalue equation for $\hat{H}_1\phi_\gamma = E_\gamma\phi_\gamma$ in terms of α_i and β_i

$$\phi_\gamma \ = \ exp\left(-i(\gamma\alpha_2 + \frac{h_1}{\hbar h_2}\alpha_1\alpha_2)\right)$$
$$E_\gamma \ = \ \frac{\hbar\lambda h_2}{m}\gamma + \frac{\lambda^2}{2m} + h_5, \quad \gamma \in R$$

- Energy spectrum of the Hamiltonian $\hat{H}$ is

$$\Psi_{(n,\gamma,\Theta,\Xi)} = |n,\gamma,\Theta,\Xi\rangle \ = \ \frac{1}{\sqrt{(2m\hbar\omega)^n n!}} exp\left(-i(\gamma\alpha_2 + \frac{h_1}{\hbar h_2}\alpha_1\alpha_2)\right)(b^\dagger)^n|0>,$$
$$E_{(n,\gamma)} \ = \ \frac{\hbar\omega}{2}(2n+1) - \frac{\hbar\lambda h_2}{m}\gamma - \frac{\lambda^2}{2m} - h_5$$

Hall conductivity

- Current operator $\hat{\vec{J}}$ on the noncommutative plane as

$$\hat{\vec{J}} = \frac{ie\rho}{\hbar}\left[\hat{H}, \hat{\vec{r}}\right]$$

- The expectation value of $\langle\hat{\vec{J}}\rangle$ wrt eigenstates $|n, \gamma, \Theta, \Xi\rangle$ is:

$$\begin{aligned}\langle\hat{J}_x\rangle &= 0,\\ \langle\hat{J}_y\rangle &= -e\rho\frac{1-\Theta\Xi}{\frac{B}{c}-\frac{eB^2\Theta}{4\hbar c^2}-\frac{\hbar\ \Xi}{e}}E.\end{aligned}$$

- Hall conductivity is:

$$\sigma_H = -e\rho\frac{1-\Theta\Xi}{\frac{B}{c}-\frac{eB^2\Theta}{4\hbar c^2}-\frac{\hbar\ \Xi}{e}}$$

- Hall conductivity is:

$$\sigma_H = -e\rho \frac{1 - \Theta\Xi}{\frac{B}{c} - \frac{eB^2\Theta}{4\hbar c^2} - \frac{\hbar\ \Xi}{e}}$$

- If Θ and Ξ equal to zero:

$$\sigma = -\rho ec/B$$

- If $\Xi = 0$:

$$\sigma_H = -e\rho \frac{1}{\frac{B}{c} - \frac{eB^2\Theta}{4\hbar c^2}}$$

- If $\Theta = 0$:

$$\sigma_H = -e\rho \frac{1}{\frac{B}{c} - \frac{\hbar\ \Xi}{e}}$$

Aharonov-Bohm effect

- Hamiltonian:

$$H = \frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 + V$$

- Gauge:

$$\vec{A} = \left(-\frac{B}{2}y, \frac{B}{2}x\right)$$

- Hamiltonian becomes

$$H(\vec{\alpha}, \vec{\beta}) = \frac{1}{2m}\left[\left(h_2\vec{\beta} - h_1\vec{\alpha}\right)^2\right] + V(\vec{\alpha}, \vec{\beta})$$

with $\vec{\beta} = (\beta_1, \beta_2)$ and $\vec{\alpha} = (-\alpha_2, \alpha_1)$.

- We now have to solve Schrodinger's equation

$$\left[\frac{1}{2m}\left(h_2\frac{\hbar}{i}\nabla - h_1\vec{\alpha}\right)^2 + V(\vec{\alpha},\vec{\beta})\right]\Psi = i\frac{\partial\Psi}{\partial t}$$

where, $\vec{\beta} \longrightarrow \frac{\hbar}{i}\nabla$.

- We write

$$\Psi = e^{ig}\psi$$

with:

$$g(\vec{\alpha}) = \frac{h_1}{h_2\hbar}\oint \vec{\alpha}' d\alpha'$$

- Then

$$\left(h_2\frac{\hbar}{i}\nabla - h_1\vec{\alpha}\right)^2\Psi = -h_2^2\hbar^2 e^{ig}\nabla^2\psi$$

- We note that $g(\vec{\alpha})$ is just a phase difference:

$$
\begin{aligned}
g(\vec{\alpha}) &= \frac{h_1}{h_2\hbar}\oint^{\alpha}\vec{\alpha}'d\alpha' \\
&= \frac{h_1}{h_2\hbar}\oint^{\alpha}(-\mathbf{i}\alpha_2' + \mathbf{j}\alpha_1')d\alpha' \\
&= \frac{h_1}{h_2\hbar}2\pi\alpha^2
\end{aligned}
$$

- Substituting h_1 and h_2:

$$
g(\vec{\alpha}) = \frac{2a^2\left[\frac{\hbar}{\Theta}\left(1 \mp \sqrt{\kappa}\right) - \left(\frac{eB}{2c}\right)\right]}{\Theta\left[\frac{\hbar}{\Theta}\left(1 \pm \sqrt{\kappa}\right) - \left(\frac{eB}{2c}\right)\right]}2\pi\alpha^2
$$

- Phase difference:

$$g(\vec{\alpha}) = \frac{2a^2\left[\frac{\hbar}{\Theta}\left(1 \mp \sqrt{\kappa}\right) - \left(\frac{eB}{2c}\right)\right]}{\Theta\left[\frac{\hbar}{\Theta}\left(1 \pm \sqrt{\kappa}\right) - \left(\frac{eB}{2c}\right)\right]} 2\pi\alpha^2$$

- If Θ and Ξ equal to zero:

$$g(\vec{\alpha}) = -e\Phi/\hbar$$

- For small Θ and Ξ:

$$g(\vec{\alpha}) = \left(-\frac{eB}{c\hbar} + \Xi - \frac{eB^2\Theta}{4c^2\hbar^2}\right)\pi\alpha^2$$

- For $\Xi = 0$:

$$g(\vec{\alpha}) = \left(-\frac{eB}{c\hbar} - \frac{eB^2\Theta}{4c^2\hbar^2}\right)\pi\alpha^2$$

- If $\Theta = 0$:

$$g(\vec{\alpha}) = \left(-\frac{eB}{c\hbar} + \Xi\right)\pi\alpha^2$$

Thank you!!!