

Finite Temperature Field Theory

Part II: The Partition Function

$$\langle f(x) \rangle = \frac{\int f(x) P(x) dx}{\int P(x) dx}$$

Classical Partition Function

$$Z = \frac{\int dp^{3N} dq^{3N} \rho e^{-\beta E}}{h_o^{3N}}$$

$$\langle N \rangle = -\frac{\partial \ln Z}{\partial \alpha} \quad \alpha = \mu\beta \quad \mu = \frac{\partial \langle E \rangle}{\partial N}$$

$$\langle E \rangle = -\frac{\partial \ln Z}{\partial \beta} + \mu \langle N \rangle \quad c_v = \frac{\partial \langle E \rangle}{\partial T}$$

$$\langle P \rangle = \beta^{-1} \frac{\partial \ln Z}{\partial V} \quad \langle PV \rangle = \beta^{-1} \ln Z$$

$$F = -\beta^{-1} \ln Z \quad S = -\frac{\partial F}{\partial T}$$

Properties of the Density Matrix

$$\begin{aligned}\langle \Psi(x, y, t) | \hat{f} | \Psi(x, y, t) \rangle &= \sum_{oo'} \langle \Phi_o(y, t) | \Phi_{o'}(y, t) \rangle \langle \phi_o(x) | \hat{f} | \phi_{o'}(x) \rangle \\ &= \sum_{oo'} \rho_{oo'} \langle \phi_o(x) | \hat{f} | \phi_{o'}(x) \rangle\end{aligned}$$

The density matrix for a system with nondegenerate energy eigenvalues E_i in contact with a thermal reservoir using the energy basis: $\exp(\beta E_i)$

$$\rho_i = e^{\beta E_j} \delta_{ji}$$

Averages

$$\langle \hat{A} \rangle = \text{Tr}(\rho \hat{A}) \quad \hat{\rho} = \sum_i w_i |i\rangle \langle i|$$

$$\begin{aligned} \langle \hat{A} \rangle &= \sum_{i'} \langle i' | \rho \hat{A} | i' \rangle = \sum_{ii'} w_i \langle i' | i \rangle \langle i | \hat{A} | i' \rangle \\ &= \sum_i w_i \langle i | \hat{A} | i \rangle \end{aligned}$$

- **Prob in state i**
- **Pure State**
- **$\rho = \rho^2$**
- **Define S?**

Quantum Mechanical Partition Function

$$E \rightarrow \hat{H} \quad \text{units} \quad \hbar = 1$$

$$Z = \int dpdq e^{-\beta \hat{H}}$$

$$Z = \sum_n \langle n | e^{-\beta \hat{H}} | n \rangle = \text{Tr}(e^{-\beta \hat{H}}) = \text{Tr}(\hat{\rho})$$

Z for SHO

1. Classical H

$$Z = \int dp dx e^{-\beta H} = \int dp dx e^{-\beta(p^2/2m + m\omega^2 x^2/2)}$$

$$= \left(\frac{2\pi m}{\beta} \right)^{1/2} \left(\frac{2\pi}{\beta m \omega^2} \right)^{1/2} = \frac{2\pi}{\beta \omega}$$

$$F_C = -\beta^{-1} \ln(\beta \omega)^{-1} = \frac{\ln(\beta \omega)}{\beta}$$

Z SHO

2. Quantum Mechanical

$$E_n = (n + 1/2)\omega$$

$$\begin{aligned} Z &= \sum_n e^{-\beta(n+1/2)\omega} = e^{-\beta\omega/2} \sum_n (e^{-\beta\omega})^n \\ &= \frac{e^{-\beta\omega/2}}{1 - e^{-\beta\omega}} = \frac{1}{2} \operatorname{csc} h\left(\frac{\beta\omega}{2}\right) \end{aligned}$$

$$F_Q = \frac{\omega}{2} + \beta^{-1} \ln(1 - e^{-\beta\omega})$$

$$\langle E \rangle = \frac{\omega}{2} + \frac{\omega e^{-\beta\omega}}{1 - e^{-\beta\omega}} = \left(\frac{1}{2} + \langle N \rangle \right) \omega$$

$$F_Q - F_C = \frac{\omega}{2}$$

One Bosonic Degree of Freedom

$$\langle n | n' \rangle = \delta_{nn'}, \quad \sum |n\rangle \langle n| = 1$$

$$[a^+, a] = 1 \quad N = a^+ a \quad \hat{H} = \frac{\omega}{2} (aa^+ + a^+ a) = (\hat{N} + \frac{1}{2})\omega$$

$$:\hat{H}: = a^+ a \omega = \hat{N} \omega$$

$$\begin{aligned} Z = \text{Tr}(\hat{\rho}) &= \text{Tr}\left(e^{-\beta(:\hat{H}: - \mu \hat{N})}\right) = \sum \langle n | e^{-\beta(\hat{H} - \mu \hat{N})} | n \rangle \\ &= \sum \langle n | e^{-\beta(E - \mu N)} | n \rangle = \sum \langle n | e^{-\beta(\omega - \mu)} | n \rangle \\ &= \sum e^{-\beta(\omega - \mu)n} = \frac{1}{1 - e^{-\beta(\omega - \mu)}} \end{aligned}$$

$$\langle N \rangle = \frac{1}{e^{\beta(\omega - \mu)} - 1}$$

$$\langle E \rangle = \langle N \rangle \omega$$

One Fermionic Degree of Freedom

$$\alpha^+|0\rangle = |1\rangle \quad \alpha|1\rangle = 0 \quad \{\alpha, \alpha^+\} = 1$$

$$\hat{H} = \frac{\omega}{2}(\alpha^+ \alpha - \alpha \alpha^+) = \omega(\hat{N} - \frac{1}{2})$$

$$Z = \text{Tr}(\hat{\rho}) = \text{Tr}\left(e^{-\beta(\hat{H} - \mu\hat{N})}\right) = \sum_{n=0}^1 \langle n | e^{-\beta(\omega - \mu)\hat{N}} | n \rangle$$

$$Z = 1 + e^{-\beta(\omega - \mu)}$$

$$\langle N \rangle = \frac{1}{e^{\beta(\omega - \mu)} + 1}$$

*****the minus sign*****

Field Theory

$$Z_M = \int D\pi D\phi e^{-i \int dt \int d^3x \mathcal{L}}$$

- Independent fields and momenta
- $[]$ or $\{\}$ relations on fields
- Construct Action

$$Z_p = \int D\pi D\phi e^{-\beta(\hat{H} - \mu\hat{N})}$$

QFT

- Legendre transformation on Lagrangian density
- Add chemical potential
- Let (i) factor with t and $dt \rightarrow \beta$
- Trace \rightarrow initial = final \rightarrow periodic in $\beta \rightarrow$ Sum

$$Z_p = \int D\pi D\phi e^{\int_0^\beta d\tau \int d^3x (i\pi \partial_t \phi - \mathcal{H} - \mu \mathcal{N})}$$

Bosonic Field

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 + U \quad \mu = 0$$

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \frac{\partial \phi}{\partial t}$$

$$\mathcal{H} = \pi \frac{\partial \phi}{\partial t} - \mathcal{L} = \frac{\pi^2}{2} + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 + U$$

$$Z = \int D\pi D\phi \exp \left(\int_0^\beta d\tau \int d^3x \left[i\pi \frac{\partial \phi}{\partial t} - \frac{1}{2} \pi^2 - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2 - U \right] \right)$$

- Integrate quadratic term
- Integrate by parts
- $U=0$

Bosonic Field

$$\int dx e^{x\hat{D}x} = \pi^{n/2} (\det D)^{-1}$$

$$Z = A \int d\phi \exp \left(-\frac{1}{2} \int_0^\beta d\tau \int d^3x \phi \left(-\frac{\partial^2}{\partial \tau^2} - \nabla^2 + m^2 \right) \phi \right)$$

$$\phi = \left(\frac{\beta}{V} \right)^{1/2} \sum_p \sum_{n=-\infty}^{\infty} e^{i(p \cdot x + \omega_n \tau)} \phi_n(p)$$

$$\omega_n = 2\pi nT$$

$$\ln Z = -\frac{1}{2} \sum_{n,p} \ln \left[\beta^2 (\omega^2 + \omega_n^2) \right]$$

****minus sign****

Two Component Boson Field

$$L = \partial_{\mu} \phi^* \partial^{\mu} \phi - m^2 \phi^* \phi - \lambda (\phi^* \phi)^2 \quad \Phi = \frac{(\phi_1 + i\phi_2)}{\sqrt{2}}$$

$$\pi_i = \frac{\partial \phi_i}{\partial t}$$

$$j^{\mu} = i(\phi^* \partial_{\mu} \phi - \phi \partial_{\mu} \phi^*) \quad Q = N = \int d^3 x j_0 = \int d^3 x (\phi_2 \pi_1 - \phi_1 \pi_2)$$

$$H = \frac{1}{2} \left[\pi_1^2 + \pi_2^2 + (\nabla \phi_1)^2 + (\nabla \phi_2)^2 + m^2 \phi_1^2 + m^2 \phi_2^2 + \frac{1}{4} \lambda (\phi_1^2 + \phi_2^2)^2 \right]$$

$$Z = \int d\pi_1 d\pi_2 d\phi_1 d\phi_2 \exp \left\{ \int_0^{\beta} d\tau \int d^3 x \left(i\pi_1 \frac{\partial \phi_1}{\partial \tau} + i\pi_2 \frac{\partial \phi_2}{\partial \tau} - H + \mu (\phi_2 \pi_1 - \phi_1 \pi_2) \right) \right\}$$

Fermions-Grassmann variables

$$L = \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi \quad \bar{\psi} = \psi^\dagger \gamma^0$$

$$j_\mu = \bar{\psi} \gamma^\mu \psi \quad N = \int d^3x \psi^\dagger \psi \quad \pi = i\psi^\dagger$$

$$Z = \int iD\psi^\dagger D\psi \exp \left\{ \int_0^\beta d\tau \int d^3x \left[\bar{\psi} \left(-\gamma^0 \frac{\partial}{\partial \tau} + i\vec{\gamma} \cdot \nabla - m + \mu\gamma^0 \right) \psi \right] \right\}$$

- Anticommuting fields
- Chemical potential

Anticommuting and Interacting Fields

- **Couple Fields**
- **Make metric terms explicit**
- **Identify independent variables**
- **Functional Derivative- $\rightarrow Z$**
- **Find thermal propagators and diagrams**