Horizons, Holography and Hydrodynamics

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A dissertation submitted in partial fulfillment
of the requirements for the degree of

Doctor of Philosophy
Physics

University of Mississippi

2012

Abstract

In this dissertation we study black-hole horizons in general relativity and in Gauss-Bonnet gravity. We generalize Wald's "black-hole entropy is the Noether charge" construction to the first-order formalism of these theories. Next we construct the "membrane paradigm" for black objects in Gauss-Bonnet gravity and demonstrate how the horizon can be viewed as a membrane with fluidlike properties. Holography is invoked to relate the transport coefficients obtained for the horizons in the membrane paradigm with the transport coefficients which describe the hydrodynamic limit of the boundary gauge theory.

Dedication

To my parents

Aliya Tasneem and Munir Uddin Siddiqui

Acknowledgments

I thank my parents and my sister for their continual support and encouragement. Not only do they motivate me, they inspire me to great human wisdom and kindness.

I am very grateful to my mentors Luca Bombelli and Marco Cavaglià for the things they taught to me, for the ideas they shared with me and for the opportunities they provided to me. Their confidence in me has been very important for me.

I am very grateful to Ted Jacobson for stimulating discussions and interesting collaborations. I express my deep gratitude to the members of the Maryland Center of Fundamental Physics for their warm hospitality and pleasant company while I was visiting there and where some of the work reported in this dissertation was done.

I am thankful to Sudipta Sarkar for very interesting discussions and collaborations and for providing me shelter for a month.

I am also thankful to the members of gravity group at Ole Miss for their patience and constructive criticism and to the faculty and staff of the Department of Physics and Astronomy for their constant support and ecouragement.

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Chapter 1

Introduction

Gravitation, being the manifestation of the curvature of spacetime, affects the causal structure of spacetime. This can lead to the existence of regions which are causally inaccessible to a class of observers. An example of such a region is the portion of spacetime inside the event horizon of a black hole, which is causally disconnected from any outside observer. Hence, the relevant physics for the observers outside the black hole must be independent of what is happening inside the black hole. This observation forms the basis of the membrane paradigm for black holes.

The membrane paradigm is an approach in which the interaction of the black hole with the outside world is modelled by replacing the black hole by a membrane of fictitious fluid "living" on the horizon (Damour, 1978, Thorne *et al.*, 1986). The mechanical interactions of the black hole with the outside world are then captured by the (theory dependent) transport coefficients of the fluid. The electromagnetic interaction of the black hole is described by endowing the horizon with conductivity and so forth. This formalism provides an intuitive and elegant understanding of the physics of the event horizon and also serves as an efficient computational tool useful in dealing with some astrophysical problems. After the advent of holography, the membrane paradigm took a new life in which the membrane fluid is conjectured to provide the long wavelength description of the strongly coupled quantum field theory at a finite temperature (Policastro *et al.*, 2001).

The original membrane paradigm was constructed for black holes in general relativity (Damour, 1978, Thorne *et al.*, 1986). The membrane fluid has the shear viscosity, $\eta =$

$1/16\pi G$. Dividing this by the Bekenstein-Hawking entropy density, $s = 1/4G$, gives a dimensionless number, $\eta/s = 1/4\pi$. The calculation which leads to this ratio relies only on the dynamics and the thermodynamics of the horizon in classical general relativity. But, interestingly enough, it was found by Policastro *et al.* (2001) that the same ratio is obtained in the holographic description of the hydrodynamic limit of the strongly coupled $\mathcal{N} = 4$, U(N) gauge theory at finite temperature which is dual to general relativity in the limit $N \to \infty$ and $\lambda_t \to \infty$, where N is the number of colors and λ_t is the 't Hooft coupling. Kovtun *et al.* (2003) conjectured that this ratio $1/4\pi$ is a universal lower bound for all materials. This bound became known as the KSS bound. The relationship between the membrane paradigm calculations and the holographically derived KSS bound was explained as a consequence of the trivial renormalization group (RG) flow from infrared (IR) to ultraviolet (UV) in the boundary gauge theory as one moves the outer cutoff surface from the horizon to the boundary of spacetime (Bredberg *et al.*, 2011, Iqbal & Liu, 2009). The universality of this bound, how it might be violated, and the triviality of the RG flow in the long wavelength limit at the level of the linear response were first clarified by Iqbal & Liu (2009).

The general theory of relativity, which is based upon the Einstein-Hilbert action functional, is the simplest theory of gravity one can write guided by the principle of diffeomorphism invariance while containing only time derivatives of the second order in the equation of motion. Although such a simple choice of the action functional has so far been adequate to explain all the experimental and observational results, there is no reason to believe that this choice is fundamental. Indeed, it is expected on various general grounds that the low-energy limit of any quantum theory of gravity will contain higher-derivative correction terms. In fact, in string theory the low-energy effective action generically contains terms which are higher order in curvature due to the stringy (α') corrections. In the context of holographic duality, such α' modifications correspond to $1/\lambda_t$ corrections. The specific form of these terms depends ultimately on the detailed features of the quantum theory. From the classical point of view, a simple modification of the Einstein-Hilbert action is to include the higher-order curvature terms preserving the diffeomorphism invariance and still leading to an equation of motion containing no more than second-order time derivatives. In fact this generalization is unique (Lovelock, 1971) and goes by the name of Lanczos-Lovelock gravity, of which the lowest order correction (second order in curvature) appears as a Gauss-Bonnet (GB) term in spacetime dimensions $D > 4$. Einstein-Gauss-Bonnet (EGB) gravity is free from ghosts (unphysical degrees of freedom) and leads to a well-defined initial value problem (Zumino, 1986, Zwiebach, 1985). Black hole solutions in EGB gravity have been studied extensively and are found to have various interesting features (Boulware & Deser, 1985, Myers & Simon, 1988). The entropy of these black holes is no longer proportional to the area of the horizon but contains a curvature dependent term (Iyer & Wald, 1995, Jacobson & Myers, 1993, Visser, 1993, Wald, 1993). Hence, unlike in general relativity, the entropy density of the horizon in EGB gravity is not a constant but depends on the horizon curvature. Now, the form of the membrane stress tensor in the fluid model of the horizon is also theory dependent and therefore the transport coefficients of the membrane fluid will change due to the presence of the GB term in the action. Hence, it is of interest to investigate the membrane paradigm and calculate the transport coefficients for the membrane fluid in the EGB gravity. The violation of the KSS bound due to the GB term in the action has already been shown by Brigante *et al.* (2008b) and Brigante *et al.* (2008a) using other methods. Also, the arguments that there really are string theories that violate the bound were presented by Kats & Petrov (2009) and Buchel *et al.* (2009).

The plan of this thesis is as follows: In Chapter 2 we discuss the entropy of black holes as given by the Noether charge corresponding to diffeomorphisms. We generalize the Noether-charge method to the theories with local Lorentz gauge freedom by introducing a gauge-covariant Lie derivative which realizes the action of diffeomorphisms on the dynamical fields in a gauge-covariant fashion. In Chapter 3 we review the holographic principle and we show how in the long-wavelength regime the linear response of the boundary gauge theory to small perturbations is very well captured by the linear response of the black-hole horizon. We introduce the membrane paradigm for the first time in this chapter which connects the two descriptions. In Chapter 4 we first review the membrane paradigm in general relativity and then construct the membrane paradigm for Gauss-Bonnet gravity. The transport coefficients for different bulk backgrounds are then intrepreted via holography as the transport coefficients of the dual gauge theory on the boundary. We conclude with the discussion of the results. The Appendix contains the relevant identities and some basics of hydrodynamics.

Chapter 2

Horizons

2.1 Introduction

A horizon is the causal boundary of the future history of an observer or a set of observers. For example, consider an observer undergoing constant acceleration in the right quadrant of the 1+1 dimensional Minkowski spacetime. In the whole history of her future she will never be able to receive any signal that originates in the region on the left of the outgoing null ray, see Figure 4.1. Hence this null ray forms the boundary of the set of events from which she can receive signals and thus it acts as a horizon for her.

We will be interested in the event horizons of black holes. What is special about this horizon is that it is the boundary of the set of events which can send signals to *any* observer who reaches the future null infinity. For an asymptotically flat spacetime, the event horizon is mathematically defined as the boundary of the causal past ($\mathcal{J}^-$) of the future null infinity $(\mathcal{I})^+$ (Wald, 1984). Physically this is the surface from beyond which even light cannot escape. Thus, classically black holes are black with zero temperature. However, Hawking showed that when quantum mechanics is taken into account black holes emit thermal radiation at a black-body temperature which is proportional to the surface gravity of the black hole (Hawking, 1974, 1975). One can then interpret the laws of black-hole mechanics as the laws of thermodynamics with the role of entropy played by a quarter of the area of the event horizon. The area theorem of Hawking, which says that the area of the horizon never decreases in a classical process, lends support to this interpretation of the area of the horizon as a measure of its entropy (Hawking & Ellis, 1973). Due to Hawking radiation though the horizon loses energy and the area of the horizon can decrease, the black hole can evaporate. The second law of thermodynamics was then promoted to the Generalized Second law by Bekenstein (1974). This says that the sum of the entropy of the horizon plus the Hawking radiation plus other matter can never decrease. To account for the microscopic degrees of freedom responsible for the black hole entropy is one of the driving forces for the research in quantum gravity. Also, the area instead of the volume dependence of the entropy of the black hole was a precursor of the holographic principle (Susskind, 1995, 't Hooft, 2001).

Notice that the entropy of the black hole is Area/4 only in general relativity, i.e., in the Einstein theory of gravity. In more general theories of gravity there could be other curvature dependent terms in the formula for the black-hole entropy. Wald has given an elegant construction of the formula for the entropy of black holes in diffeomorphism-invariant theories of gravity, Wald (1993). His insight was based upon the proof of the first law by Sudarsky & Wald (1993) and the analysis of symmetries by Lee & Wald (1990). This construction was then applied to many diffeomorphism invariant theories by Iyer & Wald (1994). But only the metric theories of gravity were treated in these references. Here we will generalize Wald's method to the theories which in addition to being diffeomorphism invariant also have another gauge invariance: the local Lorentz invariance. We will first see how a naive application of Wald's algorithm fails. We will then show why it fails and how to recover the correct expression for the entropy. We then introduce a new kind of Lie derivative which is gauge covariant and we will show how the use of this Lie derivative as the variational derivative of fields gives the correct Wald entropy. It turns out that the gauge covariant Lie derivative that we have rediscovered was recently constructed by mathematicians using the language of fibre bundles and it was called the Kosmann derivative (Fatibene & Francaviglia, 2009). We thus identify Wald's entropy as the Noether charge corresponding to the combined Lorentz-diffeomorphism symmetry.

In Section 2.2 we review the algorithm due to Wald which leads us to identify the black-hole entropy as the Noether charge corresponding to the infinitesimal diffeomorphisms generated by the Killing vector field which becomes null on the horizon. We will closely follow Wald's original paper (Wald, 1993). In the Section 2.3 we will apply Wald's algorithm to the first-order formulation of general relativity and find that the entropy seems to vanish. In this section we also propose a resolution of this problem. In Section 2.4 we construct a new representation of the diffeomorphism symmetry and realize it using a gauge-covariant derivative and show how the black-hole entropy is the Noether charge corresponding to this symmetry. Finally, in Section 2.5 we use our construction to calculate the black-hole entropy in Gauss-Bonnet gravity. The results reported in this chapter are based upon original work (Jacobson & Mohd, 2012).

2.2 Wald's algorithm

Wald's method applies to any diffeomorphism-invariant theory given by a Lagrangian D-form where D is the spacetime dimensionality. The dynamical fields in the theory are collectively denoted as ϕ. The indices on the fields in this abstract treatment will be suppressed.

The first variation of the Lagrangian under a variation of the dynamical fields ϕ can be written as

$$\delta L = E\delta\phi + \mathrm{d}\theta(\delta\phi), \tag{2.1}$$

where d is the exterior derivative on the spacetime manifold. The symbol δ in equation (2.1) stands for the partial derivative on the phase space. There is an elegant geometrical way of thinking about δ as an exterior derivative on the infinite-dimensional phase space (Ashtekar *et al.*, 1990, Crnković, 1988, Crnković & Witten, 1986). For our purposes though, we are going to stick with the definition of δ as a partial derivative on the phase space. Since the partial derivatives along two directions (δ_1 and δ_2) commute, we have $\delta_1\delta_2 = \delta_2\delta_1$.

The equation of motion satisfied by the field ϕ is given by $E = 0$. The $(D-1)$-form θ is called the symplectic potential and is locally constructed out of the dynamical fields and their first variation. The antisymmetrized variation of θ defines the $(D-1)$-form called the symplectic current as

$$\Omega(\phi, \delta_1\phi, \delta_2\phi) = \delta_1\theta(\phi, \delta_2\phi) - \delta_2\theta(\phi, \delta_1\phi). \tag{2.2}$$

Now, let ξ be any vector field on the spacetime and consider the variation of the dynamical field given by $\hat{\delta}\phi = \mathcal{L}_\xi\phi$. Diffeomorphism invariance means that the variation of the Lagrangian is given by $\hat{\delta}L = \mathcal{L}_\xi L = \mathrm{d}\,\mathrm{i}_\xi L$, where i_ξ stands for the operation of contracting the vector index of ξ with the first index of the Lagrangian form. Since the Lagrangian changes by a total derivative we learn that the vector fields on the spacetime generate infinitesimal local symmetries. Therefore, with each ξ is associated a $(D-1)$-form called the Noether current form which is defined as

$$j = \theta(\phi, \mathcal{L}_\xi\phi) - \mathrm{i}_\xi L. \tag{2.3}$$

The exterior derivative of j can be calculated using the equations (2.1) and (2.3) to be

$$\mathrm{d}j = E\mathcal{L}_\xi\phi. \tag{2.4}$$

Hence, we see that when the equation of motion $E = 0$ is satisfied, j is closed. Since this is true for all ξ, it must be that the pull-back of j to the space of solutions of the equation of motion is an exact form, $j = \mathrm{d}Q$, for some $(D-2)$-form Q called the Noether charge form corresponding to ξ. The integral of Q over a closed codimension-2 surface S is called the Noether charge of S relative to ξ (Wald, 1990).

The on-shell variation of j can be calculated to be

$$\delta j = \delta\theta(\phi, \mathcal{L}_\xi\phi) - \mathcal{L}_\xi\theta(\phi, \delta\phi) + \mathrm{d}\,\mathrm{i}_\xi\theta. \tag{2.5}$$

If the covariant derivative ∇ compatible with the unperturbed metric is Lie dragged by ξ, we can combine the first two terms of equation (2.5) to get

$$\delta j = \Omega(\phi, \delta\phi, \mathcal{L}_\xi\phi) + \mathrm{d}\,\mathrm{i}_\xi\theta. \tag{2.6}$$

Now, if the Hamiltonian generating the transformation on the phase space corresponding to the evolution with respect to ξ exists, then by its very definition, $\delta H_\xi = \int_\Sigma \Omega(\phi, \delta\phi, \mathcal{L}_\xi\phi)$, where Σ is a Cauchy surface. Hence from equation (2.6) we get

$$\delta H_\xi = \int_\Sigma \delta j - \int_\Sigma \mathrm{d}\,\mathrm{i}_\xi\theta. \tag{2.7}$$

It must be emphasized that equation (2.7) is well-defined, i.e., it does not depend on the choice of the Cauchy surface Σ when Ω is a closed form. One has to show that this is the case in the theory at hand by using the equations of motion and the linearized equations of motion. This requirement typically imposes some restriction on the allowed variations in the form of the boundary conditions. Also, if motions on the phase space generated by ξ are gauge transformations then Ω is an exact form too.

On shell, δj can be replaced by $\delta \mathrm{d}Q$ and we see from equation (2.7) that the variation of the Hamiltonian is given by the boundary integral

$$\delta H_\xi = \int_{\partial\Sigma} \left(\delta Q - \mathrm{i}_\xi\theta\right). \tag{2.8}$$

Now consider a stationary black hole with a bifurcation surface S_B and a Killing field ξ that vanishes on S_B (Raćz & Wald, 1992). The Killing field ξ can be written in terms of the stationary Killing field t and the axial Killing field φ as

$$\xi^a = t^a + \Omega\varphi^a, \tag{2.9}$$

where Ω is the angular velocity of the horizon. Now choose the hypersurface Σ to run from spatial infinity and intersecting the horizon at S_B. Since ξ is the Killing vector field we have $\mathcal{L}_\xi\phi = 0$. This implies that $\delta H_\xi = \int_\Sigma \Omega(\phi, \delta\phi, \mathcal{L}_\xi\phi) = 0$. Then using the fact that $\xi = 0$ on S_B equation (2.8) gives

$$\delta \int_{S_B} Q = \int_\infty (\delta Q_t - \mathrm{i}_t\theta) + \Omega \int_\infty \delta Q_\varphi. \tag{2.10}$$

The terms on the right-hand side of equation (2.10) coming from the boundary at infinity are the variation of the Hamiltonian corresponding to the stationary and the axial Killing vector field, respectively. These terms can be recognized as the $\delta\mathcal{E}$ and $-\delta\mathcal{J}$ where $\mathcal{E}$ and $\mathcal{J}$ are the asymptotically defined Energy and Angular Momentum, respectively. Thus, equation (2.10) is recognized as the first law of black-hole mechanics. However, the left-hand side of (2.10) is not of the form of κ times the variation of a local, geometric quantity on S_B. Besides the fields contained in the Lagrangian, it also depends on ξ and its derivatives. But all that dependence can be simplified using the fact that ξ is a Killing vector field and therefore the only degree of freedom is in its value and its first derivative. Higher-order derivatives of ξ can be written in terms of the Riemann tensor and the first-order derivative of ξ. Now on the bifurcation surface, the first-order derivative of ξ is $\nabla_a\xi_b = \kappa n_{ab}$, where n_{ab} is the binormal to S_B. Also, ξ vanishes on S_B by the definition of bifurcation surface. Hence all the dependence of Q on ξ^a is contained in κ. We take a factor of κ out of Q thus writing Q as κ times $\tilde{Q}$. Now identifying $\kappa/2\pi$ as the black-hole temperature we get the expression $2\pi\int_{S_B}\tilde{Q}$ as the entropy of the horizon. So we have the expression of the entropy as a local, geometric quantity constructed out of the dynamical fields in the Lagrangian. Finally, assuming that the zeroth law of black-hole mechanics, i.e., the constancy of the surface gravity κ on the horizon, holds true then the first law of black-hole mechanics really looks like the first law of thermodynamics

$$\frac{\kappa}{2\pi}\delta S = \delta\mathcal{E} - \Omega\delta\mathcal{J}. \tag{2.11}$$

This completes Wald's identification of the black-hole entropy as a Noether charge corresponding to the infinitesimal diffeomorphism generated by the Killing-field ξ. In the next section we apply this formalism in the first-order formulation of general relativity. The lower case Latin indices will stand for the internal Lorentz indices and the spacetime form indices are suppressed. Useful identities are collected in Appendix A and we shall use them extensively without any warning.

2.3 General relativity in the first-order formulation

The Lagrangian 4-form for General relativity in 4 dimensions in the first-order formalism is written in terms of the vierbein one forms e^a, which are so(3,1) vector-valued, and the so(3,1) valued connection one forms ω^{ab}. These are the dynamical variables of the theory. Instead of working directly with the vierbein it is more convenient to work with the field B_{ab} defined as $B_{ab} = \frac{1}{2}\epsilon_{abcd}e^c \wedge e^d$. The Lagrangian can be written as

$$L = \mathrm{tr}(B \wedge R), \tag{2.12}$$

where R is the curvature 2-form of the connection, $R = \mathrm{d}\omega + \omega \wedge \omega$. This Lagrangian is manifestly gauge and diffeomorphism invariant. We are working in units such that $16\pi G = 1$.

An arbitrary variation of L gives

$$\delta L = \mathrm{tr}(\delta B \wedge R - \mathrm{D}B \wedge \delta\omega) + \mathrm{d}\,\mathrm{tr}(B \wedge \delta\omega). \tag{2.13}$$

Here, D is the covariant exterior-derivative and we have used the identity $\delta R = \mathrm{D}\delta\omega$. The equations of motion are given by

$$\mathrm{D}B = 0, \tag{2.14}$$

$$R = 0. \tag{2.15}$$

The first one of these says that the action of D defined by ω on the Lorentz indices is the same as that of the unique derivative ∇ that annihilates the vierbein, $\nabla_\mu e^a_\nu = 0$. This equation can be solved to write the connection ω in terms of the vierbein and Christoffel symbols. When substituted in the second equation of motion, one gets the Einstein equation. When the equations of motion are satisfied, the curvature of ω is related to the curvature of the Christoffel connection, $R_{\mu\nu}{}^{\rho\sigma} e^a_\rho e^b_\sigma = R_{\mu\nu}{}^{ab}$.

From equation (2.13) we can read off the symplectic potential θ as

$$\theta = \mathrm{tr}(B \wedge \delta\omega). \tag{2.16}$$

Given a point in the phase space with coordinates (e^a, ω^{ab}) and two tangent vectors $(\delta_1 e^a, \delta_1 \omega^{ab})$ and $(\delta_2 e^a, \delta_2 \omega^{ab})$, the second variation of the symplectic potential gives the symplectic current

$$\Omega(\delta_1, \delta_2) = \mathrm{tr}\left(\delta_1 B \wedge \delta_2 \omega - \delta_2 B \wedge \delta_1 \omega\right), \tag{2.17}$$

where the argument of Ω on the left-hand side stands for two tangent vectors: $(\delta_1, \delta_2) =: \big((\delta_1 e^a, \delta_1 \omega^{ab}), (\delta_2 e^a, \delta_2 \omega^{ab})\big)$. Now, vector fields on the spacetime constitute a collection of infinitesimal diffeomorphisms. To each vector field ξ^a, associate a Noether current 3-form $j = \theta(\hat{\delta}) - \mathrm{i}_\xi L$, where $\hat{\delta}$ means the variation on the phase space induced by taking the Lie derivative of fields along ξ^a on the spacetime, i.e., $\hat{\delta} = \mathcal{L}_\xi$. This gives the Noether current corresponding to the infinitesimal diffeomorphisms as

$$j = -\mathrm{tr}(\mathrm{D}B \wedge \mathrm{i}_\xi \omega) - \mathrm{tr}(\mathrm{i}_\xi B \wedge R) + \mathrm{d}\,\mathrm{tr}(B \wedge \mathrm{i}_\xi \omega). \tag{2.18}$$

Using the equations of motion the first two terms on the right-hand side are seen to vanish and we get $j = \mathrm{d}Q$ where

$$Q = \mathrm{tr}(B \wedge \mathrm{i}_\xi \omega). \tag{2.19}$$

Thus, when the equations of motion are satisfied we have that j is an exact and hence closed form, i.e., $\mathrm{d}j = 0$ on shell. According to the general arguments due to Wald that we reviewed in Section 2.2, if ξ is taken to be the Killing field that becomes the null generator of the horizon and vanishes on the bifurcation surface, the integral of Q over the bifurcation surface gives κS, where κ is the surface gravity and S is the entropy of the black hole. But we see from equation (2.19) that $Q = 0$ on the bifurcation surface because ξ vanishes there. This seems to suggest that Wald's algorithm, when applied in the first-order formulation of general relativity, gives a zero entropy. This cannot be correct. What went wrong?

It must be that something singular is happening on the bifurcation surface. The connection is diverging and ξ is approaching zero in just such a way that the Noether charge is finite and non-zero. To see this, recall that Wald's algorithm requires that the dynamical fields should be Lie dragged by ξ. If the vierbein is Lie dragged by ξ then ω will be Lie dragged too, because ω is determined from e on shell. Now demanding that $\mathcal{L}_\xi e^a = 0$ we get

$$\mathcal{L}_\xi e^a = 0 = \mathrm{i}_\xi \mathrm{D}e^a + \mathrm{D}\mathrm{i}_\xi e^a - \mathrm{i}_\xi \omega^a{}_b \wedge e^b. \tag{2.20}$$

Solving this equation for $\mathrm{i}_\xi \omega$ we get

$$i_\xi \omega^a{}_b = e_b^\mu \mathrm{D}_\mu (\mathrm{i}_\xi e^a). \tag{2.21}$$

On substituting this in the equation (2.19) we get

$$Q = \frac{1}{2} \epsilon_{abcd} e^c \wedge e^d e^{\mu a} \mathrm{D}_\mu (\xi^\rho e_\rho^b). \tag{2.22}$$

Integrating this on the bifurcation surface we get

$$\int_{S_B} Q = \frac{1}{2} \int_{S_B} \epsilon_{abcd} e^c \wedge e^d e_\rho^a e_\sigma^b \nabla^\rho \xi^\sigma, \tag{2.23}$$

where we have replaced D_μ by ∇_μ because on shell their action is the same on internal indices. Now we get rid of the differential-form notation and write everything in terms of tensors and noting that on S_B, we have $\nabla_\mu \xi_\nu = \kappa n_{\mu\nu}$, where $n_{\mu\nu}$ is the binormal to S_B, we get

$$\int_{S_B} Q = \frac{1}{2} \int_{S_B} \epsilon_{abcd} e_\mu^c e_\nu^d e_\rho^a e_\sigma^b \kappa n^{\rho\sigma} \epsilon^{\mu\nu} \sqrt{\sigma}, \tag{2.24}$$

where $\sqrt{\sigma}$ is the volume element of S_B and $\epsilon^{\mu\nu}$ is the volume form on S_B. Now freely converting between internal and spacetime indices and noting that $n^{ab}\epsilon_{abcd} = 2\epsilon_{cd}$ we get after restoring $16\pi G$,

$$\int_{S_B} Q = \frac{1}{8\pi G} \kappa \int_{S_B} \sqrt{\sigma} = \frac{\kappa}{2\pi} \frac{\text{Area}}{4G}. \tag{2.25}$$

Thus we recover the formula for the black-hole entropy in general relativity in the first-order formulation using Wald's algorithm. In the next section we discover a new mathematical structure which will lead us to an elegant derivation of the black-hole entropy.

2.4 New symmetry and the Noether charge entropy

In the last section we mentioned that it was necessary that the dynamical fields should be Lie dragged by the Killing vector field ξ for Wald's algorithm to go through. Let us recall the reason why we needed that in the first place. The symplectic structure for a general, and not necessarily a Killing, vector field ξ is

$$\Omega(\delta, \delta_\xi) = \delta B \wedge \delta_\xi \omega - \delta_\xi B \wedge \delta\omega. \tag{2.26}$$

Using the equations of motion $\mathrm{D}B = 0$ and $R = 0$ and the linearized equations of motion $\mathrm{D}\delta B = [\delta\omega, B]$ and $\mathrm{D}\delta\omega = 0$, the symplectic structure can be written as

$$\Omega(\delta, \delta_\xi) = \mathrm{d}\left(\delta B \wedge \mathrm{i}_\xi \omega - \mathrm{i}_\xi B \wedge \delta\omega\right). \tag{2.27}$$

Thus, the symplectic structure is an exact form, as it should be for a diffeomorphism invariant theory. However, recall how the first law of black-hole thermodynamics was obtained. We obtained equation (2.10) because the dynamical fields were Lie dragged by the Killing vector field ξ so that $\Omega(\delta, \delta_\xi)$ vanished identically. Hence the surface integral at the inner bounday becomes equal to the one at the outer boundary. The outer boundary contribution gives the change in mass plus the angular velocity times the change in angular momentum of the black hole. Then comparing with the first law of black-hole thermodynamics, we identified the surface term in the inner boundary as equal to κS and we could read off the expression for S. But we see from equation (2.27) that there is no reason why $\Omega(\delta, \delta_\xi)$ should vanish if ξ is a Killing vector field. Therefore, as it stands Wald's construction will not go through and one cannot obtain the black-hole entropy from the Noether charge method.

In the last section, we forced $\Omega(\delta, \delta_\xi)$ to vanish by fixing the gauge so that $\mathcal{L}_\xi e^a = 0$. But this condition cannot be maintained at the bifurcation surface. Thus one needs an alternative method to extract the black-hole entropy.

We start by introducing the definition of a gauge-covariant Lie derivative,

$$\delta_\xi e^a = \mathcal{L}_\xi e^a + \lambda^a{}_b\, e^b, \tag{2.28}$$

$$\text{and} \qquad \delta_\xi \omega^a{}_b = \mathcal{L}_\xi \omega^a{}_b - \mathrm{D}\lambda^a{}_b. \tag{2.29}$$

where $\lambda^a{}_b = \mathrm{i}_\xi \omega^a{}_b + e^a_\mu e_{b\nu} \nabla^{[\mu}\xi^{\nu]}$. Here ∇ is the covariant extension of D. It is the unique connection which is compatible with the vierbein. The point of the definitions in equations (2.28) is that when we are on shell (i.e., $\mathrm{D}e = 0$), i.e., if the connection $\omega^a{}_b$ is torsion-free and hence determined from the vierbein e^a, then it is easy to check that

$$\delta_\xi e^a_\mu = \frac{1}{2} e^{a\rho} \mathcal{L}_\xi g_{\rho\mu}. \tag{2.30}$$

Thus, when ξ^a is a Killing vector field $\delta_\xi e^a = 0 = \delta_\xi \omega^{ab}$. The variation defined in equations (2.28) and (2.29) is called the Kosmann Lie derivative in the mathematical physics literature (see for example Fatibene & Francaviglia (2009)).

Since the new variation is the linear combination of ordinary Lie derivative and a gauge transformation, and since the Lagrangian is invariant under gauge transformation (i.e., it does not produce even a surface term under gauge transformation), it follows that the variation of the Lagrangian under the field variation defined in equations (2.28) and (2.29) produces the same surface term as it did under the old variation, which was induced by the Lie derivative of the fields. Hence the symplectic potential does not change either. The expression of the Noether current corresponding to the invariance under these field transformations is the same as before,

$$j = \mathrm{tr}(B \wedge \delta_\xi \omega) - \mathrm{tr}\ \mathrm{i}_\xi (B \wedge R). \tag{2.31}$$

On using the equation of motion and the definition of field variations it now follows immediately that j is an exact form, $j = \mathrm{d}Q$, where

$$Q = B_{ab} \nabla^{[b} \xi^{a]}, \tag{2.32}$$

which is the desired Komar form as is easily seen by comparing it with equation (2.22). Rest of the steps are then identical to those after equation (2.22). Thus, we see that the black-hole entropy is still the Noether charge but the symmetry that it is the Noether charge of is not the diffeomorphism but a certain combination of diffeomorphism and gauge transformation.

2.5 Black-hole entropy in Gauss-Bonnet gravity

The computational efficiency resulting from our use of the gauge-covariant Lie derivative will be apparent now when we calculate the black-hole entropy in Einstein-Gauss-Bonnet gravity in five dimensions. For the reasons discussed in the previous section, all we need to do to find the black-hole entropy is write down the expression of Noether charge corresponding to the gauge-covariant transformations of the fields given in equations (2.28) and (2.29).

The Gauss-Bonnet (GB) part of the Lagrangian is given by

$$L = \alpha' \; R^{ab} \wedge R^{cd} \wedge e^f \epsilon_{abcdf}, \tag{2.33}$$

where α' is the GB coupling which we shall set to unity and only in the end shall we restore it. An arbitrary variation of the Lagrangian gives

$$\delta L = 2 \; \mathrm{d}(\delta\omega^{ab} \wedge R^{cd} \wedge e^f \epsilon_{abcdf}) + 2 \; \delta\omega^{ab} \wedge R^{cd} \wedge \mathrm{D}e^f \epsilon_{abcdf} + R^{ab} \wedge R^{cd} \wedge \delta e^f \epsilon_{abcdf}. \tag{2.34}$$

Unlike general relativity, the torsion-free condition is not an equation of motion for the pure GB theory. In order to not have the torsion propagating we have to restrict ourselves to a torsion-free connection by hand. Then the equation of motion obtained by setting the coefficient of $\delta\omega$ equal to zero is automatically satisfied. The remaining equation of motion is $R^{ab} \wedge R^{cd}\epsilon_{abcdf} = 0$. The symplectic potential can be read off from the variation of the Lagrangian as

$$\theta = 2\,\delta\omega^{ab} \wedge R^{cd} \wedge e^f \epsilon_{abcdf}. \tag{2.35}$$

Given a vector field ξ, the Noether charge due to the induced variations of fields is

$$\begin{aligned} j &= \theta(\delta_\xi) - \mathrm{i}_\xi L \\ &= 2\ \delta_\xi\omega^{ab} \wedge R^{cd} \wedge e^f \epsilon_{abcdf} - \mathrm{i}_\xi(R^{ab} \wedge R^{cd} \wedge e^f \epsilon_{abcdf}). \end{aligned} \tag{2.36}$$

Using the equations of motion and the gauge-covariant Lie derivative of ω given in equation (2.29) we see that j is again an exact, and hence closed form, $j = \mathrm{d}Q$, where

$$Q = -2\ \nabla^a\xi^b\ R^{cd} \wedge e^f \epsilon_{abcdf}. \tag{2.37}$$

Integrating Q over the bifurcation surface S_B and using the fact that on S_B the Riemann tensor with all its indices projected on S_B is equal to the intrinsic Riemann tensor of S_B we get, after restoring α',

$$\int_{S_B} Q = -8\alpha'\kappa \int_{S_B} \sqrt{\sigma}\ {}^{(3)}R. \tag{2.38}$$

Hence the contribution of the GB term to the black-hole entropy depends upon the intrinsic Ricci curvature of the horizon,

$$S = -16\pi\alpha' \int_{S_B} \sqrt{\sigma}\ {}^{(3)}R, \tag{2.39}$$

which is the correct expression for the black-hole entropy in Gauss-Bonnet gravity (Jacobson & Myers, 1993).

Chapter 3

Holography

3.1 Introduction

The holographic principle states that a quantum theory of gravity in $d+1$ dimensions is dual to a non-gravitational quantum field theory in d dimensions (Susskind, 1995, 't Hooft, 2001). The AdS/CFT correspondence, where AdS stands for Anti-de Sitter space and CFT stands for Conformal Field Theory, is a concrete realization of the holographic principle (Gubser *et al.*, 1998a, Maldacena, 1998, Witten, 1998a). The correspondence relates type IIB string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ to $\mathcal{N} = 4$ Super Yang-Mills theory on a four-dimensional Minkowski spacetime. The strong form of the conjecture states the equality between the partition function of the two theories (Aharony *et al.*, 2000).

The field theory has two parameters: the number of colors N_c and the gauge coupling g_{YM}. When the number of colors is very large, the perturbation theory is governed by the 't Hooft coupling, $\lambda_t = g_{\mathrm{YM}}^2 N_c$. On the string side of the duality there are three parameters: the string coupling g_s, the string length (or inverse square-root of tension) l_s and the radius of curvature of the Anti-de Sitter space, R_{AdS}. The dimensionless parameters of the two theories are related by

$$g_{\mathrm{YM}}^2 = 4\pi g_s \tag{3.1}$$

$$\text{and} \qquad \lambda_t = g_{\mathrm{YM}}^2 N_c = \frac{R_{\mathrm{AdS}}^4}{l_s^4}. \tag{3.2}$$

We see from (3.1) that when the gauge theory coupling is small then the string theory is weakly interacting. The useful limit comes from the equation (3.2), which says that for small $g_{\rm YM}$ (to be consistent with the weakly-interacting strings) and large N_c such that the t'Hooft coupling λ_t is large, the AdS radius is much larger than the string length. In this limit the string theory can be approximated by supergravity or gravity. Therefore we learn that the holograpic duality is a strong/weak duality in the sense that when the string is weakly coupled and the AdS radius is large, such that classical gravity is a good description, the gauge theory is strongly coupled and vice versa. In fact, most of the applications of holography in condensed matter physics and hydrodynamics rest upon this crucial feature of the holographic duality.

The relation between the partition functions of the two theories in Euclidean space is

$$\langle {\rm e}^{\int {\rm d}^4x \phi_0 \mathcal{O}} \rangle_{\rm CFT} \;=\; Z_{\rm string}\left[\phi(\vec{x},z)|_{z=0} = \phi_0(\vec{x})\right], \tag{3.3}$$

where on the left-hand side we have the generating functional for the correlation functions in the field theory. Here ϕ_0 is an arbitrary function (or source in the language of field theory) and we can calculate the correlation functions of $\mathcal{O}$ by functionally differentiating the left-hand side with respect to ϕ_0. The right-hand side has the partition function of the string theory in AdS containing a bulk field $\phi(\vec{x},z)$ with a boundary value $\phi_0(\vec{x})$. The correspondence in equation (3.3) is summarized by saying that the boundary value of the bulk field $\phi(\vec{x},z)$ acts a source for the operator $\mathcal{O}$ in the field theory.

A very fruitful limit of the correspondence is obtained by considering the field theory to be strongly coupled so that classical gravity would be a good approximation for the string theory. In this case, the right-hand side of (3.3) can be calculated using the saddle-point approximation and one obtains

$$\langle {\rm e}^{\int {\rm d}^4x \phi_0 \mathcal{O}} \rangle_{\rm CFT} \;=\; {\rm e}^{-S_{\rm grav}[\phi(z\to\infty)=\phi_0]}. \tag{3.4}$$

Therefore the prescription is to find the classical solution for ϕ in the bulk which has a boundary value ϕ_0. Derivatives of the on-shell gravity action with respect to ϕ_0 will give the correlators of $\mathcal{O}$. In particular, noticing that the derivative of the classical action with respect to the boundary value of the field is equal to the canonical momentum conjugate to the field evaluated at the boundary, we get for the one-point function,

$$\langle \mathcal{O}(\vec{x}) \rangle = \lim_{z \to \infty} \Pi(\vec{x}, z). \tag{3.5}$$

Although AdS/CFT was the first example of holography to be discovered, there is a widespread belief that the holographic principle itself transcends the particulars of any string or gauge theory (Heemskerk & Polchinski, 2011, Sundrum, 2011). There exists a general holographic dictionary that is valid irrespective of whether we know the gauge theory side of the duality or not. It says that *every* d-dimensional CFT is dual to *some* AdS_{d+1} theory.

3.2 Linear response theory

It seems intuitively reasonable that if we perturb a system in equilibrium very slightly then it will deviate very slightly from the equilibrium configuration and it would relax back to the equilibrium. It also seems reasonable that the response of the system to a small perturbation will be linear in the perturbation. What is non-trivial is that this non-equilibrium behavior of the system, while it is relaxing back to the equilibrium, is in fact governed by the equilibrium correlation functions (see equation (3.16)). In this section we use linear response theory to deduce this fact and we show how the linear response of the boundary gauge theory can be calculated holographically using classical gravity. A good discussion of linear response theory is in the review by Kadanoff & Martin (1963). We here follow a shorter route to the same results as discussed in Gubser *et al.* (2008).

Consider a system at a finite temperature which at equilibrium is described by the hamiltonian H_0 and the density matrix ρ_0. Now perturb the system at time $t = 0$ with a time-independent Schrödinger operator $\mathcal{O}$ so that the new Hamiltonian becomes

$$H = H_0 + V, \qquad \text{where} \tag{3.6}$$

$$V = -\,\epsilon\,\mathcal{O}\,\delta(t), \tag{3.7}$$

where ϵ is a small parameter and $\delta(t)$ is the Dirac delta function. The equation of motion for the full density matrix $\rho = \rho_0 + \delta\rho$, where ρ_0 is the unperturbed density matrix at equilibrium, is given by

$$\mathrm{i}\frac{\partial\rho}{\partial t} = [H, \rho]\,. \tag{3.8}$$

We can solve for the time dependence of the density matrix using time-dependent perturbation theory. To linear order in the perturbation, the result is

$$\delta\rho(t) = \mathrm{i}\theta(t)e^{-\mathrm{i}H_0 t}\,[\epsilon\mathcal{O}, \rho_0]\,e^{\mathrm{i}H_0 t}. \tag{3.9}$$

For any other operator B, its expectation value at time t, $\langle B(t)\rangle$, is given by

$$\langle B(t)\rangle = \mathrm{tr}\{\rho(t)B\} \tag{3.10}$$

$$= \mathrm{tr}\{\rho_0 B\} + \delta\langle B(t)\rangle, \tag{3.11}$$

where $\delta\langle B(t)\rangle$ is given by

$$\delta\langle B(t)\rangle = \mathrm{i}\theta(t)\,\mathrm{tr}\{e^{-\mathrm{i}H_0 t}\,[\epsilon\mathcal{O}, \rho_0]\,e^{\mathrm{i}H_0 t}B\}. \tag{3.12}$$

Now using the cyclic property of the trace and changing to the Heisenberg picture we obtain

$$\delta\langle B(t)\rangle = \mathrm{i}\theta(t)\mathrm{tr}\{\rho_0\,[B(t),\epsilon\mathcal{O}(0)]\} \tag{3.13}$$

$$= \mathrm{i}\theta(t)\langle[B(t),\epsilon\mathcal{O}(0)]\rangle. \tag{3.14}$$

Since small perturbations add linearly we can now generalize the above result from the perturbation localized at $t = 0$ to perturbations sourced by a spacetime-dependent field ϕ_0 as

$$\delta\langle B(\vec{x},t)\rangle = \mathrm{i}\int \mathrm{d}t'\mathrm{d}\vec{x}'\theta(t-t')\langle[B(\vec{x},t),\epsilon\mathcal{O}(\vec{x}',t')]\rangle\phi_0(\vec{x}',t'). \tag{3.15}$$

In particular, in the context of AdS/CFT when the operator $\mathcal{O}$ is sourced by the boundary value ϕ_0 of the bulk field ϕ, we can write the linear change in the expectation value of $\mathcal{O}$ as

$$\delta\langle\mathcal{O}(x)\rangle = \int_{x'} \mathrm{d}x' G^{\mathrm{ret}}(x-x')\phi_0(x'), \tag{3.16}$$

where $G^{\mathrm{ret}}(x-x')$ is the retarded Green's function

$$G^{\mathrm{ret}}(x-x') = \mathrm{i}\theta(t-t')\langle[\mathcal{O}(\vec{x},t),\mathcal{O}(\vec{x}',t')]\rangle. \tag{3.17}$$

In the Fourier space, equation (3.16) becomes

$$\delta\langle\mathcal{O}(\omega,\vec{k})\rangle = -G^{\mathrm{ret}}(\omega,\vec{k})\phi_0(\omega,\vec{k}). \tag{3.18}$$

In the limit of long wavelength and low frequency when the hydrodynamic description of the field theory is valid, the retarded Green's function defines a transport coefficient (called susceptibility in general) as

$$\chi = -\lim_{\omega\to 0}\lim_{\vec{k}\to 0}\frac{\mathrm{Im}G^{\mathrm{ret}}(\omega,\vec{k})}{\omega}. \tag{3.19}$$

For example, if one takes for $\mathcal{O}$ the off-diagonal component of the stress-energy tensor then the transport coefficient is called the shear viscosity. The meaning of χ is that in the limit of low frequency, χ is the coefficient of proportionality between the linear response of the system to the applied source and the time rate of change of the source,

$$\langle\mathcal{O}\rangle = -\chi\partial_t\phi_0. \tag{3.20}$$

Comparing equation (3.18) with equation (3.5) we deduce a very useful expression

$$G^{\mathrm{ret}}(\omega,\vec{k}) = -\lim_{z\to\infty}\frac{\Pi(\omega,\vec{k},z)}{\phi(\omega,\vec{k},z)}, \tag{3.21}$$

where it is understood that one first evaluates the ratio of the conjugate momentum and the field in the bulk at a cut-off surface close to the boundary and then one takes the limit $z\to\infty$, where z is the coordinate in the bulk. Here $\vec{k}$ and ω are the Fourier frequencies corresponding to the field theory coordinates which are on the boundary of the bulk geometry, ω corresponds to time and $\vec{k}$ corresponds to space. The susceptibility can now be written as

$$\chi = \lim_{\omega,\vec{k}\to 0}\lim_{z\to\infty}\frac{\Pi(\omega,\vec{k},z)}{i\omega\phi(\omega,\vec{k},z)}. \tag{3.22}$$

It must be noticed that the left-hand side of equation (3.21) is calculated in the strongly coupled quantum field theory while the right-hand side is calculated in the dual classical gravity, and through the holographic principle the values must match. This is the power of holography. Calculations in strongly coupled quantum field theories are notoriously difficult. Through holography, the results of these calculations are matched to certain results which involve much simpler calculations in a classical gravitational theory.

A crucial point involved in the above manipulations is that they are all done in the Euclidean signature. In particular, in equation (3.21) the bulk solution of the classical field equations is calculated in the Euclidean signature. While this works perfectly at zero temperature, at a finite temperature this prescription runs into difficulties. For example, in Euclidean formalism the horizon of the black hole is the end of spacetime and one does not need to impose any conditions on the bulk field at the horizon. But in the Lorentzian signature horizon is not the end of spacetime and one needs to impose boundary conditions at the horizon in order to solve the classical field equation for ϕ. That is not difficult, though. Incoming boundary conditions at the horizon make perfect physical sense. However, we are interested in the hydrodynamic limit, so we need the limit of the retarded Green's functions as $\omega \to 0$ and $\vec{k} \to 0$. But in the Euclidean formalism the frequencies ω are discrete (these are called the Matsubara frequencies) and it is not clear how to take the limit $\omega \to 0$. In order to deal with these difficulties it is important to have a real-time prescription to calculate the correlators in the field theory. This is the topic of the next section.

3.3 Real-time holography at finite temperature

The prescription to calculate the retarded two-point function at finite temperature in the Lorentzian signature was first proposed by Son & Starinets (2002) and then justified by Herzog & Son (2003). In the holographic correspondence the states of the boundary gauge theory are mapped to the states of the bulk gravitational theory. At a finite temperature, the gravitational state dual to the thermal vacuum of a field theory is a black hole. Now, for the bulk field in addition to the appropriate boundary conditions at infinity one also needs to worry about the regularity conditions at the horizon. The real-time prescription to calculate the retarded two-point function in the field theory in the presence of an inner boundary goes as follows:

- Find the solution to the equations of motion that is infalling at the horizon and approaches a constant $\phi_0(\vec{x})$ at the outer boundary.

- Plug this solution into the action and integrate by parts. The action becomes a boundary term containing contributions both from the outer boundary and the horizon,

$$S = \frac{1}{2} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \phi_0(-p) f(p,z) \phi_0(p) \Big|_{z=\infty}^{z=z_{hor}}. \tag{3.23}$$

- The retarded Green's function is now given by

$$G^{\mathrm{ret}}(p) = -\lim_{z\to\infty} f(p,z). \tag{3.24}$$

On very general grounds one expects that any interacting quantum field theory at a finite temperature, when viewed at sufficiently long length scales, should be described by hydrodynamics (Son & Starinets, 2007). The UV/IR connection implied by the holographic duality then suggests that the behaviour of the thermal quantum field theory at very large length scales must be captured by the behaviour of the bulk geometry near the horizon. In particular, the hydrodynamic limit of the field theory at a finite temperature must have a description in terms of the physics close to the black-hole horizon (Kovtun *et al.*, 2003).

In the next section we will see that the real-time prescription to calculate the Green's function gives the same result as that calculated in the Euclidean AdS/CFT prescription at the level of linear response.

3.4 Lorentzian versus Euclidean prescription

In this section, we will closely follow Iqbal & Liu (2009) and Faulkner *et al.* (2011). We will see how at the level of the linear response the Lorentzian and Euclidean prescriptions give the same result. Moreover, we will see how in the long-wavelength limit the renormalization group flow of the susceptibility becomes trivial. This will provide the essential link between the horizon membrane fluid and the fluid description of the boundary gauge theory.

3.4.1 First look at the black-hole membrane

Consider a massless bulk field on a black-hole background. The bulk metric of the black hole is given by

$$\mathrm{d}s^2 = g_{rr}\mathrm{d}r^2 - g_{tt}\mathrm{d}t^2 + g_{ij}\mathrm{d}x^i\mathrm{d}x^j. \tag{3.25}$$

The action of the scalar field is given by

$$S_{\text{out}} = -\frac{1}{2\alpha}\int_{r>r_h} \mathrm{d}^{d+1}x\sqrt{-g}(\nabla_a\phi)(\nabla^a\phi), \tag{3.26}$$

where α is a coupling constant. The integration is restricted to the region outside the time-like surface $r = r_{\mathrm{h}} + \epsilon = r_0$ called the stretched horizon. This will be discussed more completely in the later chapters. But the idea is simply that for the outside observers the black-hole horizon acts as a causal boundary and nothing comes out of it. Therefore the observers should be able to discuss physics without knowing about the region interior to the horizon. The absorptive nature of the horizon is then modelled by endowing the horizon with membrane-like properties.

The momentum canonically conjugate to ϕ with respect to the r-foliation is given by

$$\Pi = \frac{\delta S_{\text{out}}}{\delta\nabla_r\phi} = -\frac{1}{\alpha}\sqrt{-g}g^{rr}\nabla_r\phi. \tag{3.27}$$

Variation of this action with respect to ϕ gives

$$\delta S_{\text{out}} = \frac{1}{\alpha}\int_{r>r_{\mathrm{h}}} \mathrm{d}^{d+1}x\sqrt{-g}\nabla^2\phi\delta\phi - \frac{1}{\alpha}\int_{r=r_0} \mathrm{d}^d x\sqrt{-g}g^{rr}\nabla_r\phi\delta\phi. \tag{3.28}$$

Up to the bulk term which yields the equation of motion, the variation of the action is a total boundary term which can be written on shell as

$$\delta S_{\text{out}} = -\int_{r=r_0} \mathrm{d}^d x \Pi(r_0, x) \delta\phi(r_0, x). \tag{3.29}$$

In order to have a viable variational principle for the outside observers we then have to add a surface term to the action which would yield the negative of the δS_{out} upon variation. This surface term is given by

$$S_{\text{surface}} = \int_{r=r_0} \mathrm{d}^d x \sqrt{-h} \left(\frac{\Pi(r_0, x)}{\sqrt{-h}} \right) \phi(r_0, x), \tag{3.30}$$

and the total action is now $S_{\text{eff}} = S_{\text{out}} + S_{\text{surface}}$. Equation (3.30) says that the observers hanging out close to the stretched horizon (called fiducial observers or FIDOs) feel that the horizon is endowed with a scalar charge given by

$$Q_\phi = \frac{\Pi(r_0, x)}{\sqrt{-h}} = -\frac{1}{\alpha} \sqrt{g^{rr}} \nabla_r \phi. \tag{3.31}$$

The condition that the horizon should be a regular place for a freely falling observer implies that the field ϕ should be non-singular on the horizon. Thus ϕ can only depend on the coordinates t and r in the form of the ingoing Eddington-Finkelstein coordinate v, which is defined by solving $dv = dt + \sqrt{g_{rr}/g_{tt}} dr$. This relates the radial and time derivatives of the field on the horizon as

$$\nabla_r \phi = \sqrt{\frac{g_{rr}}{g_{tt}}} \nabla_t \phi. \tag{3.32}$$

Thus the scalar charge perceived by the FIDOs is

$$Q_\phi = -\frac{1}{\alpha}\sqrt{g^{tt}}\nabla_t\phi(r_0). \tag{3.33}$$

Writing this in the orthonormal frame as $Q_\phi = -\frac{1}{\alpha}\nabla_{\hat{t}}\phi$ we see from equation (3.20) that the observers outside the horizon endow the horizon with a fluid-like behavior described by a transport coefficient $\chi = 1/\alpha$.

3.4.2 Equivalence at the linear level

According to the real-time prescription of the holographic calculation of the retarded Green's function summarized in equation (3.24), we first need to calculate the on-shell value of the effective action for ϕ. This can be obtained by integrating by parts S_{out} and adding the surface term S_{surface}. This gives

$$S_{\text{eff}} = -\frac{1}{2\alpha}\int_{r_\infty} \mathrm{d}^d x\sqrt{-g}\phi g^{rr}\nabla_r\phi - \frac{1}{2\alpha}\int_{r_0} \mathrm{d}^d x\sqrt{-g}\phi g^{rr}\nabla_r\phi. \tag{3.34}$$

Now writing $\phi(r,p) = F(r,p)\phi(r_0,p)$ such that $F(r\to\infty,p) = 1$ we get for the real time Green's function

$$G^{\text{ret}}(p) = \frac{1}{\alpha}\sqrt{-g}g^{rr}\nabla_r F(r,p). \tag{3.35}$$

However, from the linear response theory and the Euclidean prescription for calculating the correlator in equation (3.21) we can calculate the imaginary time correlator by plugging in the value of Π from equation (3.27) and we get

$$G^{\text{ret}}(\omega,\vec{k}) = -\lim_{r\to\infty}\frac{\Pi(\omega,\vec{k},r)}{\phi(\omega,\vec{k},r)} \tag{3.36}$$

$$= \frac{1}{\alpha}\sqrt{-g}g^{rr}\nabla_r F(r,\omega,\vec{k}) \tag{3.37}$$

which is precisely the same as equation (3.35). The only difference is in the notation where p earlier stands for $(\omega, \vec{k})$ now.

To summarize, we have seen that the real-time prescription to calculate the retarded Green's function gives the same result as that calculated in the Euclidean AdS/CFT prescription at the level of linear response. This equivalence is crucial to relate the transport coefficients calculated in the membrane paradigm for black holes with the physical quantities describing the hydrodynamic behaviour of the boundary gauge field theory.

3.5 Hydrodynamic limit

We saw in Section 3.4.1 that a scalar field in the bulk induces a scalar charge at the horizon and the FIDOs perceive the horizon to have a linear response given by the susceptibility $\chi = 1/\alpha$. On the other hand, the linear response of the boundary gauge theory, when it is perturbed by a source which is the boundary value of the scalar field, is given by equation (3.22) as

$$\chi = \lim_{\omega, \vec{k} \to 0} \lim_{z \to \infty} \frac{\Pi(\omega, \vec{k}, z)}{i\omega\phi(\omega, \vec{k}, z)} \tag{3.38}$$

Note that these two quantities are calculated by the same formula, one is evaluated on the horizon while the other is evaluated at the boundary at ∞. Now we will see that in the hydrodynamic limit, i.e., in the limit that $\omega, \vec{k} \to 0$, equation (3.38) can in fact be evaluated at any value of the radial coordinate r. To this end, let us write down the Hamilton equations of motion for ϕ,

$$\begin{aligned} \Pi &= -\frac{1}{\alpha}\sqrt{-g}g^{rr}\nabla_r\phi \\ \text{and} \qquad \nabla_r\Pi &= \frac{1}{\alpha}\sqrt{-g}g^{rr}g^{\mu\nu}k_\mu k_\nu\phi, \end{aligned} \tag{3.39}$$

where $k^\mu = (\omega, \vec{k})$. Now in the limit that $\omega, \vec{k} \to 0$ we see from equations (3.39) that

$$\nabla_r \Pi \sim 0 \tag{3.40}$$

$$\text{and} \qquad \nabla_r(\omega\phi) \sim 0. \tag{3.41}$$

Thus in the limit that $\omega, \vec{k} \to 0$ equation (3.38) can be evaluated at any radial coordinate. Therefore the hydrodynamic transport coefficients for the boundary gauge theory can be holographically calculated from the membrane at the black-hole horizon.

This indicates the importance of the membrane paradigm in the classical gravity theories. The hydrodynamic transport coefficients describing the evolution of the horizon in the membrane paradigm, through holography, also describe the hydrodynamic limit of the boundary gauge theory at the linear response level.

Chapter 4

Hydrodynamics

In this chapter, we use the action-principle formalism of the membrane paradigm as constructed by Parikh & Wilczek (1998) and we extend it to the simplest generalization of general relativity which is quadratic in the curvature tensor called the Einstein-Gauss-Bonnet (EGB) gravity. We derive the membrane stress-energy tensor on the stretched horizon for EGB theory. After regularization and restriction to the linearized perturbations of static black-hole backgrounds with horizon cross-sections of constant curvature, we express the membrane stress tensor in the form of a Newtonian viscous fluid described by certain transport coefficients.

We find the ratio of the shear viscosity to the entropy density for the membrane fluid corresponding to the black-brane solution in EGB theory with cosmological constant $\Lambda = -(D-1)(D-2)/2l^2$ to be

$$\frac{\eta}{s} = \frac{1}{4\pi}\left[1 - 2\frac{(D-1)}{(D-3)}\frac{\lambda}{l^2}\right], \tag{4.1}$$

where $\lambda = (D-3)(D-4)\alpha'$ and α' is the GB coupling constant. This matches with the result found by Brigante *et al.* (2008b). Notice that the calculation of Brigante *et al.* (2008b) is done at the boundary of the spacetime at infinity while our calculation refers to the horizon. This naturally leads to the conclusion that in the EGB theory, as in general relativity, the ratio η/s is universal, i.e., scale independent in the sense of Bredberg *et al.* (2011). That this result is true is argued by Iqbal & Liu (2009), and our calculation combined with the result of Brigante *et al.* (2008b) provides a support for their conclusion. The same result was obtained by Banerjee & Dutta (2009), Myers *et al.* (2009), Paulos (2010) and it was also reported in a recent study of Cai *et al.* (2011) following the proposal of Bredberg *et al.* (2011). Also, we will see that the presence of the GB term in the bulk gravitational theory violates the KSS bound, $\eta/s \geq 1/4\pi$, for any $\alpha' > 0$.

Besides the black brane, our method also provides the value of this ratio for the five dimensional spherically symmetric AdS black hole in EGB gravity,

$$\frac{\eta}{s} = \frac{r_h^4}{4\pi\left(r_h^2+\alpha'\right)\left(r_h^2+3\alpha'\right)}\left(1-\frac{2\alpha\prime}{l^2}\right), \tag{4.2}$$

where r_h is the radius of the horizon and l is related to the cosmological constant by $\Lambda = -6/l^2$. As far as we are aware of this is a new result. Our method thus predicts that the hydrodynamic limit of the conformal field theory (CFT) state dual to the 5D-spherically symmetric AdS black hole in EGB theory with AdS boundary conditions has the viscosity-to-entropy ratio given by equation (4.2), where r_h is understood to be a function of the Hawking temperature of the black hole. As in the case of the black brane, the KSS viscosity bound is violated for any $\alpha' > 0$. The difference between the black-brane and black-hole results is further discussed in section 4.7.

This chapter is organized as follows: we first review in Section 4.1 Damour's derivation of the equations describing the evolution of the horizon and their hydrodynamical interpretation. Then in Section 4.2 we present the geometric setup of the membrane paradigm. In Section 4.3 we furnish a simple example of the stretched horizon and the FIDOs in Rindler space. In Section 4.4 we review the action-based membrane paradigm approach for the black holes in general relativity. This construction is then generalized in Section 4.5 where we construct the membrane paradigm for the black objects in EGB gravity. In this section we obtain the stress tensor for the membrane fluid and we derive the expressions for the transport coefficients of the fluid. In Section 4.6 we evaluate these transport coefficients for some specific black geometries. Finally, we conclude with a summary and discussion in Section 4.7. An elementary introduction to the basic equations of the first-order hydrodynamics is given in Appendix B.

All the symbols that we will be using in this chapter are defined when introduced for the first time. For the convenience of the readers we have also included Table 4.1 at the end of this chapter to summarize these symbols and their meanings.

4.1 Horizon hydrodynamics *à la* Damour

In this chapter we will derive the equations governing the dynamics of the horizon as was first done by Damour in his prescient thesis in 1979. We will closely follow the derivation given by Damour (1978), Damour & Lilley (2008). A very good reference on the geometry of null hypersurfaces is a review by Gourgoulhon & Jaramillo (2006).

Let l^a denote the null geodesic generator of the horizon, H, in a d-dimensional spacetime. Introduce a null *rigging* vector field n^a transverse to H such that it is nowhere vanishing and $l^a n_a = -1$. Since H is a null hypersurface, the spacetime metric g_{ab} induces a degenerate metric on H, l^a being the degenerate direction of the induced metric. Define a transverse projector γ^a_b as

$$\gamma^a_b = g^a_b + n^a l_b + l^a n_b. \tag{4.3}$$

It is easy to see that γ^a_b projects the indices in a direction transerse to both n^a and l^a. The failure of the geodesic deviation vector to be parallel transported along the null geodesic congruence is captured by a tensor called the Weingarten map and it is simply defined as the covariant derivative of the horizon generator,

$$\chi_a{}^b = \nabla_a l^b. \tag{4.4}$$

The Weingarten map, $\chi_a{}^b$, is orthogonal to the null generator l_b in the upper index but it has components in the direction of n^a. Information contained in $\chi_a{}^b$ is captured in its various components as

$$\chi_a{}^b l^a = \kappa l^b, \tag{4.5}$$

$$\chi_a{}^b l_b = 0, \tag{4.6}$$

$$\chi_a{}^b \gamma_c^a n_b =: \Omega_c, \tag{4.7}$$

where κ is a function on H that measures the non-affinity of l^a and Ω_c is a measure of rotation of the generators as time evolves.

Let us now define the purely transverse part of χ by projecting both of its indices transverse to the two null vectors,

$$\theta_{ab} := \gamma_a^c \gamma_b^d \chi_{cd}. \tag{4.8}$$

The purely transverse tensor θ_{ab} can be decomposed into its irreducible parts as

$$\theta_{ab} = \sigma_{ab} + \frac{1}{(d-2)}\theta\gamma_{ab} + \omega_{ab}, \tag{4.9}$$

where σ_{ab} is symmetric traceless, θ is the trace and ω_{ab} is the antisymmetric component of θ_{ab}, respectively,

$$\begin{aligned} \theta &= \theta_{ab}\gamma^{ab} = \nabla_c l^c - \kappa, \\ \sigma_{ab} &= \theta_{(ab)} - \frac{1}{(d-2)}\theta\gamma_{ab}, \\ \omega_{ab} &= \theta_{[ab]}. \end{aligned} \tag{4.10}$$

For future reference we note that ω_{ab} can be written as

$$\omega_{ab} = \chi_{[ab]} + n^c \chi_{c[b} l_{a]} + l_{[b}\chi_{a]c} n^c, \tag{4.11}$$

which can be used to show that if l^a is hypersurface orthogonal then the Frobenius theorem implies that ω_{ab} vanishes.

We are interested in the equations governing the evolution of θ and Ω_c. These equations will then be interpreted as describing the hydrodynamics of a fictitious fluid living on the horizon. To this end, let us first calculate the evolution of θ.

$$\begin{aligned}\mathcal{L}_l\theta &= l^a\nabla_a\theta \\ &= l^a\nabla_a\left(\nabla_c l^c - \kappa\right).\end{aligned}$$

The order of the two derivatives on l^c can be reversed using the Riemann tensor. A simple calculation then gives

$$\mathcal{L}_l\theta = -R_{ab}l^a l^b + \kappa(\kappa+\theta) - \chi_b{}^a\chi_a{}^b. \tag{4.12}$$

One can prove the following identity in order to simplify the last term, $\theta_{ab}\theta^{ba} = \chi_{ab}\chi^{ba} - \kappa^2$. Making this replacement and writing θ_{ab} in terms of its irreducible components we get the null focussing equation

$$\mathcal{L}_l\theta = \kappa\theta - \frac{\theta^2}{(d-2)} - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}l^a l^b. \tag{4.13}$$

Derivation of the equation governing the evolution of Ω_c is much more complicated and we refer the reader to Damour's thesis (Damour, 1978). It is given by

$$\mathcal{L}_l\Omega_a = -\Omega_a\theta - \mathrm{D}_a\kappa + \frac{(3-d)}{(d-2)}\mathrm{D}_a\theta + \mathrm{D}_c\sigma_a^c - R_{cd}\gamma_a^c l^d, \tag{4.14}$$

where D is the $(d-2)$ dimensional covariant derivative compatible with γ_{ab} on the horizon cross-sections. If we identify $-\theta$ as the energy-density ρ and l^μ as the fluid velocity field u^μ then (4.13) can be recognized as the continuity equation and (4.14) as the non-relativistic

Navier-Stokes equation that we have reviewed in Appendix B (see 5.2). Transport coefficients of the fluid can be read off from any of these equations and we get

$$\text{Pressure}: \quad p = \frac{\kappa}{8\pi} \tag{4.15}$$

$$\text{Shear viscosity}: \quad \eta = \frac{1}{16\pi} \tag{4.16}$$

$$\text{Bulk viscosity}: \quad \xi = \frac{(3-d)}{8\pi(d-2)}. \tag{4.17}$$

It should be noted that the bulk viscosity has a negative value for $d \geq 4$. For normal fluids the negative bulk viscosity would mean that the fluid is unstable to perturbations and would never relax to the equilibrium configuration once it suffers even a slight deviation. The reason for the horizon to have this feature can be traced back to its teleological nature. The future boundary conditions have to be imposed in order to have the horizon relax to a final stationary state.

We will obtain the results quoted in this section using the membrane paradigm, to which we turn to now. This is based upon the present authors original work (Jacobson *et al.*, 2011).

4.2 Geometric setup

In this section we elaborate on the geometric setup necessary to construct the membrane paradigm. The interested reader can find a detailed discussion in the monograph by Thorne *et al.* (1986).

The event horizon, H, of the black hole in D spacetime dimensions is a (D–1)-dimensional null hypersurface generated by the null geodesics l^a. We choose a non-affine parameterization such that the null generators satisfy the geodesic equation $l^a \nabla_a l^b = \kappa\, l^b$, where κ is a constant non-affine coefficient. For a stationary spacetime, l^a coincides with the null limit of the timelike Killing vector and κ can then be interpreted as the surface gravity of the horizon.

Next we introduce a timelike surface positioned just outside H which is called the stretched horizon and denoted by $\mathcal{H}_s$. One can think of $\mathcal{H}_s$ as the world-tube of a family of fiducial observers (FIDOs) just outside the black-hole horizon. The four-velocity of these fiducial observers is denoted by u^a. Just as H is generated by the null congruence l^a, $\mathcal{H}_s$ is generated by the timelike congruence u^a. The unit normal to $\mathcal{H}_s$ is denoted by n^a and is taken to point away from the horizon into the bulk. We relate the points on $\mathcal{H}_s$ and H by ingoing light rays parametrized by an affine parameter γ, such that $\gamma = 0$ is the position of the horizon and $(\partial/\partial\gamma)^a\, l_a = -1$ on the horizon. Then, in the limit $\gamma \to 0$, when the stretched horizon approaches the true one, $u^a \to \delta^{-1} l^a$ and $n^a \to \delta^{-1} l^a$, where $\delta = \sqrt{2\kappa\gamma}$ (Thorne *et al.*, 1986).

The induced metric h_{ab} on $\mathcal{H}_s$ can be expressed in terms of the spacetime metric g_{ab} and the covariant normal n_a as $h_{ab} = g_{ab} - n_a n_b$. Similarly, the induced metric γ_{ab} on the $(D{-}2)$-dimensional spacelike cross-section of $\mathcal{H}_s$ orthogonal to u_a is given by $\gamma_{ab} = h_{ab} + u_a u_b$. The extrinsic curvature of $\mathcal{H}_s$ is defined as $K^a_b = h^c_b \nabla_c n^a$. Using the limiting behavior of n^a and u^a it is easy to verify that in the $\delta \to 0$ limit various components of the extrinsic curvature behave as follows (Thorne *et al.*, 1986):

$$\begin{aligned} \text{As } \delta \to 0 \quad : \quad & K^u_u = K^a_b u_a u^b = g \sim \frac{\kappa}{\delta}, \\ & K^u_A = K^a_b u_a \gamma^b_B = 0, \\ & K^A_B = K^a_b \gamma^A_a \gamma^b_B \sim \frac{k^A_B}{\delta}, \\ & K = K_{ab} g^{ab} \sim \frac{(\theta + \kappa)}{\delta}, \end{aligned} \tag{4.18}$$

where θ is the expansion scalar of l^a and k^A_B is the extrinsic curvature of the $(D-2)$-dimensional space-like cross-section[1] of the true horizon H. Note that a priori the projection of the extrinsic curvature of $\mathcal{H}_s$ on the cross-section of $\mathcal{H}_s$ has nothing to do with the extrinsic

[1] A, B denote the indices on the cross-section of the horizon.

curvature of the cross-section (orthogonal to u^a) *as embedded in* $\mathcal{H}_s$, i.e., there is, in general, no relationship between the pull-back of $\nabla_a n_b$ and $\nabla_a u_b$ to the cross-section of the stretched horizon. However, in the null limit ($\delta \to 0$) both u^a and n^a map to the same null vector l^a and we have $K^A_B \to \delta^{-1} k^A_B$. Finally, we decompose k_{AB} into its trace-free and trace-full part as

$$k_{AB} = \sigma_{AB} + \frac{1}{(D-2)}\theta\,\gamma_{AB}, \tag{4.19}$$

where σ_{AB} is the shear of l^a. It is clear from equation (4.18) that in the null limit, various components of the extrinsic curvature diverge and we need to regularize them by multiplying by a factor of δ. The physical reason behind such infinities is that, as the stretched horizon approaches the true one, the fiducial observers experience more and more gravitational blue shift; on the true horizon, the amount of blue shift is infinite.

This completes the description of the geometric setup of the membrane paradigm. Before reviewing the derivation of the black-hole membrane paradigm in standard Einstein gravity we pause to study a simple example of the stretched horizon and how is it that both u^a and n^a map to l^a in the limit that the stretched horizon goes to the true horizon.

4.3 The stretched horizon

We now study a simple example of the stretched horizon in 1+1 dimensions. A Rindler observer in the Rindler spacetime provides a good example of the FIDO. We will see explicitly how the FIDO's velocity vector and the vector normal to her world-line both approach the null generator of the Rindler horizon in an appropriate limit. We follow the discussion of Rindler spacetime given by Wald (1995).

4.3.1 Rindler space

Consider a 2-dimensional spacetime whose metric in Cartesian coordinates is given by

$$\mathrm{d}s^2 = -x^2\mathrm{d}t^2 + \mathrm{d}x^2, \tag{4.20}$$

where $0 < t < \infty$ and $0 < x < \infty$. Null geodesics are obtained by putting $ds^2 = 0$, which gives

$$t = \pm \ln x. \tag{4.21}$$

Now define the ingoing & outgoing null coordinates as

$$\begin{aligned} u &= t - \ln x, \\ v &= t + \ln x, \end{aligned} \tag{4.22}$$

where u is the parameter along the ingoing null geodesic $(-\infty < u < \infty)$ and v is the parameter along the outgoing null geodesic $(-\infty < v < \infty)$. In terms of these coordinates the metric becomes

$$\mathrm{d}s^2 = -e^{(v-u)}\mathrm{d}v\mathrm{d}u. \tag{4.23}$$

The vector field tangent to the outgoing null geodesic is $(\frac{\partial}{\partial v})^a$ and the corresponding integral curves are $u = \mathit{constant}$. The vector field tangent to the ingoing null geodesic is $(\frac{\partial}{\partial u})^a$ and the corresponding integral curves are $v = \mathit{constant}$. These vector fields are not affinely parametrized. The affinely parametrized null geodesics are

$$k^a = \mathrm{e}^{-v}\left(\frac{\partial}{\partial v}\right)^a = \left(\frac{\partial}{\partial \lambda_{\mathrm{out}}}\right)^a,$$
$$n^a = \mathrm{e}^{u}\left(\frac{\partial}{\partial u}\right)^a = \left(\frac{\partial}{\partial \lambda_{\mathrm{in}}}\right)^a, \tag{4.24}$$

where $\lambda_{\mathrm{out}} = \mathrm{e}^v$ and $\lambda_{\mathrm{in}} = -\mathrm{e}^{-u}$ are the affine parameters of the outgoing and ingoing null geodesics, respectively. These affine parameters cover only half of the real line, which shows that the spacetime is geodesically incomplete. Now, define the new coordinates $U = -\mathrm{e}^{-u}$ and $V = \mathrm{e}^v$ so that U and V become the affine parameters of the ingoing and outgoing null geodesics, respectively. We can then analytically extend the range of U and V from $-\infty$ to ∞, thus making the resulting spacetime geodesically complete. The metric in the new coordinates is

$$\mathrm{d}s^2 = -\mathrm{d}U\mathrm{d}V. \tag{4.25}$$

Now define the new Cartesian coordinates

$$T = \frac{(V+U)}{2},$$
$$X = \frac{(V-U)}{2}, \tag{4.26}$$

in terms of which the metric is recognized to be a Minkowski space,

$$\mathrm{d}s^2 = -\mathrm{d}T^2 + \mathrm{d}X^2, \tag{4.27}$$

and both the coordinates T and X range from $-\infty$ to ∞. The relation between the original coordinates x and t and the new coordinates X and T is

$$\begin{aligned} x &= \sqrt{X^2 - T^2}, \\ t &= \tanh^{-1}\left(\frac{T}{X}\right). \end{aligned} \tag{4.28}$$

We see that the old coordinates cover only a part of Minkowski spacetime: the right wedge in the right quadrant (see Figure 4.1).

4.3.2 Rindler observer

In order to understand better the meaning of the Rindler spacetime as a wedge in Minkowski spacetime let us do a simple exercise. We want to know the trajectory of an observer undergoing a uniform acceleration in the Minkowski spacetime. Let us parametrize the worldline of the observer with a parameter λ which is a function of the proper time τ along the trajectory of the observer

$$x^\mu(\lambda) = (T(\lambda), X(\lambda)), \tag{4.29}$$

where $\lambda \equiv \lambda(\tau)$. The four-velocity of the observer is given by

$$\begin{aligned} u^\mu =& \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \\ =& \left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}T}{\mathrm{d}\lambda}, \frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\frac{\mathrm{d}X}{\mathrm{d}\lambda}\right). \end{aligned} \tag{4.30}$$

Normalizing $u^\mu u_\mu = -1$ we get

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^2 \left[-\left(\frac{\mathrm{d}T}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}X}{\mathrm{d}\lambda}\right)^2\right] = -1. \tag{4.31}$$

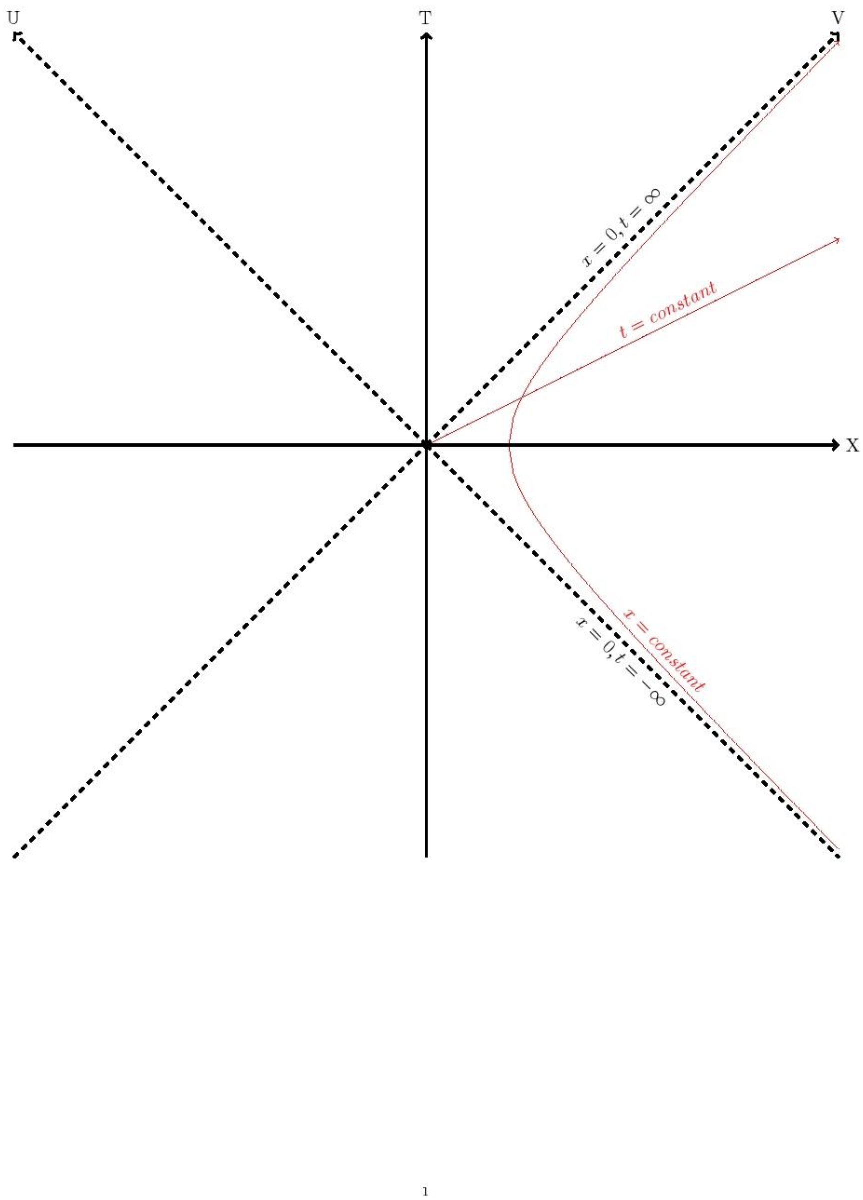

Figure 4.1. Rindler space is the right wedge of Minkowski Space. The observer following the integral curve $x = 1/a$ is undergoing a constant acceleration a.

Therefore we can write the tangent vector in terms of the hyperbolic functions as

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}\lambda} &= \frac{\cosh\lambda}{(\mathrm{d}\lambda/\mathrm{d}\tau)},\\ \frac{\mathrm{d}X}{\mathrm{d}\lambda} &= \frac{\sinh\lambda}{(\mathrm{d}\lambda/\mathrm{d}\tau)},\end{aligned} \tag{4.32}$$

in terms of which the four-velocity of the observer becomes $u^\mu = (\cosh\lambda, \sinh\lambda)$. The acceleration of the observer is given by

$$\begin{aligned}a^\mu &= \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}\\ &= \left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\sinh\lambda, \frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\cosh\lambda\right).\end{aligned} \tag{4.33}$$

Requiring that the magnitude of acceleration be constant, i.e., $\sqrt{a^\mu a_\mu} = \text{constant} \equiv a$ then gives $\mathrm{d}\lambda/\mathrm{d}\tau = a$. Therefore the parameter λ along the trajectory of a uniformly accelerated observer is given by $\lambda = a\tau$ and the trajectory can be described by

$$\begin{aligned}x^\mu(\tau) &= \frac{1}{a}\left(\sinh a\tau, \cosh a\tau\right),\\ u^\mu(\tau) &= \left(\cosh a\tau, \sinh a\tau\right),\\ a^\mu(\tau) &= \left(a\sinh a\tau, a\cosh a\tau\right).\end{aligned} \tag{4.34}$$

From the parametrization of the trajectory, $T(\tau) = (1/a)\sinh a\tau$, $X(\tau) = (1/a)\cosh a\tau$, we read that $X^2 - T^2 = 1/a$. Then from the relation between the Rindler coordinates and the Cartesian coordinates we get that the observer undergoing a constant acceleration a is at a constant Rindler coordinate $x = 1/a$. This gives physical meaning to the Rindler spacetime. The Rindler coordinate x is the worldline of an observer in flat space undergoing a constant acceleration and the Rindler spacetime is a right wedge of Minkowski spacetime bounded by the two null lines. The null line $U = 0$ acts as the future causal horizon for the Rindler observer and the null line $V = 0$ acts as the past causal horizon.

4.3.3 The singular limit

Consider a congruence of the Rindler observers in Minkowski spacetime who are undergoing a constant acceleration, a.[2] These Rindler observers are then the FIDOs we alluded to above and their worldtube is the stretched horizon $\mathcal{H}_s$. The normalized vector normal to the worldtube of FIDOs, i.e., normal to $\mathcal{H}_s$ is given by

$$n^\mu = (\sinh a\tau, \cosh a\tau) \,. \tag{4.35}$$

Now consider the timelike Killing vector field in the Rindler space, $\xi^\mu = (\partial/\partial t)^\mu$. In the (T, X) coordinates it is given by

$$\begin{aligned} \xi^\mu &= \frac{\mathrm{d}T}{\mathrm{d}t}\left(\frac{\partial}{\partial T}\right)^\mu + \frac{\mathrm{d}X}{\mathrm{d}t}\left(\frac{\partial}{\partial X}\right)^\mu \\ &= X\left(\frac{\partial}{\partial T}\right)^\mu + T\left(\frac{\partial}{\partial X}\right)^\mu , \end{aligned} \tag{4.36}$$

which shows that the timelike Killing vector field of the Rindler space is the boost Killing vector field of Minkowski space. In the second step of this equation we have used the relation between different coordinates

$$\begin{aligned} X &= \frac{V+U}{2} = \frac{\mathrm{e}^v - \mathrm{e}^{-u}}{2} = \frac{\mathrm{e}^{t+\ln x} - \mathrm{e}^{-t+\ln x}}{2} = x\frac{\mathrm{e}^t - \mathrm{e}^{-t}}{2} \\ T &= \frac{V-U}{2} = \frac{\mathrm{e}^v + \mathrm{e}^{-u}}{2} = \frac{\mathrm{e}^{t+\ln x} + \mathrm{e}^{-t+\ln x}}{2} = x\frac{\mathrm{e}^t + \mathrm{e}^{-t}}{2} . \end{aligned} \tag{4.37}$$

At any point on the trajectory of a particular FIDO parametrized by its proper time τ and

[2] In 2-dimensional spacetime this congruence consists of a tangent vector to one curve. But one can easily imagine a 4-dimensional spacetime with $\mathbf{S}^2$ as the other two spatial dimensions in which case it really is a congruence, say constant r surfaces in Schwarzchild spacetime.

acceleration a, ξ can be written as

$$\xi^\mu = \frac{1}{a}\left(\sinh a\tau, \cosh a\tau\right). \tag{4.38}$$

Now in the $a \to \infty$ limit the stretched horizon $\mathcal{H}_s$ approaches the causal horizon H. Since $\sinh a\tau \to \cosh a\tau$ as $a \to \infty$, it is clear from the equations (4.34),(4.35) and (4.38) that

$$\lim_{a\to\infty} \frac{u^\mu}{a} = \lim_{a\to\infty} \frac{n^\mu}{a} = \lim_{a\to\infty} \xi^\mu, \tag{4.39}$$

which is precisely the condition that we require in Section 4.2 with $\delta = \frac{1}{a}$. Therefore in the limit that the stretched horizon $\mathcal{H}_s$ reaches the true horizon H, both the tangent and normal to $\mathcal{H}_s$ approach the non-affinely parametrized null geodesic generator of the true horizon H.

4.4 The membrane paradigm in Einstein gravity

In this section we construct the membrane paradigm in Einstein gravity in four spacetime dimensions. Our construction will closely follow the action approach of Parikh & Wilczek (1998). Our purpose is to fix the notation and emphasize the points in the construction which will be of importance for the corresponding construction in the EGB gravity. We will highlight the steps which will be different in the EGB case and where one has to make assumptions. For the construction of the membrane stress tensor we will work exclusively with differential forms and only in the end do we go back to the metric formalism.

In the rest of this chapter, unless otherwise explicitly written, we will work with units such that $16\pi G = 1$. The small Roman letters on the differential forms are the Lorentz indices while in the spacetime tensors we will not differentiate between the Lorentz and the world indices, this being understood that one can always use the vierbeins to convert between the Lorentz and the world indices.

In the Cartan formalism, the Einstein-Hilbert Lagrangian is written in terms of the vector-valued vierbein one-form e^a, related to the metric by $g = \eta_{ab} e^a \otimes e^b$, and the Lorentz

Lie-algebra valued torsion-free connection one-form $\omega^{ab}(e)$ defined by the equation $\mathcal{D}e^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b = 0$. The action is then

$$S_{\mathrm{EH}} = \frac{1}{2!}\int_M \Omega^{ab} \wedge e^c \wedge e^d \epsilon_{abcd} \pm \frac{1}{2!}\int_{\partial M} \theta^{ab} \wedge e^c \wedge e^d \epsilon_{abcd}, \tag{4.40}$$

where Ω^{ab} is the curvature of the torsion-free connection given by $\Omega^{ab} = \mathrm{d}\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}$, and θ^{ab} is the second fundamental form on the boundary ∂M of M (Myers, 1987).[3] It is related to the extrinsic curvature by

$$\theta^{ab} = (n.n)(n^a K^b_c - n^b K^a_c)e^c. \tag{4.41}$$

In our case, the boundary ∂M of M consists of the outer boundary at spatial infinity and the inner boundary at the stretched horizon, $\mathcal{H}_s$. The variation of the action with respect to e^a can be separated into the contribution from $\omega(e)$ and the rest. The variation with respect to ω of the bulk part of the action yields a total derivative which, after integration by parts, gives a contribution identical to the negative of the variation of the boundary part. Thus the variation of the action with respect to ω vanishes identically. In the absence of the inner boundary, variation of S_{EH} with respect to the vierbein, e^a, under the Dirichlet boundary condition (holding the vierbein fixed on the outer boundary) yields the equation of motion for the vierbein. But when the inner boundary is present, there is no natural way to fix the vierbein there. The physical reason for this is simply that the horizon acts as a boundary only for a class of observers, and surely the metric is not fixed there. However, because it is a causal boundary the dynamics outside the horizon is not affected by what happens inside. Hence, for the consideration of the outside dynamics one can imagine some fictitious matter living on the stretched horizon whose contribution to the variation of the action cancels that

[3] Our sign convention is such that the plus (minus) signs apply to a timelike (spacelike) boundary. It should be noted that this is different from the convention in Myers (1987) because their definition of extrinsic curvature differs from ours by a negative sign.

of the inner boundary. This is the basic idea of the construction of stress-tensor *à la* Brown & York (1993). Since we will be interested in the boundary term on the stretched horizon which is a timelike hypersurface, from now on we will put $n \cdot n = 1$.

Hence, variation of the action with respect to the vierbein gives

$$\delta S_{\text{EH}} = \int_M \Omega^{ab} \wedge e^c \wedge \delta e^d \epsilon_{abcd} + \int_{\partial M} \theta^{ab} \wedge e^c \wedge \delta e^d \epsilon_{abcd}. \tag{4.42}$$

The bulk term just gives the equation of motion. Using the Dirichlet boundary condition on the outer boundary and the expression of θ given in equation (4.41), the surviving contribution of the variation of the total action $S_{\text{total}} = S_{\text{EH}} + S_{\text{matter}}$ comes solely from the inner boundary and is given by

$$\delta S_{\text{total}} = \int_{\mathcal{H}_s} K^b_m e^m \wedge e^c \wedge \delta e^d \epsilon_{bcd}, \tag{4.43}$$

where $\epsilon_{bcd} = -n^a \epsilon_{abcd}$. This surviving contribution can be interpreted as due to a fictitious matter source residing on $\mathcal{H}_s$ whose stress tensor is given by

$$t^{ab} = 2(Kh^{ab} - K^{ab}). \tag{4.44}$$

In terms of t_{ab} the on-shell variation of S_{EH} becomes

$$\delta S_{\text{EH}} = -\frac{1}{2} \int_{\mathcal{H}_s} t_{ab} \delta h^{ab} + \frac{1}{2} \int_M T_{ab} \delta g^{ab}, \tag{4.45}$$

where $T_{ab} = (-2/\sqrt{-g})(\delta S_{matter}/\delta g^{ab})$ is the external matter's stress energy tensor. Now consider the variation δ_ξ induced by a vector field ξ which is arbitrary in the bulk and behaves in a prescribed fashion on the boundary. We take ξ to be such that it is tangential to the inner-boundary and vanishes on all the other boundaries. Then the diffeomorphism invariance of the theory ensures that

$$\delta_\xi S_{\rm EH} = -\frac{1}{2}\int_{\mathcal{H}_s} t_{ab}\delta_\xi h^{ab} + \frac{1}{2}\int_M T_{ab}\delta_\xi g^{ab} = 0,$$
$$\Rightarrow -\int_{\mathcal{H}_s} t_{ab}D^a\xi^b + \int_M T_{ab}\nabla^a\xi^b = 0, \tag{4.46}$$

where D^a is the covariant derivative of the induced metric on the inner boundary $\mathcal{H}_s$, and ∇^a is the covariant derivative of the spacetime metric. Using integration by parts and the afore-mentioned conditions on ξ^a, equation (4.46) gives

$$D^a t_{ab} = -T_{ac}n^a h^c{}_b, \tag{4.47}$$

where the negative sign on the right-hand side arises because we have chosen n^a as pointing away from the stretched horizon into the bulk. The right-hand side of this equation can be interpreted as the flux of external matter crossing the horizon from the bulk. Then equation (4.47) has the interpretation of the continuity equation satisified by the fictitious matter living on the stretched horizon.

At this stage, we would like to point out a difference between our approach and that of Parikh & Wilczek (1998). In Parikh & Wilczek (1998), the Gibbons-Hawking boundary term is considered to be only on the outer boundary. Therefore, one needs to show that a certain contribution containing the derivatives of the variation of the metric on the stretched horizon vanishes in the limit as the stretched horizon reaches the true horizon. In our approach there is no such requirement because we have the boundary term on the entire boundary, which includes the stretched horizon. Since both the approaches finally yield the same horizon stress tensor, there is no difference in the physics in our approach and that of Parikh & Wilczek (1998). However, an added advantage of our approach is that it also works beyond general relativity and, in particular, in any Lovelock gravity. If one does not include the Gibbons-Hawking term for Lovelock gravity, one needs to show that all the extra terms in the variation of the action vanish when the limit to the true horizon is taken. We have simply avoided this difficulty by adding the boundary term on the stretched horizon as well. Consequently, although our approach and that of Parikh & Wilczek (1998) ultimately give the same result for the horizon stress tensor, we believe the inclusion of the Gibbons-Hawking term makes the calculations easier. Also, it is worth pointing out that if we were to put the membrane at some finite distance from the horizon then we need the Hawking-Gibbons term in order to derive the stress tensor and the approach of Parikh & Wilczek (1998) just won't work. We believe that our approach is conceptually transparent and computationally simpler than the one by Parikh & Wilczek (1998).

We have derived the form of the membrane stress tensor for the particular case of $D = 4$ spacetime dimensions, but it is easy to check that the form of the stress tensor in equation (4.44) remains unchanged for a general D-dimensional spacetime. Then the components of the membrane stress tensor t_{ab}, evaluated on the stretched horizon in the basis (u^a, x^A) are given by

$$\begin{aligned} t_{uu} &= \rho = -2\theta_s, \\ t^{AB} &= 2\left(-\sigma_s^{AB} + \frac{(D-3)}{(D-2)}\theta_s\gamma^{AB} + g\,\gamma^{AB}\right), \end{aligned} \tag{4.48}$$

where $g = \kappa/\delta$. In deriving this expression, we have replaced K^{AB} by the expression: $\sigma_s^{AB} + \frac{\theta_s}{(D-2)}\gamma^{AB}$, where θ_s is the expansion and σ_s^{AB} is the shear of the congruence generated by the time-like vector field u^a on $\mathcal{H}_s$. As pointed out by Damour (1978) and Thorne *et al.* (1986), this replacement is valid only up to $\mathcal{O}(\delta)$. Since we are ultimately interested in the limit $\delta \to 0$, any $\mathcal{O}(\delta)$ error does not contribute. Although this is certainly true for general relativity, in EGB gravity such $\mathcal{O}(\delta)$ terms will play an important role and they actually contribute in the limit that the stretched horizon becomes the true horizon.

The particular form of the components of the membrane stress tensor in equation (4.48) has an interpretation: the fictitious matter on the stretched horizon can be regarded as a

$(D-2)$-dimensional viscous fluid (Landau & Lifshitz, 1959) with the energy density and transport coefficients given by

$$\begin{aligned} \text{Energy density}: \quad & \rho_s &= -2\theta_s, \\ \text{Pressure}: \quad & p_s &= 2g, \\ \text{Shear Viscosity}: \quad & \eta_s &= 1, \\ \text{Bulk Viscosity}: \quad & \zeta_s &= -2\left(\frac{D-3}{D-2}\right). \end{aligned}$$

Hence the entire t_{ab} on the stretched horizon can be expressed as

$$t_{ab} = \rho_s u_a u_b + \gamma_a^A \gamma_b^B \left(p_s \gamma_{AB} - 2\eta_s \sigma_{s\,AB} - \zeta_s \theta_s \gamma_{AB}\right). \tag{4.49}$$

Substituting these quantities in the conservation equation (4.47) then gives the evolution equation for the energy density,

$$\mathcal{L}_u \rho_s + \rho_s \theta_s = -p_s \theta_s + \zeta_s \theta_s^2 + 2\eta_s \sigma_s^2 + T_{ab} n^a u^b. \tag{4.50}$$

The evolution equation matches exactly with the energy conservation equation of a viscous fluid. We stress the fact that equation (4.50) is a direct consequence of the conservation equation (4.47) and the form of the stress tensor in equation (4.49). In fact, in any theory of gravity once we can express the stress tensor of the fictitious matter obeying the conservation equation on the stretched horizon in a form analogous to the one in equation (4.49), the conservation equation will automatically imply an evolution equation of the form (4.50). Notice that the conservation equation is only valid on shell, which means that the equations of motion of the theory have to be satisfied for it to hold.

From the analysis of Section 4.2 it is evident that as the stretched horizon approaches the true horizon the membrane stress tensor in equation (4.48) diverges as δ^{-1} due to the large blueshift near the horizon. This divergence is regulated by simply multiplying it by δ. This limiting and regularization procedure then yields a stress tensor of a fluid living on the cross sections of the horizon itself in terms of the quantities intrinsic to the horizon. This stress tensor is

$$t^{(H)}_{AB} = \left(p\gamma_{AB} - 2\eta\sigma_{AB} - \zeta\theta\gamma_{AB}\right), \tag{4.51}$$

and the energy density and the transport coefficients become

$$\begin{aligned} \text{Energy density :} \quad & \rho && = -2\theta, \\ \text{Pressure :} \quad & p && = 2\kappa, \\ \text{Shear Viscosity :} \quad & \eta && = 1, \\ \text{Bulk Viscosity :} \quad & \zeta && = -2\left(\frac{D-3}{D-2}\right), \end{aligned}$$

where θ is the expansion of the null generator of the true horizon as discussed in section 4.2. Similarly, the regularization of the evolution equation (4.50) gives

$$\mathcal{L}_l\rho + \rho\theta = -p\theta + \zeta\theta^2 + 2\eta\sigma^2 + T_{ab}l^a l^b, \tag{4.52}$$

which is just the Raychaudhuri equation of the null congruence generating the horizon. It should be noted that our approach for deriving the evolution equation is different from the one of Parikh & Wilczek (1998). In principle, one can just take the Lie derivative of the energy density with respect to the generator of the horizon to obtain the Raychaudhuri equation as in Parikh & Wilczek (1998) and then use the Einstein equation to replace the curvature dependence in terms of the matter energy-momentum tensor. We have followed an indirect approach in which we derive the evolution equation via the continuity equation (4.47) which is valid on shell, i.e., the equation of motion has already been used in its derivation. This approach is particularly useful when the equations of motion are complicated as in the EGB gravity and it becomes difficult to replace the curvature dependence in terms of the matter energy-momentum tensor.

For general relativity, the Bekenstein-Hawking entropy of the horizon is $4\pi\mathcal{A}$, where $\mathcal{A}$ is the area of the horizon, hence the ratio of the shear viscosity to the entropy density is,

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{4.53}$$

Note that this is a dimensionless constant independent of the parameters of the horizon. Comparing this with the KSS bound, $\eta/s \geq 1/4\pi$, we see that the bound is saturated in general relativity. Interestingly, for any gravity theory with a Lagrangian depending on the Ricci scalar only, the value of this ratio is the same as that in general relativity and therefore the KSS bound is saturated in these theories (Chatterjee *et al.*, 2012).

Another important fact is that the bulk viscosity associated with the horizon is negative. Clearly, the fluid corresponding to the horizon is not an ordinary fluid and, as explained in Thorne *et al.* (1986), the negative bulk viscosity is related to the teleological nature of the event horizon.

4.5 The membrane paradigm in Einstein-Gauss-Bonnet gravity

EGB gravity is a natural generalization of general relativity which includes terms of higher order in the curvature, but in just such a way that the equation of motion remains second order in time.

The action of the theory is given by

$$S_{\text{total}} = S_{\text{EH}} + \alpha' S_{\text{GB}} + S_{\text{matter}} \tag{4.54}$$

where S_{EH} and S_{matter} are the contribution of the Einstein-Hilbert and the matter, respectively, while S_{GB} is the Gauss-Bonnet addition to the action. From the analysis in Section

4.4 we know how to take care of the $S_{\rm EH}$ term. So, we can exclusively work with the GB term now. The GB contribution to the total action, in $D = 5$, is given by[4] (Myers, 1987)

$$S_{GB} \; = \int_M \Omega^{ab}\wedge\Omega^{cd}\wedge e^f\epsilon_{abcdf} + 2\int_{\partial M}\theta^{ab}\wedge(\Omega - \tfrac{2}{3}\theta\wedge\theta)^{cd}\wedge e^f\epsilon_{abcdf}. \tag{4.55}$$

As in the case of general relativity discussed in Section 4.4 the variation of $S_{\rm GB}$ with respect to the connection $\omega^a{}_b$ vanishes identically. The variation with respect to vierbein e^a under the Dirichlet boundary condition yields the equation of motion. Using the torsion-free condition $\mathcal{D}e^a = 0$ on the connection, this equation can be shown to be the same as that obtained in the metric formalism. In the presence of the inner boundary at the stretched horizon, $\mathcal{H}_s$, we need the variation of the boundary part of the S_{GB} due to the variation of the vierbein on the inner boundary. This is obtained from equation (4.55), which after variation with respect to e^a and *then* using the relations $\theta^{ab} = (n^aK^b_c - n^bK^a_c)e^c$ and $\Omega^{ab} = \frac{1}{2}R^{ab}{}_{mn}e^m\wedge e^n$, gives

$$\delta S_{\rm GB}|^{\rm bndy} = 4\int_{\partial M} K^s_a\left(\frac{1}{2}h^c_p\,h^d_q\,h^r_m\,h^s_n\,R^{pq}{}_{rs} + \frac{2}{3}K^c_m\,K^d_n\right)4!\,\delta^{[amnb]}_{s\,c\,d\,f}\,\delta\,e^f_b. \tag{4.56}$$

The projections of the spacetime Riemann tensor can be written in terms of the Riemann tensor intrinsic to $\mathcal{H}_s$ and the extrinsic curvature of $\mathcal{H}_s$ using the Gauss-Codazzi equation

$$h^c_ph^d_qh^r_ah^s_bR^{pq}{}_{rs} = \hat{R}^{cd}{}_{ab} - K^c_aK^d_b + K^c_bK^d_a, \tag{4.57}$$

where $\hat{R}^{cd}{}_{ab}$ is the Riemann tensor intrinsic to $\mathcal{H}_s$. Thus the variation of the boundary term

[4]For general $D \geq 4$, GB action is given by $S_{\rm GB} = \int_M \mathcal{L}_m \pm \int_{\partial M}\mathcal{Q}_m$, where m is the greatest integer $\leq D/2$, $\mathcal{L}_m = \Omega^{a_1b_1}\wedge\ldots\wedge\Omega^{a_mb_m}\wedge\Sigma_{a_1b_1\ldots a_mb_m}$ and $\mathcal{Q}_m = m\int_0^1 {\rm ds}\;\theta^{a_1b_1}\wedge\Omega_s^{a_2b_2}\wedge\ldots\wedge\Omega_s^{a_mb_m}\wedge\Sigma_{a_1b_1\ldots a_mb_m}$. Here, $\Sigma_{a_1b_1\ldots a_mb_m} = \frac{1}{(D-2m)!}\epsilon_{a_1b_1\ldots a_mb_mc_1\ldots c_{D-2m}}e^{c_1}\wedge\cdots\wedge e^{c_{D-2m}}$ and Ω_s is the curvature of the connection $\omega_s = \omega - s\theta$, $\Omega_s = d\omega_s + \omega_s\wedge\omega_s$. The sign of the surface term depends upon the timelike or spacelike nature of the boundary. The surface term was first constructed in Myers (1987) and more details can be found there. The GB coupling in the total action is still denoted by α'.

can be written in terms of quantities intrinsic to the boundary,

$$\delta S_{\rm GB}|^{\rm bndy} = 4\int_{\partial M} K^s_a \left(\frac{1}{2}\hat{R}^{cd}{}_{mn} - \frac{1}{3}K^c_m K^d_n\right) 4!\, \delta^{[amnb]}_{s\,c\,d\,f}\delta e^f_b. \tag{4.58}$$

This can be evaluated to be

$$\delta S_{\rm GB}|^{\rm bndy} = 4\int_{\partial M} \left(2K_{mn}\hat{P}^{amn}{}_b - J^a_b + \frac{1}{3}J\delta^a_b\right)\delta e^b_a, \tag{4.59}$$

where we have defined,

$$\begin{aligned}
\hat{P}^{amn}{}_b &= \hat{R}^{amn}{}_b + 2\hat{R}^{n[m}h^{a]}_b + 2\hat{R}^{[a}_b h^{m]n} + \hat{R}h^{n[a}h^{m]}_b, & (4.60)\\
J^a_b &= K^2K^a_b - K^{cd}K_{cd}K^a_b + 2K^a_cK^c_dK^d_b - 2KK^a_cK^c_b, & (4.61)\\
J &= K^3 - 3KK^{cd}K_{cd} + 2K^a_bK^b_cK^c_a. & (4.62)
\end{aligned}$$

As in the case of general relativity, we can interpret the variation $\delta S_{\rm total}$ as due to a fictitious matter living on the membrane whose stress-energy tensor is given by the coefficient of δe^a_b. Thus from the equation (4.59) we can read off the membrane stress tensor due to the GB term (now including the GB coupling α' and a negative sign arising from the fact that we are defining the stress tensor with covariant indices) and we get

$$t_{ab}|_{\rm (GB)} = -4\alpha'\left(2K_{mn}\hat{P}_a{}^{mn}{}_b - J_{ab} + \frac{1}{3}Jh_{ab}\right). \tag{4.63}$$

Adding to this the contribution coming from the Einstein-Hilbert action we get the total stress tensor for the membrane,

$$t_{ab} = 2\left(Kh_{ab} - K_{ab}\right) - 4\alpha'\left(2K_{mn}\hat{P}_a{}^{mn}{}_b - J_{ab} + \frac{1}{3}Jh_{ab}\right). \tag{4.64}$$

Although we have derived the membrane stress tensor for the particular case of $D = 5$, the result can be easily generalized to arbitrary dimensions and the form in equation (4.64) remains unchanged. By the same arguments as discussed in the case of general relativity, this t_{ab} also satisfies the continuity equation (4.47). This can also be verified explicitly by taking the divergence of t_{ab} and using the appropriate projections of the equation of motion (Davis, 2003).

Note that a crucial difference between the membrane stress tensors for general relativity and EGB gravity is that in the former the stress tensor is linear in the extrinsic curvature while in the latter the stress tensor contains terms cubic in the extrinsic curvature of the stretched horizon. Since, as we take the limit to the true horizon, the extrinsic curvature of the stretched horizon diverges as δ^{-1}, one would expect higher-order divergences in the case of EGB gravity coming from the contribution of the GB term to the stress tensor. The regularization procedure used to tame the divergence coming from the Einstein-Hilbert term involves multiplication with δ, which does not tame the cubic order divergence coming from the GB term. Clearly one needs either a new regularization procedure or some well-motivated prescription which justifies neglect of the terms that lead to higher-order divergences in the limit when the stretched horizon approaches the true one. In this paper, we will adopt the latter approach.

We will restrict attention to background geometries which are static so that the expansion and the shear of the null generators of the true horizon are zero. Next, we will consider some arbitrary perturbation of this background which may arise due to the flux of matter flowing into the horizon. As a result, the horizon becomes time dependent and acquires expansion and shear. We will assume this perturbation of the background geometry to be small so that we can work in the linear order of perturbation and ignore all the higher-order terms. Essentially, our approximation mimics a slow-physical-process version of the dynamics of the horizon (Jacobson & Parentani, 2003). This essentially means that we are restricting ourselves to the terms proportional to the first derivative of the observer's four velocity u^a, which plays the role of the velocity field for the fluid. We will discard all higher-order derivatives of the velocity except the linear one so that we can write the membrane stress tensor as a Newtonian viscous fluid. In this limited setting we will see that the only divergence that survives is of $\mathcal{O}(\delta^{-1})$ which can be regularized in the same fashion as in the case of general relativity. Therefore, when we encounter a product of two quantities X and Y, we will always express such a product as

$$XY \approx \mathring{X}\,\mathring{Y} + \mathring{X}\,\delta Y + \mathring{Y}\,\delta X, \tag{4.65}$$

where $\mathring{X}$ is the value of the quantity X evaluated on the static background and δX is the perturbed value of X linear in the perturbation.

In order to implement this scheme, we first define a quantity Q_{ab}, whose importance will be apparent later, as

$$Q_{ab} = KK_{ab} - K_{ac}K^{c}_{b}, \tag{4.66}$$

in terms of which we write

$$J_{ab} = K_{ab}Q - 2K_{ac}Q^{c}_{b}, \tag{4.67}$$

where Q is the trace of Q_{ab}.

Now we observe the following facts. First, the components of the extrinsic curvature of $\mathcal{H}_s$ in the backgrounds that we are interested in are $\mathcal{O}(\delta)$. In particular, for the static spacetimes one can choose the cross-sections of the $\mathcal{H}_s$ such that the pull-back of the extrinsic curvature to these cross-sections is $K_{AB} = \frac{\delta}{r}\gamma_{AB}$. Secondly, for these backgrounds Q_{ab} and Q defined as above are finite on $\mathcal{H}_s$ and remain finite in the limit as $\mathcal{H}_s$ reaches the true horizon. In fact, Q_{AB} for the background, in the limit that $\mathcal{H}_s$ approaches the true horizon, is simply $Q_{AB} = \frac{\kappa}{r}\gamma_{AB}$. Finally, the most singular terms of the linearized Q_{AB} and Q are given by

$$\delta Q_{AB} \sim \frac{1}{\delta^2}\kappa\left(\sigma_{AB} + \frac{\theta}{(D-2)}\gamma_{AB}\right), \tag{4.68}$$

$$\delta Q \sim \frac{1}{\delta^2}2\kappa\theta. \tag{4.69}$$

Notice that the perturbation of the non-affine coefficient κ can always be gauged away by choosing a suitable parametrization of the horizon. This remark means that we can put $\delta\kappa$ equal to zero. So, without any loss of generality, we set κ equal to the surface gravity of the background black geometry. Now, as an illustration of the perturbation scheme mentioned in equation (4.65) and the reason for the definition of the Q_{ab}, we use the facts mentioned in the previous paragraph to evaluate a term $K_{AC}Q^C_D$ contributed by J_{AD} which we encounter in the projection of t_{ab} on the cross-section of $\mathcal{H}_s$. We evaluate this term as follows :

$$\begin{aligned} K_{AC}Q^C_D &\approx \mathring{K}_{AC}\mathring{Q}^C_D + \delta K_{AC}\mathring{Q}^C_D + \mathring{K}_{AC}\delta Q^C_D \\ &= \frac{\delta}{r}\gamma_{AC}\mathring{Q}^C_D + \frac{1}{\delta}\left(\sigma_{AC} + \frac{\theta}{(D-2)}\gamma_{AC}\right)\mathring{Q}^C_D + \frac{\delta}{r}\gamma_{AC}\frac{1}{\delta^2}\kappa\left(\sigma^C_D + \frac{\theta}{(D-2)}\gamma^C_D\right) \\ &\sim \frac{1}{\delta}\left(\sigma_{AC}\mathring{Q}^C_D + \frac{\theta}{(D-2)}\mathring{Q}_{AD} + \frac{\kappa}{r}\sigma_{AD} + \frac{\kappa}{r}\frac{\theta}{(D-2)}\gamma_{AD}\right) \\ &\sim \frac{1}{\delta}\left(\frac{\kappa}{r}\sigma_{AD} + \frac{\theta}{(D-2)}\frac{\kappa}{r}\gamma_{AD} + \frac{\kappa}{r}\sigma_{AD} + \frac{\kappa}{r}\frac{\theta}{(D-2)}\gamma_{AD}\right). \end{aligned} \tag{4.70}$$

In the steps above we have dropped the terms which are of $\mathcal{O}(1)$ or $\mathcal{O}(\delta)$ because after regularization (i.e., multiplying by δ) those terms will make no contribution. In this way one sees that our method of approximation is consistent. At linear order in the perturbation the only divergence that comes up in the membrane stress-energy tensor is of $\mathcal{O}(\delta^{-1})$ and therefore the whole stress tensor can be regularized simply by multiplying with δ exactly as in general relativity. The sample calculation above elaborates on how it is done for one particular term. Using the Gauss-Codazzi equation and the Raychaudhuri equation it can be shown that the projection of the curvature of the $\mathcal{H}_s$ on the cross-section of $\mathcal{H}_s$ is equal to the curvature of the cross-section in the limit $\delta \to 0$.

Before writing down the stress tensor there is one more restriction that we are going to put on the background geometry. We require that the cross-section of the horizon of the background geometry be a space of constant curvature, i.e., ${}^{(D-2)}\mathring{R}_{ABCD} = c\,(\gamma_{AC}\gamma_{BD} - \gamma_{AD}\gamma_{BC})$, where c is a constant of dimension $length^{-2}$ which is related to the intrinsic Ricci scalar of the horizon cross-section as

$$c = \frac{{}^{(D-2)}\mathring{R}}{(D-3)(D-2)}. \tag{4.71}$$

This assumption regarding the intrinsic geometry of the horizon cross-section is necessary for the stress tensor to be of the form of an isotropic viscous fluid. Using these observations and approximations the contribution of different terms in the membrane stress-energy tensor due to the GB term in the action are obtained as :

$$\begin{aligned}
K_{mn}\hat{P}_a{}^{mn}{}_b\gamma^a_A\gamma^b_B &\sim -\frac{1}{\delta}\frac{(D-3)(D-4)}{2}c\kappa\gamma_{AB} + \frac{1}{\delta}\frac{(D-5)(D-4)}{2}c\left[\sigma_{AB} - \frac{(D-3)}{(D-2)}\theta\gamma_{AB}\right],\\
J_{ab}\gamma^a_A\gamma^b_B &\sim \frac{1}{\delta}\sigma_{AB}(D-4)\frac{2\kappa}{r} + \frac{1}{\delta}\gamma_{AB}\frac{(D-3)}{(D-2)}\frac{4\kappa\theta}{r},\\
J &\sim \frac{1}{\delta}(D-3)\frac{6\kappa\theta}{r},\\
K_{mn}\hat{P}_a{}^{mn}{}_b u^a u^b &\sim -\frac{1}{\delta}\frac{(D-3)(D-4)}{2}c\kappa\theta,\\
J_{ab}u^a u^b &\sim -\frac{1}{\delta}(D-3)\frac{2\kappa\theta}{r}.
\end{aligned} \tag{4.72}$$

All the steps required to obtain the regularized membrane stress tensor are laid out now. Our perturbative strategy and the restriction that the horizon's cross-section be a space of constant curvature then yield the membrane stress tensor as

$$t^{(H)}_{AB} = p\gamma_{AB} - 2\eta\sigma_{AB} - \zeta\theta\gamma_{AB}, \tag{4.73}$$

where

$$p = 2\kappa + 4\alpha'\kappa c(D-4)(D-3), \tag{4.74a}$$

$$\eta = 1 - 4\alpha'\left[-\frac{(D-5)(D-4)}{2}c + (D-4)\frac{\kappa}{r}\right], \tag{4.74b}$$

$$\zeta = -2\frac{(D-3)}{(D-2)} + 4\alpha'\left[-\frac{(D-5)(D-4)(D-3)}{(D-2)}c + \frac{(D-4)(D-3)}{(D-2)}\frac{2\kappa}{r}\right]. \tag{4.74c}$$

We stress that while we are working only at the linear order in the perturbations, the α' corrections in the transport coefficients given in equations (4.74) are non-perturbative. This simply means that the theory is exactly the EGB theory and the background spacetimes of interest are exact static solutions of this theory. We are perturbing these backgrounds slightly and therefore we care about only the linear order terms in the perturbations. This approach differs from some of the other work in the literature, see for example Brustein & Medved (2009), where one considers the effect of the GB term in the action as a small perturbation of general relativity. Also, note that the bulk viscosity ζ and shear viscosity η of the membrane fluid are related as

$$\zeta = -2\frac{(D-3)}{(D-2)}\eta. \tag{4.75}$$

This relationship holds in general relativity and interestingly enough it survives in EGB gravity. In fact, this relationship is also true for any gravity theory with a Lagrangian which is an arbitrary function of the Ricci scalar (Chatterjee *et al.*, 2012) and it shows that the bulk viscosity of the membrane fluid is always negative as long as the shear viscosity is positive.

The energy density of the fluid is given by regularizing (i.e., multiplying by δ) the component of t_{ab} along fiducial observers,

$$\rho = -2\theta - 4\alpha'\theta\left[(D-4)(D-3)c\right]. \tag{4.76}$$

The continuity equation (4.47) yields the equation describing the evolution of the energy density along the null generators of the horizon. Note that we are actually applying the linearized continuity equation and in order to derive the evolution equation we have to keep the induced metric on the stretched horizon fixed. As in the case of general relativity, we first write down the equation on the stretched horizon keeping the terms which are linear in perturbations and have $\mathcal{O}(\delta^{-1})$ divergence in t_{ab}, which gives a $\mathcal{O}(\delta^{-2})$ divergence in the continuity equation. This is regularized by multiplying the whole equation by δ^2. This again has the same form as in the equation (4.50) with p, η and ζ now given by the equations (4.74).

4.6 Specific background geometries

In this section we will study some specific static background geometries. We will calculate the transport coefficients and the ratio η/s for the membrane fluid.

4.6.1 Black-brane background

The D-dimensional black-brane solution of EGB gravity is given by (Brigante *et al.*, 2008b, Cai, 2002)

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + \frac{r^2}{l^2}\left(\sum_{i=1}^{D-2}\mathrm{d}x_i^2\right), \tag{4.77}$$

where the function $f(r)$ is given by

$$f(r) = \frac{r^2}{2\lambda}\left[1 - \sqrt{1 - \frac{4\lambda}{l^2}\left(1 - \left(\frac{r_h}{r}\right)^{D-1}\right)}\right], \tag{4.78}$$

with λ defined in terms of the Gauss-Bonnet coupling α' as $\lambda = (D-3)(D-4)\alpha'$. The location of the horizon is denoted by r_h, in terms of which the Hawking temperature and the entropy density (in units $16\pi G = 1$)[5] are given by

$$T = \frac{(D-1)r_h}{4\pi l^2}, \quad (4.79)$$

$$s = 4\pi. \quad (4.80)$$

Note that the intrinsic curvature of the cross-section of the planar horizon is zero, i.e., $c = 0$. Also, the entropy density of the planar horizon in EGB theory is the same as that in general relativity (Jacobson & Myers, 1993). From equation (4.76), the energy density for the fluid is just -2θ which is same as that in general relativity, i.e., there is no correction in the energy density due to the Gauss-Bonnet coupling. The transport coefficients for the fluid description of the horizon can now be obtained from the equations (4.74) as

$$p = (D-1)\frac{r_h}{l^2}, \quad (4.81)$$

$$\eta = 1 - 2\frac{(D-1)}{(D-3)}\frac{\lambda}{l^2}, \quad (4.82)$$

$$\zeta = -2\frac{(D-3)}{(D-2)} + \frac{4(D-1)}{(D-2)}\frac{\lambda}{l^2}. \quad (4.83)$$

The dimensionless ratio η/s is

$$\frac{\eta}{s} = \frac{1}{4\pi}\left[1 - 2\frac{(D-1)}{(D-3)}\frac{\lambda}{l^2}\right]. \quad (4.84)$$

We see that the value of η/s calculated from the membrane paradigm matches the one found by other methods in the literature, for example Brigante *et al.* (2008b), where one of the calculations utilizes the Kaluza-Klein reduction to express the transverse metric perturbation in D dimensions as the vector potential in $(D-1)$ dimensions and then uses the membrane paradigm results for the electromagnetic interaction of the black hole in $(D-1)$ dimensions.

[5] Note that our notation is different from Brigante *et al.* (2008b). We define the entropy density as the entropy per unit cross-section area of the horizon. For planar black holes the total entropy is infinite because the horizon cross-section has an infinite area, but the entropy density is well defined.

It is evident that the KSS bound is violated for any $\lambda > 0$. Also, the bulk viscosity becomes positive when $\lambda/l^2 > (D-3)/2(D-1)$ and for the same range of λ the shear viscosity is negative. In fact, as pointed out in Brigante *et al.* (2008b) and Iqbal & Liu (2009), the gravitons in the theory become strongly coupled as $\lambda/l^2 \to (D-3)/2(D-1)$. For $\lambda/l^2 > (D-3)/2(D-1)$ the theory becomes unstable.

4.6.2 Boulware-Deser black-hole background

The Boulware-Deser black hole (Boulware & Deser, 1985, Myers & Simon, 1988) is a spherically symmetric, asymptotically flat solution of the vacuum EGB theory. It is a natural generalization of the Schwarzschild spacetime in general relativity. The metric is given by

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{4.85}$$

where $\mathrm{d}\Omega^2$ is the metric on a $(D-2)$-dimensional sphere and[6]

$$f(r) = 1 + \frac{r^2}{2\lambda}\left[1 - \sqrt{1 + \frac{4\lambda M}{r^{D-1}}}\right]. \tag{4.86}$$

The constant M is proportional to the ADM mass of the spacetime. This spacetime is asymptotically flat and reduces to the D-dimensional Schwarzschild solution when $\alpha' \to 0$. The location of the horizon is obtained from the solution of the equation

$$r_h^{D-3} + \lambda\, r_h^{D-5} = M. \tag{4.87}$$

[6] The spherically symmetric solution of Einstein-Gauss-Bonnet gravity in vacuum admits two different branches. For our purposes, we are only interested in the branch which has a general-relativity limit when $\lambda \to 0$.

The surface gravity of the horizon is obtained as $\kappa = f'(r_h)/2$, which gives the Hawking temperature of the horizon as (Myers & Simon, 1988)

$$T = \frac{(D-3)}{4\pi r_h}\left[\frac{r_h^2 + \frac{(D-5)}{(D-3)}\lambda}{(r_h^2+2\lambda)}\right]. \tag{4.88}$$

For the Boulware-Deser black hole, the entropy associated with the horizon is no longer proportional to the area $\mathcal{A}$ but is given by (in the units $16\pi G = 1$) (Myers & Simon, 1988),

$$S = 4\pi\mathcal{A}\left[1 + \frac{(D-2)}{(D-4)}\frac{\lambda}{r_h^2}\right]. \tag{4.89}$$

We can now evaluate the transport coefficients of the membrane fluid when the background is chosen as the D-dimensional Boulware-Deser black hole. Note that, unlike the black-brane case, here the curvature of the horizon cross-section is non-zero and will contribute to the transport coefficients. Using equation (4.74), these coefficients are calculated to be

$$p = \frac{(D-3)}{r_h} + \frac{(D-5)}{r_h^3}\lambda, \tag{4.90}$$

$$\eta = 1 - \frac{4}{(D-3)}\frac{\lambda}{(r_h^2+2\lambda)} + \frac{2(D-5)}{(D-3)}\frac{\lambda^2}{r_h^2(r_h^2+2\lambda)}, \tag{4.91}$$

$$\zeta = -2\frac{(D-3)}{(D-2)}\frac{r_h^2}{(r_h^2+2\lambda)} - \frac{4(D-5)}{(D-2)}\frac{\lambda}{(r_h^2+2\lambda)} - \frac{4(D-5)}{(D-2)}\frac{\lambda^2}{r_h^2(r_h^2+2\lambda)}. \tag{4.92}$$

The ratio of the shear viscosity to the entropy density is

$$\frac{\eta}{s} = \frac{(D-4)}{4\pi(D-3)\,(r_h^2+2\lambda)}\left[\frac{(D-3)r_h^4 + 2\lambda(D-5)r_h^2 + 2\lambda^2(D-5)}{(D-4)r_h^2 + 2(D-2)\lambda}\right]. \tag{4.93}$$

For $\lambda = 0$ this reduces to the familiar result found in general relativity in equation (4.53). The ratio violates the KSS bound for any $\lambda > 0$.

In the particular case of $D = 5$, the transport coefficients and the ratio η/s are

$$p = \frac{2}{r_h}, \;\; \eta = \frac{r_h^2}{(r_h^2 + 2\lambda)}, \;\; \text{and} \;\; \zeta = -\frac{4r_h^2}{3(r_h^2 + 2\lambda)}, \tag{4.94}$$

$$\frac{\eta}{s} = \frac{r_h^4}{4\pi\,(r_h^2 + 2\lambda)\,(r_h^2 + 6\lambda)}. \tag{4.95}$$

4.6.3 Boulware-Deser-AdS black-hole background

The spherically symmetric solution of EGB gravity with a negative cosmological constant $\Lambda = -(D-1)(D-2)/2l^2$ is given by (Cai, 2002)

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{4.96}$$

where $\mathrm{d}\Omega^2$ is the metric on a $(D-2)$-dimensional sphere and

$$f(r) = 1 + \frac{r^2}{2\lambda}\left[1 - \sqrt{1 + \frac{4\lambda M}{r^{D-1}} - \frac{4\lambda}{l^2}}\right]. \tag{4.97}$$

The position of the horizon is determined by the equation

$$r_h^{D-3} + \lambda\, r_h^{D-5} + \frac{r_h^{D-1}}{l^2} = M. \tag{4.98}$$

The Hawking temperature associated with this horizon is

$$T = \frac{1}{4\pi r_h l^2\,(r_h^2 + 2\lambda)}\left[(D-1)r_h^4 + (D-3)r_h^2\, l^2 + (D-5)\lambda\, l^2\right], \tag{4.99}$$

and the entropy density is given by

$$s = 4\pi\left[1 + \frac{(D-2)}{(D-4)}\frac{\lambda}{r_h^2}\right]. \tag{4.100}$$

Using equations (4.74), the various transport coefficients are obtained as

$$
\begin{aligned}
p &= \frac{(D-3)}{r_h} + \frac{(D-1)}{l^2}\frac{1}{r_h} + \frac{(D-5)}{r_h^3}\lambda, && (4.101)\\
\eta &= 1 - \frac{4}{(D-3)}\frac{\lambda}{(r_h^2+2\lambda)} - \frac{2(D-1)}{(D-3)}\frac{\lambda\, r_h^2}{l^2(r_h^2+2\lambda)}\\
&\quad + \frac{2(D-5)}{(D-3)}\frac{\lambda^2}{r_h^2(r_h^2+2\lambda)}, && (4.102)\\
\zeta &= -2\frac{(D-3)}{(D-2)}\frac{r_h^2}{(r_h^2+2\lambda)} - \frac{4(D-5)}{(D-2)}\frac{\lambda}{(r_h^2+2\lambda)}\\
&\quad - \frac{4(D-5)}{(D-2)}\frac{\lambda^2}{r_h^2(r_h^2+2\lambda)} + \frac{4(D-1)}{(D-2)}\frac{\lambda\, r_h^2}{l^2(r_h^2+2\lambda)}. && (4.103)
\end{aligned}
$$

Note that in the limit $\Lambda \to 0$, i.e., $l \to \infty$, these coefficients reduce to those for the vacuum case discussed in the Section 4.6.2.

The ratio of shear viscosity and entropy density is

$$
\frac{\eta}{s} = \frac{(D-4)}{4\pi(D-3)l^2\,(r_h^2+2\lambda)} \times \left[\frac{(D-3)l^2 r_h^4 + 2\lambda(D-5)l^2 r_h^2 + 2\lambda^2 l^2(D-5) - 2\lambda(D-1)r_h^4}{(D-4)r_h^2 + 2(D-2)\lambda}\right]. \qquad (4.104)
$$

In particular, for $D = 5$ the ratio is

$$
\frac{\eta}{s} = \frac{r_h^4}{4\pi\,(r_h^2+2\lambda)\,(r_h^2+6\lambda)}\left(1 - \frac{4\lambda}{l^2}\right). \qquad (4.105)
$$

The ratio is positive and violates the KSS bound as long as $0 < \lambda < l^2/4$.

The result in equation (4.105) can be viewed as a prediction that the long-wavelength hydrodynamic limit of the dual CFT (at the temperature given by (4.99) in terms of r_h) living on the boundary $S^3 \times R$ has the ratio η/s given by (4.105). Here we are assuming that the ratio η/s has a trivial renormalization group flow even for the Boulware-Deser-AdS background.[7] Also, in the limit of high black-hole temperature (which corresponds to the

[7] We thank R. Myers for pointing out that the triviality of the flow on this background does not obviously

limit $r_h \to \infty$), the ratio η/s for Boulware-Deser-AdS black hole reduces to the ratio for the black brane (4.84).

4.7 Summary and Discussion

In this chapter we have discussed our proposal of a perturbative scheme to derive the membrane stress tensor and the transport coefficients for the membrane fluid in Einstein-Gauss-Bonnet gravity. We used the action principle formalism to determine the membrane stress tensor on the stretched horizon. Our derivation is slightly different from the one given in Parikh & Wilczek (1998), since we include the Gibbons-Hawking boundary term at infinity as well as on the stretched horizon. As a result, all the contribution from the bulk part under the variation with respect to ω vanishes automatically. (It was argued by Parikh & Wilczek (1998) that without the Gibbons-Hawking term, these contributions vanish in the limit when the stretched horizon approaches the true horizon.) Our method can be easily generalized to the higher-order Lovelock theories.

The membrane stress tensor in EGB gravity has terms cubic in the extrinsic curvature and in the limit that the stretched horizon approaches the true horizon these terms are cubically divergent in δ^{-1}. In order to tame these divergences we studied only the perturbations about static black geometries. We found that restricting to the linear order in perturbations on the stretched horizon, the membrane stress tensor is in fact only linearly divergent. Therefore, at the linear order the divergence structure of the GB contribution to the stress tensor is identical to that of the Einstein contribution. Hence the whole membrane stress tensor could be regularized in the same way as in general relativity: simply absorb one power of the divergent factor in the definition of the stress tensor. We expect that this method can be generalized to the higher-order Lovelock theories. We also believe that our method to obtain the evolution equation for the energy density is particularly easy to adapt to the Lovelock theories because it avoids manipulating the equation of motion in order to

follow from the triviality of flow on the black brane background.

write the curvature term in the Raychaudhuri equation in terms of the energy-momentum tensor of the matter.

In order to write the membrane stress tensor in the form of that of an isotropic viscous fluid we had to restrict the background geometries to those which have a constant-curvature horizon cross-section. This looks like a severe restriction, and it should be possible to bypass this requirement by modelling the horizon as an anisotropic fluid with tensorial transport coefficients. But even with this restriction we are able to study interesting cases discussed in section 4.6. Nevertheless, it would be interesting to study the more general case by lifting this restriction and study backgrounds with non-constant curvature horizon cross-section.

The transport coefficients of the membrane fluid for the horizon in EGB gravity are given in equations (4.74). We notice a relation between the shear viscosity and the bulk viscosity,

$$\zeta = -2\frac{(D-3)}{(D-2)}\eta,$$

which says that if the shear viscosity is positive then the bulk viscosity is negative and vice-versa. The negative value of ζ of the membrane fluid is related to the teleological nature of the horizon. This relation also gives a consistency check that the allowed range of the GB coupling for the fluid description to make sense is the same whether calculated from the positivity of η or the negativity of ζ. It is quite possible that the same relationship between these two transport coefficients holds for the higher-order Lovelock theories and this question is worth investigating.

Some particular static spacetimes are analyzed in Section 4.6. For the black-brane solution of EGB gravity, the membrane paradigm calculation gives a value of η/s which agrees with that found in the literature; see for example Brigante *et al.* (2008b), which calculates the ratio from both linear response theory and a Kaluza-Klein compactified version of the membrane paradigm. The KSS bound is violated for any positive value of the GB coupling. From the other cases we have studied it appears that the violation of the KSS bound is a generic feature of EGB gravity except for the case of an extremal ($\kappa = 0$) black-brane solution, where all the GB corrections vanish.

From the purely classical point of view via the membrane paradigm and an input from black-hole thermodynamics, we have seen that in general relativity the values of viscosity and entropy density are constants independent of the solution. In particular, they are the same for the black brane and the Boulware-Deser-AdS black hole. When the α' corrections are taken into account, the entropy density for the black brane does not change (because the horizon is flat), but the entropy density for the black hole increases (see (4.89)). Also, the viscosity of both the black brane (4.82) and the black hole (4.102) decreases as α' is turned on. Consequently, in both cases α' affects the ratio η/s and causes it to fall below $1/4\pi$, the value in general relativity.

Let us try to understand the reason for the difference between the black-brane and black-hole entropy densities and shear viscosity from a dual CFT viewpoint. If a dual CFT exists, it is not the same one that is dual to Type IIB string theory, since in that case the leading correction to the gravitational action is an $O(\alpha'^3 \mathrm{R}^4_{\mathrm{Ads}})$ term (Gubser *et al.*, 1998b), not an $O(\alpha')$ Gauss-Bonnet term. But let us suppose that some dual CFT does exist. The black hole in the bulk is then dual to a finite-temperature CFT on the conformal boundary $\mathbb{S}^3 \times \mathbb{R}$ of the bulk, and the black brane in the bulk is dual to the same CFT on $\mathbb{R}^4 \subset \mathbb{S}^3 \times \mathbb{R}$ and is thermal with respect to a different conformal Killing generator.

The α' corrections to entropy density and shear viscosity presumably correspond to corrections due to finite 't Hooft coupling λ_t (or something analogous) in the dual CFT. In the limit of infinite λ_t (i.e., when the holographic dual to the boundary theory is just classical general relativity[8]), the hydrodynamic limit of the boundary theory has the same entropy density and viscosity for both the black-hole and the black-brane duals. When the effects due to finite λ_t are taken into account they become different. Evidently the corrections due

[8]We assume that, if the duality entails other fields in the bulk, those fields do not affect the black branes and black holes, at least at the order we are working.

to finite 't Hooft coupling must be sensitive to whether the dual CFT is on $\mathbb{S}^3 \times \mathbb{R}$ or $\mathbb{R}^4$. Why might that be?

First note that even with infinite 't Hooft coupling, there is a difference, in that on $\mathbb{S}^3 \times \mathbb{R}$ there is a critical temperature, set by the radius of the $\mathbb{S}^3$, below which the CFT is confining and is dual to AdS without a black hole (Witten, 1998b). However, once above the critical temperature, the leading order entropy density and viscosity are strictly identical to those on $\mathbb{R}^4$. Perhaps we may interpret this as being due to an infinitely sharp phase transition in the presence of an infinite 't Hooft coupling.

Now consider the situation at finite 't Hooft coupling. When $r_h \gg l$, the black-hole temperature (4.99) is $T \sim r_h/l^2 \gg 1/l$. (Here we use the fact that for a stable theory we must have $\lambda < l^2/4$, so also $r_h \gg \sqrt{\lambda}$.) In this high-temperature regime, s and η become indistinguishable from their values for the black brane. In the dual CFT, this is to be expected since the thermal wavelength is much smaller than the size $\sim l$ of the $\mathbb{S}^3$. Away from this limit, the thermal state on $\mathbb{S}^3 \times \mathbb{R}$ is different, in its local properties, from the thermal state dual to the black brane.

It remains somewhat puzzling however, from the dual CFT viewpoint, that there would be no α' correction to the entropy density of the black brane, given that there is such a correction to its viscosity. In fact, in the case of the Super-Yang-Mills theory dual to Type IIB string theory, there is indeed an α'^3 correction to the black-brane entropy, seen from an $O(\alpha'^3 \mathrm{R}^4_{\mathrm{AdS}})$ correction to the gravitational action (Gubser *et al.*, 1998b). While it has not been computed in the dual CFT, the duality asserts that this correction arises from a finite 't Hooft coupling effect in the CFT. By contrast, the absence of an α' correction to the black-brane entropy in EGB theory suggests that, if there indeed is a field theory dual to EGB gravity, for some reason it has no $O(\alpha')$, finite 't Hooft coupling correction to the thermal entropy on $\mathbb{R}^4$.

Let us spell this out a little more explicitly. To obtain the CFT entropy density from the horizon entropy density one must multiply by $(r_h/l)^{D-2}$, the ratio of horizon 'area' to spatial volume ($\int \Pi_{i=1}^{D-2} dx_i$) of the CFT. Although the black-brane metric (4.78) receives corrections from α', the ratio $r_h/l = 4\pi Tl/(D-1)$ in equation (4.79) is independent of α' when expressed in terms of T. Thus, for $D = 5$, the entropy density of the CFT is $\propto (Tl)^3(l_P)^{-3} = T^3(l/l_P)^3$, where l_P is the 5-dimensional Planck length. For the black hole, by contrast, equation (4.99) shows that r_h/l depends on α' at fixed temperature, and also the horizon entropy receives an α' correction. Hence, the entropy density in the CFT must also receive an α' correction. This discrepancy is puzzling, since although the finite size of the $\mathbb{S}^3$ could naturally affect the temperature dependence of the entropy at low temperature, it would seem that the mere presence or absence of finite-coupling effects in the CFT entropy density should be independent of the finite size of the $\mathbb{S}^3$.

As far as other transport coefficients are concerned, the relation between the bulk viscosity of the membrane fluid and that of the dual gauge theory is not clear. The bulk viscosity of the membrane fluid is negative while for the gauge theory it is necessarily non-negative (and is zero if the gauge theory is conformal). It might be tempting to conjecture that the bulk viscosity undergoes a non-trivial renormalization group flow as one takes the cutoff surface from the membrane to infinity (where the gauge theory lives). But the calculation depicting the relation between bulk viscosity and the flux of gravitons (Gubser *et al.*, 2008) suggests that this flow is trivial. In fact, that this flow is trivial has very recently been shown in Eling & Oz (2011). It would be interesting to see what exactly, if any, is the relation between the bulk viscosity of the membrane fluid and the dual gauge-theory fluid.

We reiterate before closing that our results are derived for linear-order perturbations of the static background geometries where the cross-section of the horizon is a space of constant curvature, and we have evaluated only the first-order transport coefficients keeping only the terms of first order in derivatives of the velocity. It would be interesting to generalize this procedure to calculate the higher-order transport coefficients.

Table 4.1. Symbols and their meanings

Symbol	Meaning
H	True horizon, a $(D-1)$-dimensional null hypersurface
$\mathcal{H}_s$	Stretched horizon, a $(D-1)$-dimensional timelike hypersurface with tangent u^a and normal n^a
$(a, b, c, ...)$	Spacetime indices
$(A, B, C, ...)$	Indices on the $(D-2)$-dimensional cross-section of the true/stretched horizon
l^a	Null generator of the true horizon parametrized by a non-affine parameter. Obeys the geodesic equation: $l^a \nabla_a l^b = \kappa\, l^b$
h_{ab}	Induced metric on the stretched horizon
γ_{ab}	Induced metric on the cross-section of the stretched horizon, which in the null limit is identified with the metric on the cross-section of the true horizon
K_{ab}	Extrinsic curvature of the stretched horizon, defined as $K_{ab} = \frac{1}{2}\mathcal{L}_n h_{ab}$
k_{AB}	Extrinsic curvature of the cross-section of the true horizon, defined as $k_{AB} = \frac{1}{2}\mathcal{L}_l \gamma_{AB}$
θ, θ_s	Expansion of the true/strectched horizon
σ^{ab}, σ_s^{ab}	Shear of the true/stretched horizon
$\hat{R}_{abcd}$	Riemann tensor intrinsic to the stretched horizon
${}^{(D-2)}\mathring{R}_{ABCD}$	Intrinsic Riemann tensor of the $(D-2)$-dimensional cross-section of the stretched horizon in the background geometry, which in the null limit is identified with the intrinsic Riemann tensor of the cross-section of the true horizon
${}^{(D-2)}\mathring{R}$	Intrinsic Ricci scalar of the $(D-2)$-dimensional cross-section of the stretched horizon in the background geometry, which in the null limit is identified with the intrinsic Ricci scalar of the cross-section of the true horizon
δ	The parameter which measures the deviation of the stretched horizon from the true horizon
c	Defined as $c = \frac{{}^{(D-2)}\mathring{R}}{(D-3)(D-2)}$ (For the planar horizon, $c = 0$)
α'	Gauss-Bonnet coupling constant
λ	Constant of dimension *length*^{2} related to the Gauss-Bonnet coupling α' as $\lambda = (D-3)(D-4)\alpha'$

Chapter 5

Appendix

5.1 A: Useful identities

D is a connection on a G-bundle E. A is the connection one-form and F denotes its curvature two-form.

η and ρ are E-valued differential forms with degrees p and q, respectively.

ω and μ are $end(E)$-valued differential p and q forms, respectively.

The Graded Commutator is defined as $[\omega, \mu] = \omega \wedge \mu - (-1)^{pq} \mu \wedge \omega$.

- $\mathrm{D}\eta = \mathrm{d}\eta + \omega \wedge \eta$.
- $\mathrm{D}\omega = \mathrm{d}\omega + [A, \omega]$.
- $\mathrm{D}^2\eta = F \wedge \eta$.
- $\mathrm{D}^2\omega = [F, \omega]$.
- $\mathrm{tr}(\omega \wedge \mu) = (-1)^{pq}\, \mathrm{tr}(\mu \wedge \omega)$. (graded cyclic property).
- $\mathrm{tr}[\omega, \mu] = 0$.
- $\mathrm{tr}\, \mathrm{D}(\omega) = \mathrm{d}\, \mathrm{tr}(\omega)$.
- $\int_M \mathrm{tr}(\mathrm{D}\omega \wedge \mu) = \int_{\partial M} \mathrm{tr}(\omega \wedge \mu) + (-1)^{p+1} \int_M \mathrm{tr}(\omega \wedge \mathrm{D}\mu)$.
- $\int_M \mathrm{tr}(\omega \wedge *\mu) = \int_M \mathrm{tr}(\mu \wedge *\omega)$.

- $\mathrm{i}_\xi(\eta \wedge \rho) = \mathrm{i}_\xi\eta \wedge \rho + (-1)^p \eta \wedge \mathrm{i}_\xi\rho$.

- $\mathrm{i}_\xi[\mu, \omega] = [\mathrm{i}_\xi\mu, \omega] + (-1)^p[\mu, \mathrm{i}_\xi\omega]$.

- $\mathcal{L}_\xi A = \mathrm{i}_\xi F + \mathrm{D}(i_\xi A)$.

- $\mathcal{L}_\xi \eta = \mathrm{i}_\xi \mathrm{D}\eta + \mathrm{D}\mathrm{i}_\xi\eta - \mathrm{i}_\xi A \wedge \mu$.

- $\mathcal{L}_\xi \mu = \mathrm{i}_\xi \mathrm{D}\mu + \mathrm{D}\mathrm{i}_\xi\mu - [\mathrm{i}_\xi A, \mu]$.

5.2 B: Review of hydrodynamics

We give here a concise but useful tutorial on the basic equations of hydrodynamics. For this discussion we have mainly relied on Romatschke (2010) and Rangamani (2009).

5.2.1 Non-relativistic fluid dynamics

The degrees of freedom for an ideal, neutral, uncharged one-component fluid are: velocity $v(t, \vec{x})$, pressure $p(t, \vec{x})$ and mass density $\rho(t, \vec{x})$. The dynamics is governed by the Euler equation, which is the hydrodynamic version of Newton's second law, and the continuity equation, which is the hydrodynamic version of conservation of energy,

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\partial})\vec{v} = -\frac{1}{\rho}\vec{\partial}p \qquad \text{Euler equation} \tag{5.1}$$

$$\partial_t \rho + \vec{\partial} \cdot (\rho \vec{v}) = 0 \qquad \text{Continuity equation} \tag{5.2}$$

The number of unknown variables is $3+1+1 = 5$, while the number of equations is $3+1 = 4$. In order to close the system we need one more relation, namely the equation of state $p \equiv p(\rho)$. Then we have a complete description of the fluid dynamical system.

For a non-ideal, i.e., dissipative fluid the Euler equation generalizes to the Navier-Stokes equation

$$\partial_t v^i + v^k \partial_k v^i = -\frac{1}{\rho}\partial_i p - \frac{1}{\rho}\partial_k \Pi^{ki}, \tag{5.3}$$

where the dissipative term Π^{ki} is given by

$$\Pi^{ki} = -2\eta \left(\partial^{(k} v^{i)} - \frac{1}{3}\delta^{ik}\partial_j v^j \right) - \xi \delta^{ik} \partial_j v^j. \tag{5.4}$$

The term in the parenthesis is just the symmetric traceless part of $\partial^i v^k$ and its coefficient η is called the shear viscosity of the fluid. The coefficient of the trace part, ξ, is called the bulk viscosity of the fluid.

5.2.2 Relativistic fluid dynamics

The variables for the relativistic fluid are the energy density $\rho(x)$ and a velocity field $u^\mu = \frac{dx^\mu}{d\tau}$. From the metric $d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + d\vec{x}^2$ we get

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\vec{v}^2}}(1,\vec{v}) = \gamma(\vec{v})(1,\vec{v}). \tag{5.5}$$

In the non-relativistic limit $u^\mu \sim (1,\vec{v})$.

Let us first construct the ideal relativistic fluid dynamics. In this case the stress-energy tensor of the fluid contains no derivatives of the fluid velocity u^μ. The most general symmetric and generally covariant tensor that one can build is

$$T^{\mu\nu} = \rho u^\mu u^\nu + p h^{\mu\nu} \tag{5.6}$$

where $h^{\mu\nu}$ is the projector on the hypersurface orthogonal to the velocity u^μ, such that $u_\mu h^{\mu\nu} = u_\nu h^{\mu\nu} = 0$. The projector is given by

$$h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu. \tag{5.7}$$

The equations of fluid dynamics are nothing but the statement of the conservation of the stress energy tensor (and charges in the case of a non-zero chemical potential) of the fluid,

$$\nabla_\mu T^{\mu\nu} = 0. \tag{5.8}$$

Using the ideal form of the stress tensor in equation (5.6) and taking the projection of equation (5.8) parallel to u_μ, $u_\mu \nabla_\nu T^{\mu\nu}$, gives

$$(\rho + p)\nabla_\mu u^\mu + u^\mu \nabla_\mu \rho = 0, \tag{5.9}$$

and taking the projection orthogonal to u^μ, $h^\mu_\beta \nabla_\lambda T^{\beta\lambda}$, gives

$$(\rho + p) u^\lambda \nabla_\lambda u^\mu + h^{\mu\lambda} \nabla_\lambda p = 0. \tag{5.10}$$

In order to have a compact notation we define the derivative operator on the hypersurface orthogonal to u^μ as $\mathcal{D}_\alpha = h^\mu_\alpha \nabla_\mu$, where it is assumed that $\mathcal{D}_\alpha$ acts only on tensors whose components are tangential to the hypersurface orthogonal to u^μ. Also, the derivative along u^μ will be denoted by $\mathcal{D}$, i.e., $u^\mu \nabla_\mu = \mathcal{D}$. Using these operator definitions, equation (5.9) can be written as

$$\mathcal{D}\rho + (\rho + p) \nabla_\mu u^\mu = 0, \tag{5.11}$$

and equation (5.10) can be written as

$$(\rho + p) \mathcal{D} u^\mu + \mathcal{D}^\mu p = 0. \tag{5.12}$$

Equation (5.11) is the relativistic version of the continuity equation and equation (5.12) is the relativistic version of the Euler equation.

Away from the ideal limit the fluid does have energy dissipation. This is because of the internal friction and is phenomenologically due to the different fluid elements moving with different velocities. Thus the dissipative effects are due to the velocity gradients. This can be incorporated in the stress tensor of the fluid by introducing terms containing the derivatives of the velocity and respecting the symmetries of the system. In general the stress tensor can be written as

$$T^{\mu\nu} = \rho u^\mu u^\nu + p h^{\mu\nu} + \Pi^{\mu\nu}. \tag{5.13}$$

where $\Pi^{\mu\nu}$ is the viscous contribution to the stress tensor. When there are no conserved charges we can assume that the momentum density is due to the flow of energy density, i.e., $u_\mu T^{\mu\nu} = \rho u^\nu$. This implies that $u_\mu \Pi^{\mu\nu} = 0$. This really amounts to a choice of rest frame of the fluid. In general, one can choose the velocity of the fluid to be either in the direction of energy flow (Landau frame) or in the direction of charge flow (Eckart frame). One can then show that the charge diffusion in one frame is related to the heat flow in the other frame. Here we don't have any conserved charges so the Landau frame is the only possibility.

As before, taking the projection of the conservation equation of the stress tensor (5.8) in the direction parallel and orthogonal, respectively, to the fluid velocity we get

$$\mathcal{D}\rho + (\rho + p)\nabla_\mu u^\mu + \Pi^{\mu\nu}\mathcal{D}_\mu u_\nu = 0, \tag{5.14}$$

$$(\rho + p)\mathcal{D}u^\mu + \mathcal{D}^\mu p + h^\mu_\lambda \nabla_\nu \Pi^{\nu\lambda} = 0. \tag{5.15}$$

We still need to find what the dissipative contribution $\Pi^{\mu\nu}$ to the stress tensor is. So far we have not used the fact that we are dealing with the fluids which are in local thermodynamic equilibrium. Local energy density ρ, local pressure p and local temperature T of the fluid are related to its entropy density s by the following thermodynamical relations,

$$\begin{aligned} sT &= \rho + p, \\ Tds &= d\rho. \end{aligned} \tag{5.16}$$

We can derive what the viscous stress $\Pi^{\mu\nu}$ looks like by asserting that the fluid should follow the second law of thermodynamics, i.e., defining the entropy current in equilibrium as $s^\mu = su^\mu$, we demand that $\nabla_\mu s^\mu \geq 0$. Using equation (5.16) in (5.14) gives

$$\nabla_\mu (su^\mu) = -\frac{1}{T}\Pi^{\mu\nu}\mathcal{D}_\mu u_\nu \tag{5.17}$$

Now decompose $\Pi^{\mu\nu}$ into it's trace-free and the trace parts as

$$\Pi^{\mu\nu} = \pi^{\mu\nu} + \frac{1}{3}\Pi h^{\mu\nu}, \tag{5.18}$$

where $\Pi = \Pi^{\mu\nu}h_{\mu\nu}$ is the trace of $\Pi^{\mu\nu}$. Now define an operator $P^{\mu\nu}_{\alpha\beta}$ that projects a tensor of second rank to its symmetric traceless part as

$$P^{\mu\nu}_{\alpha\beta}t_{\mu\nu} = \frac{(t_{\alpha\beta} + t_{\beta\alpha})}{2} - \frac{1}{3}h_{\alpha\beta}t^{\lambda}_{\lambda}. \tag{5.19}$$

Thus the divergence of the equilibrium entropy current can be written as

$$\nabla_\mu (su^\mu) = -\frac{1}{T}\pi^{\mu\nu}P^{\alpha\beta}_{\mu\nu}\mathcal{D}_\alpha u_\beta - \frac{1}{T}\Pi\mathcal{D}_\alpha u^\alpha. \tag{5.20}$$

The second law of thermodynamics $\nabla_\mu s^\mu \geq 0$ will be ensured if

$$\begin{aligned} \pi^{\mu\nu} =& \ -2\eta P^{\mu\nu}_{\alpha\beta}\mathcal{D}^\alpha u^\beta \quad \text{and} \\ \Pi =& \ -\xi\mathcal{D}_\alpha u^\alpha, \\ \text{with} \quad \eta \geq& 0\ ,\ \ \xi \geq 0. \end{aligned} \tag{5.21}$$

Comparing with the non-relativistic case discussed earlier, we recognize η and ξ as the coefficients of shear and bulk viscosity, respectively. The entropy balance law can now be written as

$$\nabla_\mu (su^\mu) = \frac{2\eta}{T}P^{\alpha\beta}_{\mu\nu}\mathcal{D}_\alpha u_\beta\mathcal{D}^\mu u^\nu + \frac{\xi}{T}(\mathcal{D}_\alpha u^\alpha)^2\,. \tag{5.22}$$

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