Neutron stars:
Their global properties and the properties of orbits around them

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ma per seguir virtute e canoscenza
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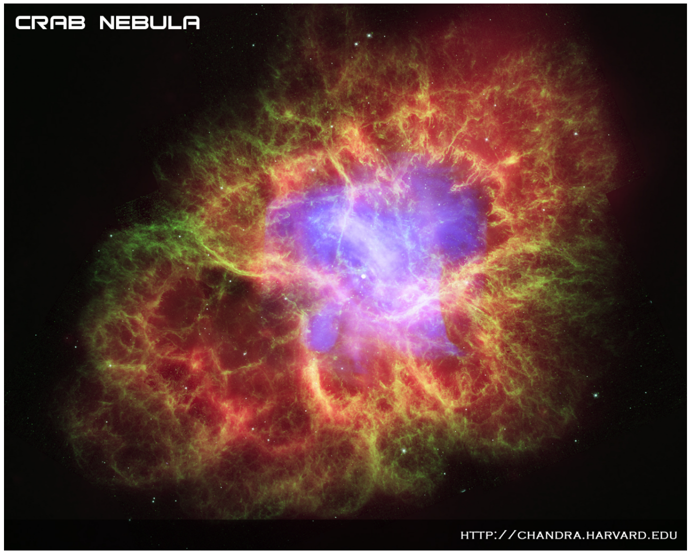
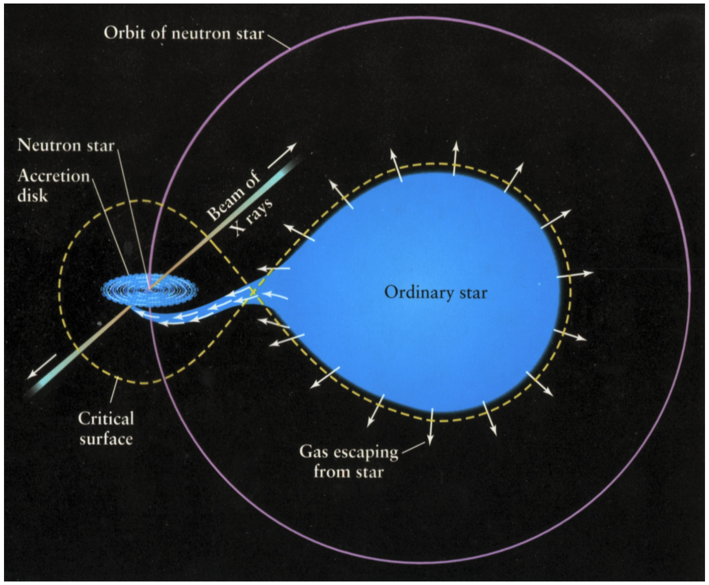
- Why compact objects?
- Compact objects:
  - Fluid configurations in Newtonian theory.
  - Static fluid configurations in General Relativity.
  - Rotating fluid configurations in General Relativity (and their spacetime).
- Multipole moments (characterizing the spacetime)
  - Newtonian multipole moments.
  - Relativistic multipole moments.
- Properties of neutron star moments (Interesting new results and properties that are independent of the equation of state).
- Astrophysical phenomena around neutron stars: Accretion disks and QPOs (and their relation to the spacetime properties).
- More spacetime geometry around neutron stars.
  - An analytic spacetime for the exterior of NSs in brief.
  - Geodesics and relativistic precession frequencies.
    - Kerr frequencies.
    - Neutron star frequencies.
    - Neutron star properties from frequencies.



Compact objects are the results of stellar evolution. We can see them in stellar remnants. A typical example is the Crab nebula that hosts the Crab pulsar[a].

[a]APOD 2006 October 26



Very often we find rapidly rotating pulsars at the end of stellar evolution. The fastest rotating known pulsar (PSR J1748-2446ad) spins at 716Hz and it is part of a binary system[a].

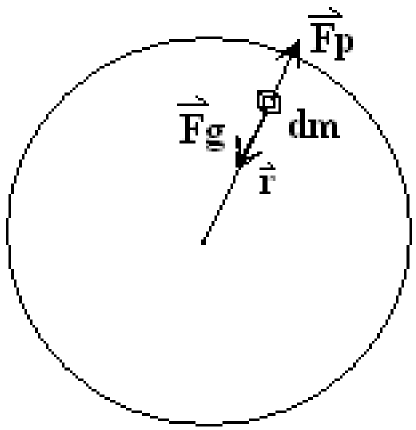
Low mass X-ray binaries are systems that are comprised by a compact object and a regular star companion. The main source of the X-rays is the accretion disk that forms around the compact object.

[a]J. W. T. Hessels et al., Science **311** 1901 (2006)

A lot of interesting astrophysics takes place around compact objects.

Compact objects have strong enough gravitational fields that can test unexplored aspects of our theories of gravity.

Compact Objects: Fluid configurations that are in equilibrium by the action of their self-gravity and their internal forces.



Newtonian Stars
Hydrostatic equilibrium (spherical symmetry):

$$\nabla P = -\rho \nabla \Phi \Rightarrow \frac{dP}{dr} = -\frac{d\Phi}{dr}\rho = -G\frac{m(r)}{r^2}\rho$$

Mass (spherical symmetry): $\frac{dm}{dr} = 4\pi\rho r^2$

Field equations: $\nabla^2 \Phi = 4\pi G \rho$, Equation of state for the fluid: $P = P(\rho)$.

A fluid configuration is in equilibrium if the particular configuration minimizes the total energy. Assuming a polytropic equation of state, $P = K\rho^{\Gamma} = K\rho^{1+1/n} \Rightarrow u = K\frac{\rho^{\Gamma-1}}{\Gamma-1} = Kn\rho^{1/n}$,

$$E = E_{int} + W = \int u dm - \int \frac{Gmdm}{r} = < u > M - a_1 G\frac{M^2}{R} = a_2 K M \rho_c^{\Gamma-1} - a_3 G \rho_c^{1/3} M^{5/3}$$

$$\left.\frac{dE}{d\rho_c}\right|_{M=const.} = 0 \Rightarrow M \propto \rho_c^{(\Gamma-4/3)(3/2)} = \rho_c^{\frac{3-n}{2n}} \Rightarrow$$

$$M \propto R^{\frac{n-3}{n-1}}$$

Relativistic non-rotating Stars

Instead of a gravitational field Φ, gravity is described by a metric g_{ab}. In spherical symmetry the metric can take the form $ds^2 = -e^{2\Phi}dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$.

Field equations: $G^{ab} = 8\pi G T^{ab}$,

Equation for the field Φ: $\frac{d\Phi}{dr} = \frac{m(r)+4\pi r^3 P}{r(r-2m(r))}$,

Definition of the Mass: $\frac{dm}{dr} = 4\pi\rho r^2$,

Hydrostatic equilibrium: $\frac{dP}{dr} = -(\rho + P)\frac{d\Phi}{dr}$.

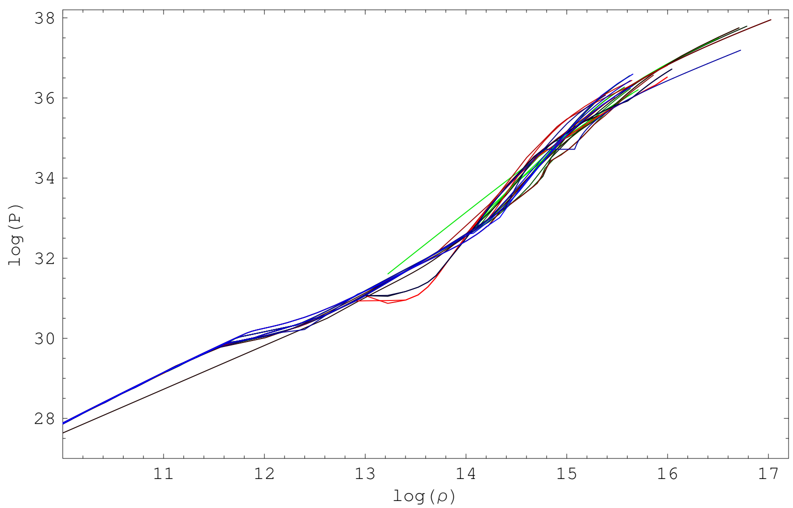
$$\frac{dP}{dr} = -\frac{\rho m(r)}{r^2}\left(1 + \frac{P}{\rho}\right)\left(1 + \frac{4\pi P r^3}{m(r)}\right)\left(1 - \frac{2m(r)}{r}\right)^{-1},$$

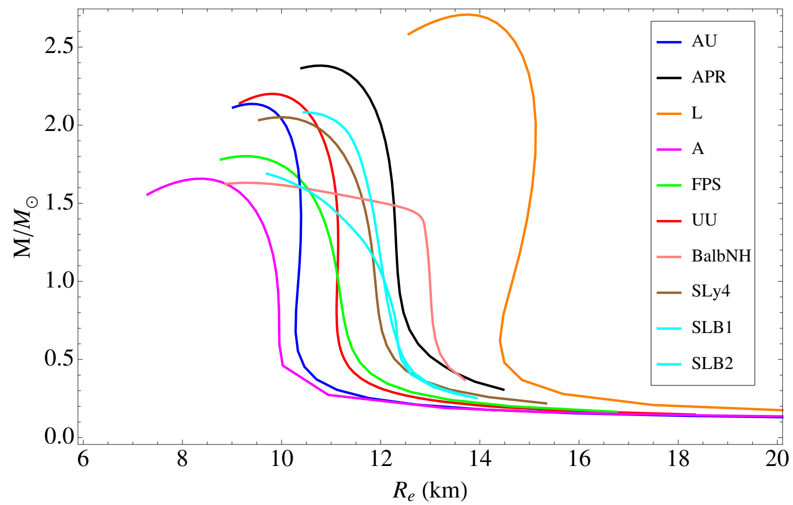
Equation of state for the fluid: $P = P(\rho)$.

The spacetime outside the star is the Schwarzschild spacetime:

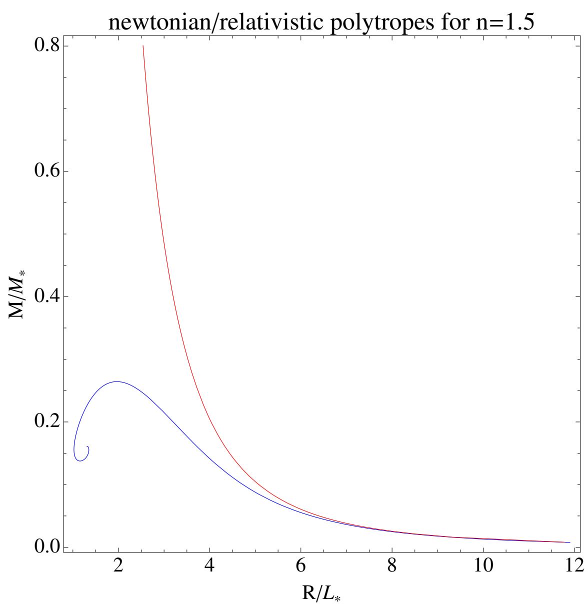
$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2.$$

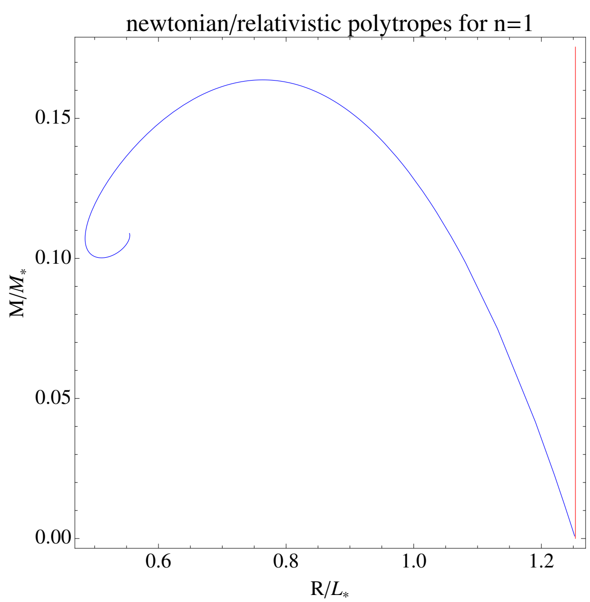
Realistic equations of state

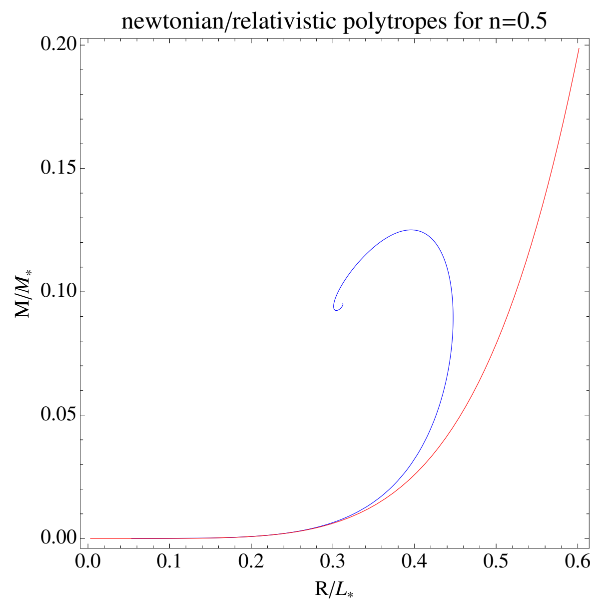




Polytropic equations of state (newtonian polytropes: $M \propto R^{\frac{n-3}{n-1}}$)







* SLB1,2 are observationally inferred EoSs

(A.W. Steiner, J.M. Lattimer, and E.F. Brown, Astrophys. J. **722** 33 (2010)).

Relativistic rotating Stars

The line element for a stationary and axially symmetric spacetime (the spacetime admits a timelike, ξ^a, and a spacelike, η^a, killing field, i.e., it has rotational symmetry and symmetry in translations in time) is [1],

$$ds^2 = -e^{2\nu}dt^2 + r^2 \sin^2\theta B^2 e^{-2\nu}(d\phi - \omega dt)^2 + e^{2(\zeta-\nu)}(dr^2 + r^2 d\theta^2).$$

Field equations in the frame of the ZAMOs:

$$\mathbf{D}\cdot(B\mathbf{D}\nu) = \frac{1}{2} r^2 \sin^2\theta B^3 e^{-4\nu}\mathbf{D}\omega\cdot\mathbf{D}\omega + 4\pi B e^{2\zeta-2\nu}\left[\frac{(\epsilon+p)(1+u^2)}{1-u^2} + 2p\right],$$

$$\mathbf{D}\cdot(r^2\sin^2\theta B^3 e^{-4\nu}\mathbf{D}\omega) = -16\pi r \sin\theta B^2 e^{2\zeta-4\nu}\frac{(\epsilon+p)u}{1-u^2},$$

$$\mathbf{D}\cdot(r\sin\theta\mathbf{D}B) = 16\pi r\sin\theta B e^{2\zeta-2\nu}p,$$

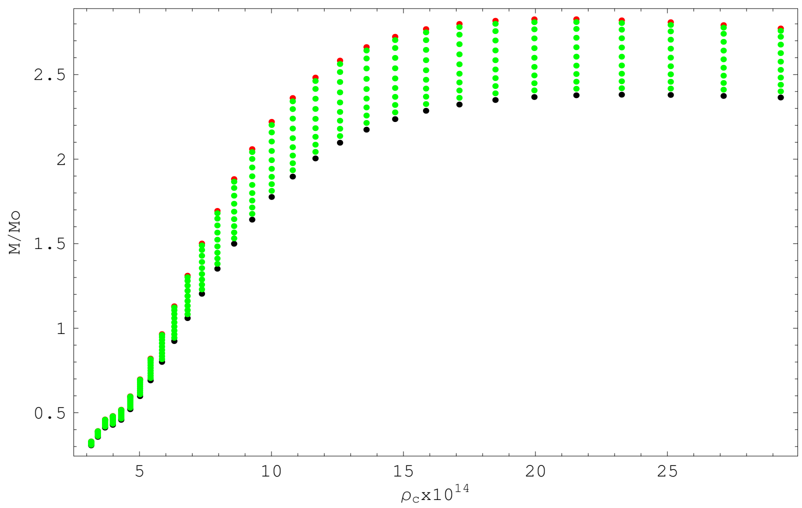
Komatsu, Eriguchi, and Hechisu[2] proposed a scheme for integrating the field equations using Green's functions. This scheme is implemented by the RNS numerical code to calculate rotating neutron stars [3].

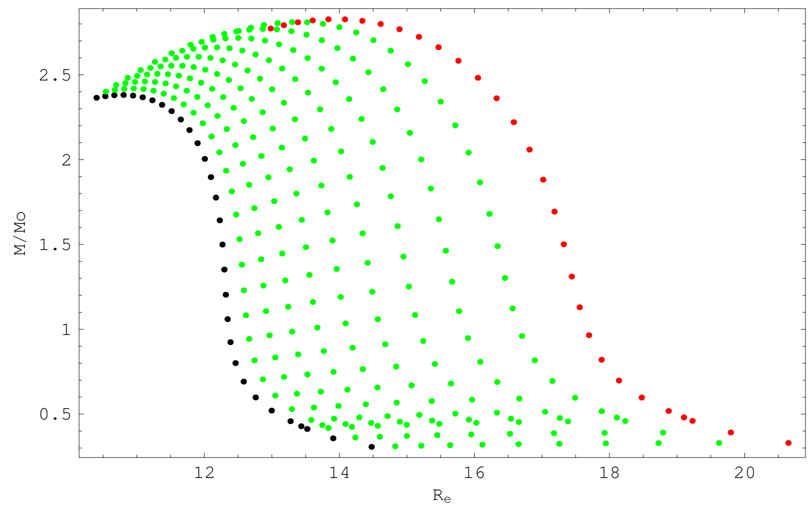
[1] E. M. Butterworth and J. R. Ipser, ApJ **204**, 200 (1976).

[2] H. Komatsu, Y. Eriguchi, and I. Hechisu, MNRAS **237**, 355 (1989).

[3] N. Stergioulas, J.L. Friedman, ApJ, **444**, 306 (1995).

One can use RNS to calculate models of rotating neutron stars for a given equation of state. For example we show here some models for the APR EOS:





The models with the fastest rotation have a spin parameter, $j = J/M^2$, around 0.7 and a ratio of the polar radius over the equatorial radius, r_p/r_e, around 0.56.

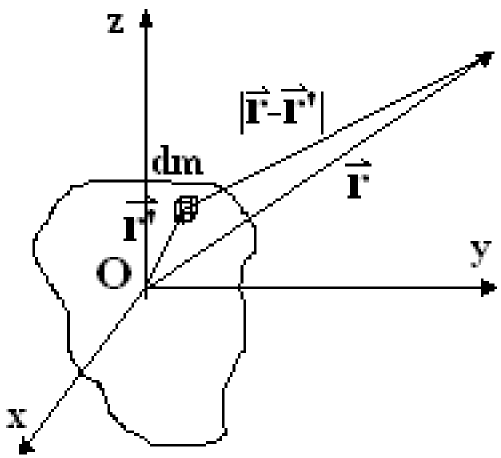
The code, except from the various physical characteristics of the neutron stars, provides the metric functions in a grid on the coordinates x and μ in the whole space (for values from 0 to 1 for both variables), where $\mu = cos\theta$, $r = \frac{xr_e}{1-x}$ and r_e is a length scale.

The spacetime outside the neutron star is calculated by evaluating numerically the coefficients in the expansion of the metric functions:

$$\begin{aligned}
\nu &= \left\{-\frac{M}{r}+\frac{1}{3}\tilde{B}_0\frac{M}{r^3}+\frac{J^2}{r^4}+\left[-\frac{\tilde{B}_0^2}{5}+\frac{\tilde{B}_2^2}{15}-\frac{12J^2}{5}\right]\frac{M}{r^5}+\ldots\right\} \\
&\quad +\left\{\frac{Q_2}{r^3}-2\frac{J^2}{r^4}+[\ldots]\frac{1}{r^5}+\ldots\right\}P_2(\mu) \\
&\quad +\left\{\frac{Q_4}{r^5}+\ldots\right\}P_4(\mu)+\ldots \\
\omega &= \left\{\frac{2J}{r^3}-\frac{6JM}{r^4}+\frac{6}{5}\left[8-3\frac{\tilde{B}_0}{M^2}\right]\frac{JM^2}{r^5}+(\ldots)\frac{J}{r^6}+\ldots\right\}P_{1,\mu}(\mu) \\
&\quad +\left\{\frac{W_2}{r^5}+(\ldots)\frac{1}{r^6}-\ldots\right\}P_{3,\mu}(\mu)+\ldots \\
B &= \left(\frac{\pi}{2}\right)^{1/2}\left(1+\frac{\tilde{B}_0}{r^2}\right)T_0^{1/2}(\mu)+\left(\frac{\pi}{2}\right)^{1/2}\frac{\tilde{B}_2}{r^4}T_2^{1/2}(\mu)+\ldots
\end{aligned}$$

where P_l are the Legendre polynomials, $\mu=\cos\theta$, and $T_l^{1/2}$ are the Gegenbauer polynomials.

The metric function ν is analogus to the Newtonian gravitational potential, while the metric function ω is the cause of the relativistic effect of dragging of the inertial frames. All metric functions are given numerically on a grid.



As we said, Newtonian gravity is described by the potential

$$\Phi(r) = G\int \frac{\rho(r')dV'}{|\vec{r}-\vec{r}'|}$$

Multipolar Expansions
(the multipoles characterise the field)
In spherical coordinates:

$$\begin{aligned}\frac{1}{|\vec{r}-\vec{r}'|} &= \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta'}} = \frac{1}{r\sqrt{1+\varepsilon^2-2\varepsilon\cos\theta'}}\\ &= \frac{1}{r}+\frac{\cos\theta'\varepsilon}{r}+\frac{(1+3\cos 2\theta')\varepsilon^2}{4r}+\ldots\end{aligned}$$

$$\Phi(r) = G\left(\frac{1}{r}\int\rho(r')dV' + \frac{1}{r^2}\int r'\cos\theta'\rho(r')dV' + \frac{1}{r^3}\int\frac{1}{2}(3\cos^2\theta'-1)(r')^2\rho(r')dV'+\ldots\right)$$

In cartesian coordinates:

$$\begin{aligned}\frac{1}{|\vec{r}-\vec{r}'|} &= \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} = \frac{1}{r\sqrt{(x^a/r-\varepsilon^a)(x^b/r-\varepsilon^b)\delta_{ab}}}\\ &= \frac{1}{r}+\frac{x^a\varepsilon_a}{r^2}+\frac{\frac{1}{2}(3x^ax^b-r^2\delta^{ab})\varepsilon_a\varepsilon_b}{r^3}+\ldots = \frac{1}{r}+\frac{x^ax'_a}{r^3}+\frac{\frac{1}{2}(3x'_ax'_b-r'^2\delta_{ab})x^ax^b}{r^5}+\ldots\end{aligned}$$

$$\Phi(r) = G\left(\frac{1}{r}\int\rho(r')dV' + \frac{x^a}{r^3}\int x'_a\rho(r')dV' + \frac{x^ax^b}{r^5}\int\frac{1}{2}(3x'_ax'_b-r'^2\delta_{ab})\rho(r')dV'+\ldots\right)$$

Newtonian multipole moments:

$$\Phi(r) = G\left(\frac{\mathcal{Q}}{r} + \frac{\mathcal{Q}_a x^a}{r^3} + \frac{\mathcal{Q}_{ab} x^a x^b}{r^5} + ...\right) \tag{1}$$

where, $\mathcal{Q}$, $\mathcal{Q}_a$, $\mathcal{Q}_{ab}$, are some integrals on the source

$$\mathcal{Q} = \int \rho(r')d^3x',\ \mathcal{Q}_a = \int x'_a \rho(r')d^3x',\ \mathcal{Q}_{ab} = \int \frac{3}{2}(x'_a x'_b - \frac{1}{3}r'^2\delta_{ab})\rho(r')d^3x'... \tag{2}$$

The multipole moments are generally tensorial quantities.

Definition of the moments at infinity:

$x^a \to \tilde{x}^a = r^{-2}x^a$: $\tilde{r}^2 = \tilde{x}^a \tilde{x}_a = r^{-2}$

$$\Phi(r) = \tilde{r}\left(\mathcal{Q} + \mathcal{Q}_a \tilde{x}^a + \mathcal{Q}_{ab}\tilde{x}^a \tilde{x}^b + ...\right) \tag{3}$$

If we define the potential at infinity $\tilde{\Phi} = \tilde{r}^{-1}\Phi$ then the moments are

$$P_{a_1...a_n} = \tilde{D}_{a_n} P_{a_1...a_{n-1}} = \tilde{D}_{a_1}...\tilde{D}_{a_n}\tilde{\Phi} \tag{4}$$

In General Relativity instead of a gravitational field Φ, gravity is described by a metric g_{ab}.

Relativistic multipole moments:

- Generalization of the Newtonian moments,
- Defined for asymptotically flat spacetimes at infinity from a "potential", that is related to the metric, by a recursive relation,
- There are two sets of moments, the Mass moments and the Rotation moments,
- For the two sets of moments we have two generating potentials, Φ_M, Φ_J,

The multipole moments for stationary and axisymmetric spacetimes can be reduced from tensors to scalars, because of the rotation symmetry.

The moments characterize the structure of the source and of the spacetime.

$$
\begin{aligned}
P &= \tilde{\Phi}, \\
P_a &= \tilde{D}_a P, \\
&\vdots \\
P_{a_1 \ldots a_{s+1}} &= \mathcal{C}\left[\tilde{D}_{a_1} P_{a_2 \ldots a_{s+1}} - \frac{s(2s-1)}{2} \tilde{R}_{a_1 a_2} P_{a_3 \ldots a_{s+1}}\right],
\end{aligned}
$$

Multipole moments of numerical spacetimes

The RNS code can calculates the first non-zero multipole moments, i.e., M, $S_1 \equiv J$, M_2, S_3 and M_4[4].

$$\begin{aligned}
M_0 &= M, \\
S_1 &= jM^2, \\
M_2 &= -\frac{1+4b_0+3q_2}{3}M^3, \\
S_3 &= -\frac{3(2j+8jb_0-5w_2)}{10}M^4, \\
M_4 &= \frac{19-18j^2+160b_0+120q_2+336b_0^2+360b_0q_2-105q_4-192b_2}{105}M^5
\end{aligned}$$

where $j \equiv J/M^2$, and the various parameters are given by the integrals,

$$\begin{aligned}
Q_{2l} = M^{2l+1}q_{2l} &= -\frac{r_e^{2l+1}}{2}\int_0^1 \frac{ds' s'^{2l}}{(1-s')^{2l+2}}\int_0^1 d\mu' P_{2l}(\mu')\tilde{S}_\rho(s',\mu'), \\
W_{2l-2} = M^{2l}w_{2l-2} &= -\frac{r_e^{2l}}{4l}\int_0^1 \frac{ds' s'^{2l}}{(1-s')^{2l+2}}\int_0^1 d\mu' \sin\theta' P^1_{2l-1}(\mu')\tilde{S}_{\hat{\omega}}(s',\mu'), \\
\tilde{B}_{2l} = M^{2l+2}b_{2l} &= -\frac{16\sqrt{2\pi}r_e^{2l+4}}{2l+1}\int_0^{1/2} \frac{ds' s'^{2l+3}}{(1-s')^{2l+5}}\int_0^1 d\mu' \sin\theta' p(s',\mu')Be^{2(\zeta-\nu)}T^{1/2}_{2l}(\mu'),
\end{aligned}$$

where in the second integral $l \geq 1$, in contrast to the other two integrals where $l \geq 0$.

[4]G.P. and T. A. Apostolatos, Phys. Rev. Lett. **108** 231104 (2012),
K. Yagi, K. Kyutoku, G. P., N. Yunes, and T.A. Apostolatos, Phys.Rev. D **89** 124013 (2014).

Neutron star multipole moments properties

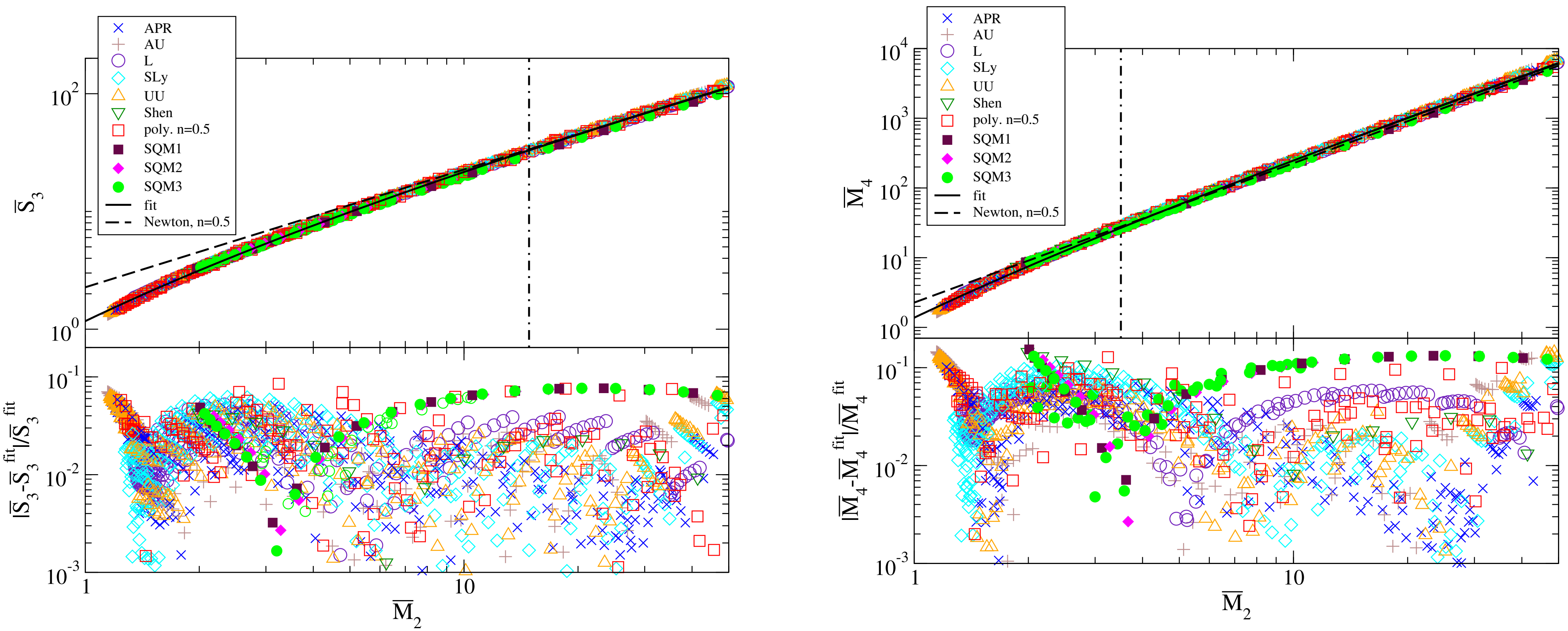
Black Hole-like behaviour of the moments[5]:

Kerr moments	Neutron star moments
$$\begin{aligned} M_0 &= M, \\ S_1 &= J = jM^2, \\ M_2 &= -j^2 M^3, \\ S_3 &= -j^3 M^4, \\ M_4 &= j^4 M^5, \\ &\vdots \\ M_{2n} &= (-1)^n j^{2n} M^{2n+1}, \\ S_{2n+1} &= (-1)^n j^{2n+1} M^{2n+2} \end{aligned}$$	$$\begin{aligned} M_0 &= M, \\ S_1 &= jM^2, \\ M_2 &= -a(EoS, M) j^2 M^3, \\ S_3 &= -\beta(EoS, M) j^3 M^4, \\ M_4 &= \gamma(EoS, M) j^4 M^5, \\ &\vdots \\ M_{2n} &= ?, \\ S_{2n+1} &= ? \end{aligned}$$

[5] W.G. Laarakkers and E. Poisson, Astrophys. J. **512** 282 (1999).
G.P. and T. A. Apostolatos, Phys. Rev. Lett. **108** 231104 (2012).
K. Yagi, K. Kyutoku, G. P., N. Yunes, and T.A. Apostolatos, Phys.Rev. D **89** 124013 (2014).

Neutron star multipole moments properties

EoS independent behaviour of the moments[6]:



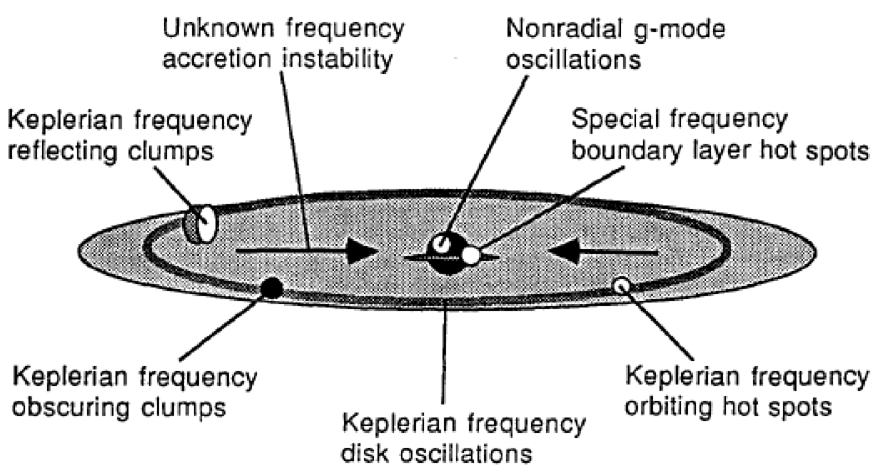
$$\bar{M}_{2n} = |M_{2n}/(j^{2n}M^{2n+1})|,\ \bar{S}_{2n+1} = |S_{2n+1}/(j^{2n+1}M^{2n+2})|$$

All these are properties that characterize the spacetime around neutron stars as well as the stars themselves.

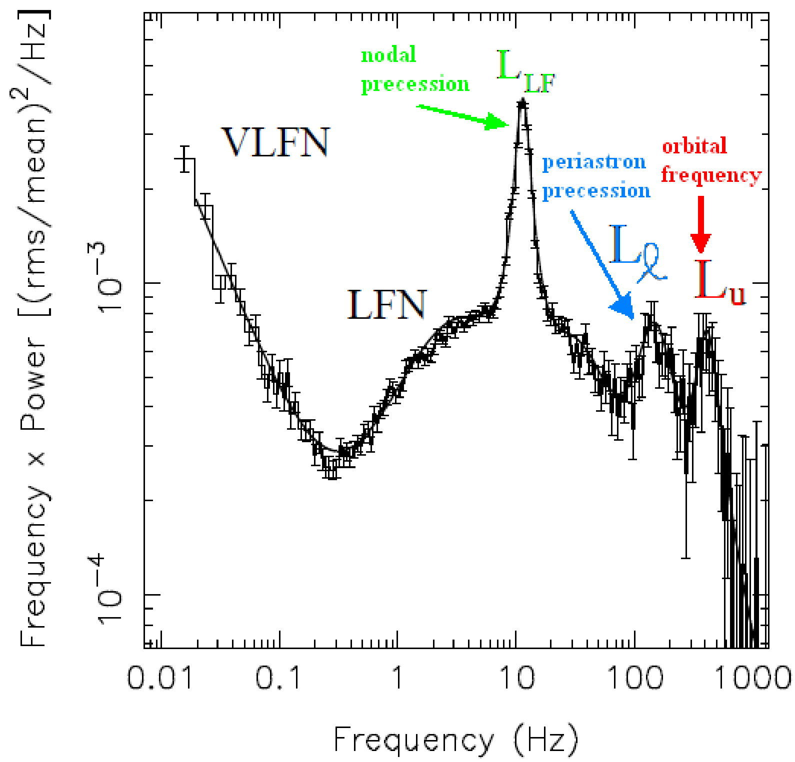
[6]G.P. and T. A. Apostolatos, Phys.Rev.Lett. **112** 121101 (2014)].
K. Yagi, K. Kyutoku, G. P., N. Yunes, and T.A. Apostolatos, Phys.Rev. D **89** 124013 (2014).

There is a variety of phenomena that take place in the spacetime around neutron stars. A class of these that can be related to orbits in an accretion disc (and geodesic motion) are the quasi-periodic oscillations (QPOs) of the spectrum[7] of the disc,

Mechanisms for producing QPOs[8] from orbital motion



Typical X-Ray spectrum[9]



Apart from orbiting hot spots and oscillations on the disc, one could also have precessing rings or misaligned precessing discs, which either themselves have a modulated emission or they are eclipsing the emission from the central object.

[7]Stella & Vietri, 1998, ApJ, 492, L59.

[8]F.K. Lamb, Advances in Space Research, 8 (1988) 421.

[9]Boutloukos et al., 2006, ApJ, 653, 1435-1444.

An analytic spacetime would be very useful to study geodesics.

The vacuum region of a stationary and axially symmetric space-time can be described by the Papapetrou line element[10],

$$ds^2 = -f\left(dt - w d\phi\right)^2 + f^{-1}\left[e^{2\gamma}\left(d\rho^2 + dz^2\right) + \rho^2 d\phi^2\right],$$

where f, w, and γ are functions of the Weyl-Papapetrou coordinates (ρ, z).

By introducing the complex potential $\mathcal{E}(\rho, z) = f(\rho, z) + i\psi(\rho, z)$ [11], the Einstein field equations take the form,

$$(Re(\mathcal{E}))\nabla^2\mathcal{E} = \nabla\mathcal{E}\cdot\nabla\mathcal{E},$$

where, $f = \xi^a\xi_a$ and ψ is defined by, $\nabla_a\psi = \varepsilon_{abcd}\,\xi^b\nabla^c\xi^d$.

An algorithm for generating solutions of the Ernst equation was developed by Sibgatullin and Manko [12]. A solution is constructed from a choice of the Ernst potential along the axis of symmetry in the form of a rational function

$$\mathcal{E}(\rho = 0, z) = e(z) = \frac{P(z)}{R(z)},$$

where $P(z), R(z)$ are polynomials of z of order n with complex coefficients in general.

[10] A. Papapetrou, Ann. Phys., **12**, 309 (1953).

[11] F.J. Ernst, Phys. Rev., **167**, 1175 (1968); Phys. Rev., **168**, 1415 (1968).

[12] V.S. Manko, N.R. Sibgatullin, Class. Quantum Grav., **10**, 1383 (1993).

Two-Soliton spacetime: This is a 4-parameter analytic spacetime which can be produced if one chooses the Ernst potential on the axis to have the form:

$$e(z) = \frac{(z - M - ia)(z + ib) - k}{(z + M - ia)(z + ib) - k}$$

The parameters a, b, k of the spacetime can be related to the first non-zero multipole moments through the equations,

$$J = aM, \quad M_2 = -(a^2 - k)M, \quad S_3 = -\left[a^3 - (2a - b)k\right] M,$$

where M is the mass.

One can use the multipole moments M, J, M_2, and S_3 of a numerically calculated neutron star and produce an analytic two-soliton spacetime that reproduces very accurately the numerically calculated spacetime around the neutron star, as well as the relevant properties of the geodesics in that spacetime, such as the various orbital and precession frequencies, the position of the ISCO and the energetics of the orbits [13].

[13] G. P., and T. A. Apostolatos, MNRAS, **429**, 3007 (2013); Other analytic spacetimes have been proposed in the past, see: E. Berti, and N. Stergioulas, MNRAS, **350**, 1416 (2004)

In general, we assume that the spacetime around compact objects has symmetry with respect to time translations and with respect to rotations around an axis of symmetry, which is associated to the axis of rotation of the compact object.

Particle motion in a spacetime with symmetries:

The symmetry in time translations implies the existence of a timelike Killing vector ξ^a and the symmetry in rotations implies the existence of a spacelike Killing vector η^a.
The four-momentum of a particle is defined as $p^a = mu^a = m\frac{dx^a}{d\tau}$.

Symmetry in time translations is associated to an integral of motion, energy E

$$E = -p_a\xi^a = -p_t = -g_{tt}p^t - g_{t\phi}p^\phi = m\left(-g_{tt}\frac{dt}{d\tau} - g_{t\phi}\frac{d\phi}{d\tau}\right) \tag{5}$$

Symmetry in rotations is again associated to an integral of motion, angular momentum L

$$L = p_a\eta^a = p_\phi = g_{t\phi}p^t + g_{\phi\phi}p^\phi = m\left(g_{t\phi}\frac{dt}{d\tau} + g_{\phi\phi}\frac{d\phi}{d\tau}\right) \tag{6}$$

From the measure of the four-momentum, $p^a p_a = -m^2$, we have the equation,

$$-1 = g_{tt}\left(\frac{dt}{d\tau}\right)^2 + 2g_{t\phi}\left(\frac{dt}{d\tau}\right)\left(\frac{d\phi}{d\tau}\right) + g_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2 + g_{\rho\rho}\left(\frac{d\rho}{d\tau}\right)^2 + g_{zz}\left(\frac{dz}{d\tau}\right)^2 \tag{7}$$

Circular equatorial orbits:

If we define the angular velocity, $\Omega \equiv \frac{d\phi}{dt}$, for the circular and equatorial orbits equation (7) defines the redshift factor (have you seen "Interstellar"?),

$$\left(\frac{d\tau}{dt}\right)^2 = -g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2, \tag{8}$$

and the energy and the angular momentum for the circular orbits take the form,

$$\tilde{E} \equiv E/m = \frac{-g_{tt} - g_{t\phi}\Omega}{\sqrt{-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2}}, \quad \tilde{L} \equiv L/m = \frac{g_{t\phi} + g_{\phi\phi}\Omega}{\sqrt{-g_{tt} - 2g_{t\phi}\Omega - g_{\phi\phi}\Omega^2}}. \tag{9}$$

From the conditions, $\frac{d\rho}{dt} = 0$, $\frac{d^2\rho}{dt^2} = 0$ and $\frac{dz}{dt} = 0$, and the equations of motion obtained assuming the Lagrangian, $L = \frac{1}{2}g_{ab}\dot{x}^a\dot{x}^b$, the angular velocity can be calculated to be,

$$\Omega = \frac{-g_{t\phi,\rho} + \sqrt{(g_{t\phi,\rho})^2 - g_{tt,\rho}g_{\phi\phi,\rho}}}{g_{\phi\phi,\rho}}. \tag{10}$$

This is the orbital frequency of a particle in a circular orbit on the equatorial plane.

More general orbits:

Equation (7) can take a more general form in terms of the constants of motion,

$$-g_{\rho\rho}\left(\frac{d\rho}{d\tau}\right)^2 - g_{zz}\left(\frac{dz}{d\tau}\right)^2 = 1 - \frac{\tilde{E}^2 g_{\phi\phi} + 2\tilde{E}\tilde{L}g_{t\phi} + \tilde{L}^2 g_{tt}}{(g_{t\phi})^2 - g_{tt}g_{\phi\phi}} = V_{eff}, \quad (11)$$

With equation (11) we can study the general properties of the motion of a particle from the properties of the effective potential.

Small perturbations from circular equatorial orbits:

If we assume small deviations from the circular equatorial orbits of the form, $\rho = \rho_c + \delta\rho$ and $z = \delta z$, then we obtain the perturbed form of (11),

$$-g_{\rho\rho}\left(\frac{d(\delta\rho)}{d\tau}\right)^2 - g_{zz}\left(\frac{d(\delta z)}{d\tau}\right)^2 = \frac{1}{2}\frac{\partial^2 V_{eff}}{\partial\rho^2}(\delta\rho)^2 + \frac{1}{2}\frac{\partial^2 V_{eff}}{\partial z^2}(\delta z)^2,$$

This equation describes two harmonic oscillators with frequencies,

$$\bar{\kappa}_\rho^2 = \frac{g^{\rho\rho}}{2}\frac{\partial^2 V_{eff}}{\partial\rho^2}\bigg|_c, \quad \bar{\kappa}_z^2 = \frac{g^{zz}}{2}\frac{\partial^2 V_{eff}}{\partial z^2}\bigg|_c,$$

The differences of these frequencies (corrected with the redshift factor (8)) from the orbital frequency, $\Omega_a = \Omega - \kappa_a$, define the precession frequencies.

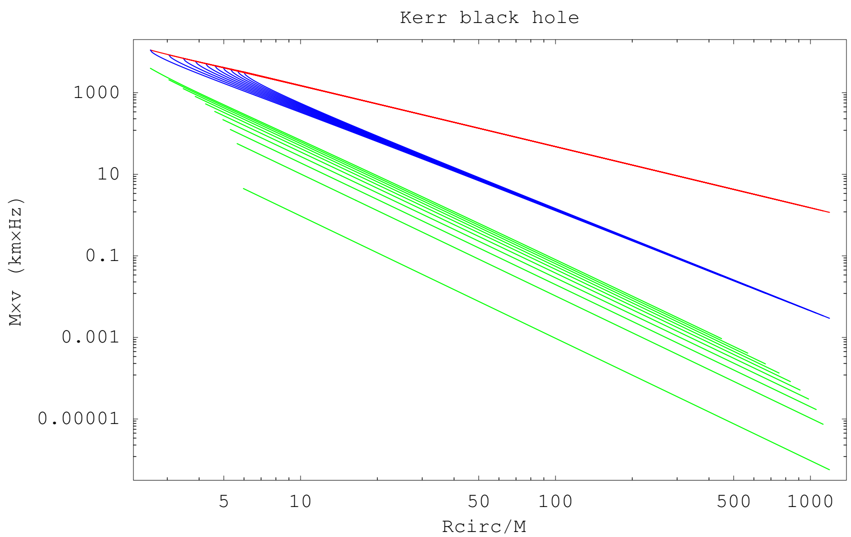
"Scaled" Frequencies for the Kerr spacetime:

Scaled Orbital frequency: $M\Omega = \frac{299790\sqrt{1/r^3}}{1+j(1/r)^{3/2}}$

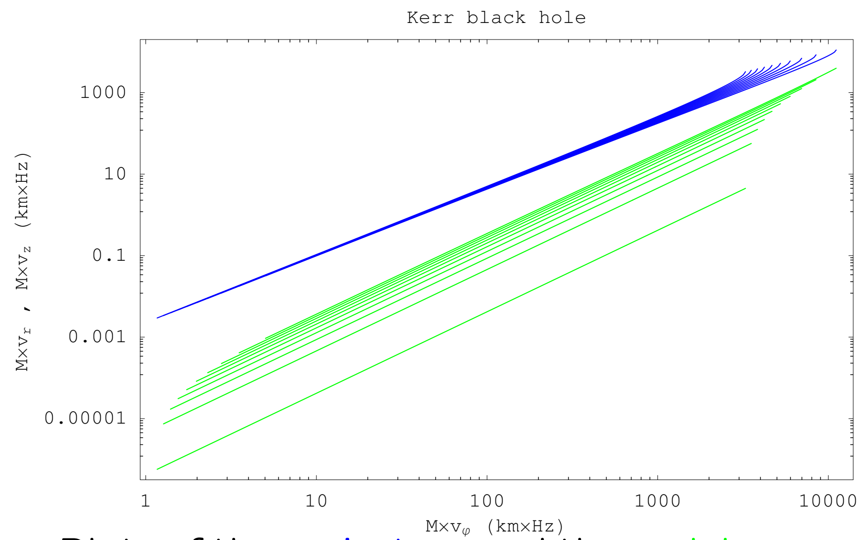
Scaled Radial frequency: $M\kappa_\rho = M\Omega\left(1 - 6(1/r) + 8j(1/r)^{3/2} - 3j^2(1/r)^2\right)^{1/2}$

Scaled Vertical frequency: $M\kappa_z = M\Omega\left(1 - 4j(1/r)^{3/2} + 3j^2(1/r)^2\right)^{1/2}$

Orbital, $M\nu_\phi$, and precession, $M\nu_a$, "scaled frequencies" for Kerr black holes for various j (0.01-0.91).



Plots of the orbital, periastron and nodal precession "scaled frequencies".



Plots of the periastron and the nodal precession "scaled frequencies" against the orbital "scaled frequency".

The general effect of rotation is to increase the observed frequencies (and reduce the ISCO radius; for $j \sim 1$ the horizon and ISCO radii tend to $1M$).

Frequencies for the spacetime around neutron stars:

Orbital frequency: $\Omega = \frac{-g_{t\phi,\rho} + \sqrt{(g_{t\phi,\rho})^2 - g_{tt,\rho} g_{\phi\phi,\rho}}}{g_{\phi\phi,\rho}}$

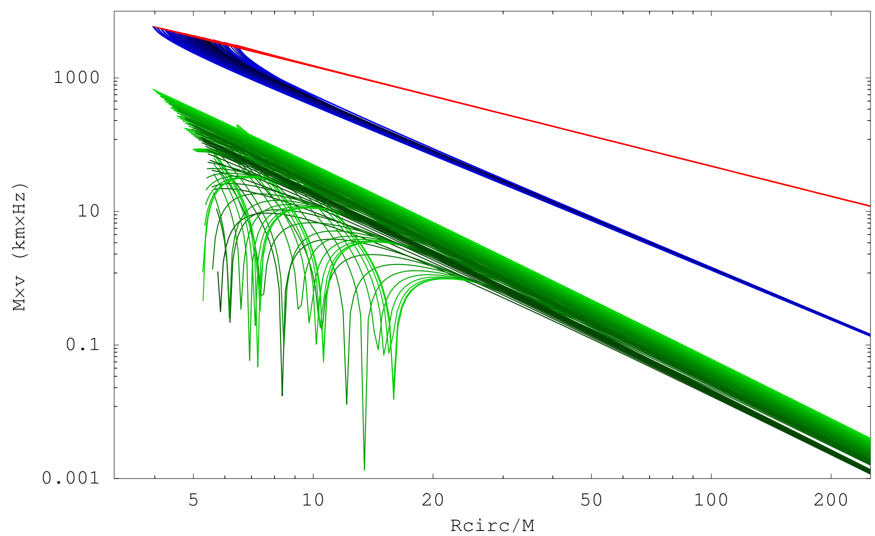
Radial frequency:

$$\kappa_\rho = \left(-\frac{g^{\rho\rho}}{2} \left\{ (g_{tt} + g_{t\phi}\Omega)^2 \left(\frac{g_{\phi\phi}}{\rho^2}\right)_{,\rho\rho} - 2(g_{tt} + g_{t\phi}\Omega)(g_{t\phi} + g_{\phi\phi}\Omega) \left(\frac{g_{t\phi}}{\rho^2}\right)_{,\rho\rho} + (g_{t\phi} + g_{\phi\phi}\Omega)^2 \left(\frac{g_{tt}}{\rho^2}\right)_{,\rho\rho} \right\} \right)^{1/2}$$

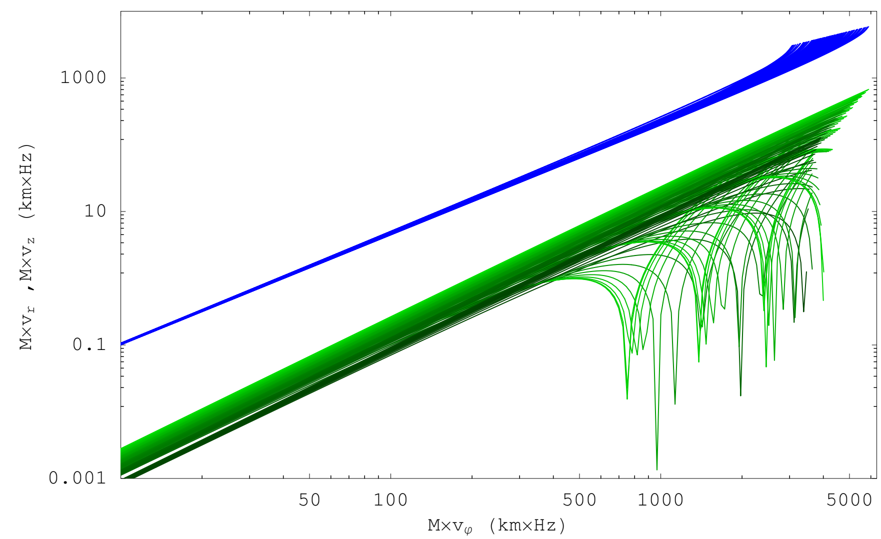
Vertical frequency:

$$\kappa_z = \left(-\frac{g^{zz}}{2} \left\{ (g_{tt} + g_{t\phi}\Omega)^2 \left(\frac{g_{\phi\phi}}{\rho^2}\right)_{,zz} - 2(g_{tt} + g_{t\phi}\Omega)(g_{t\phi} + g_{\phi\phi}\Omega) \left(\frac{g_{t\phi}}{\rho^2}\right)_{,zz} + (g_{t\phi} + g_{\phi\phi}\Omega)^2 \left(\frac{g_{tt}}{\rho^2}\right)_{,zz} \right\} \right)^{1/2}$$

Orbital, $M\nu_\phi$, and precession, $M\nu_a$, "scaled frequencies" for neutron star models (132) constructed with the APR EOS for various j up to the Kepler limit (0-0.7) and masses from $1M_\odot$ up to $2.5M_\odot$.



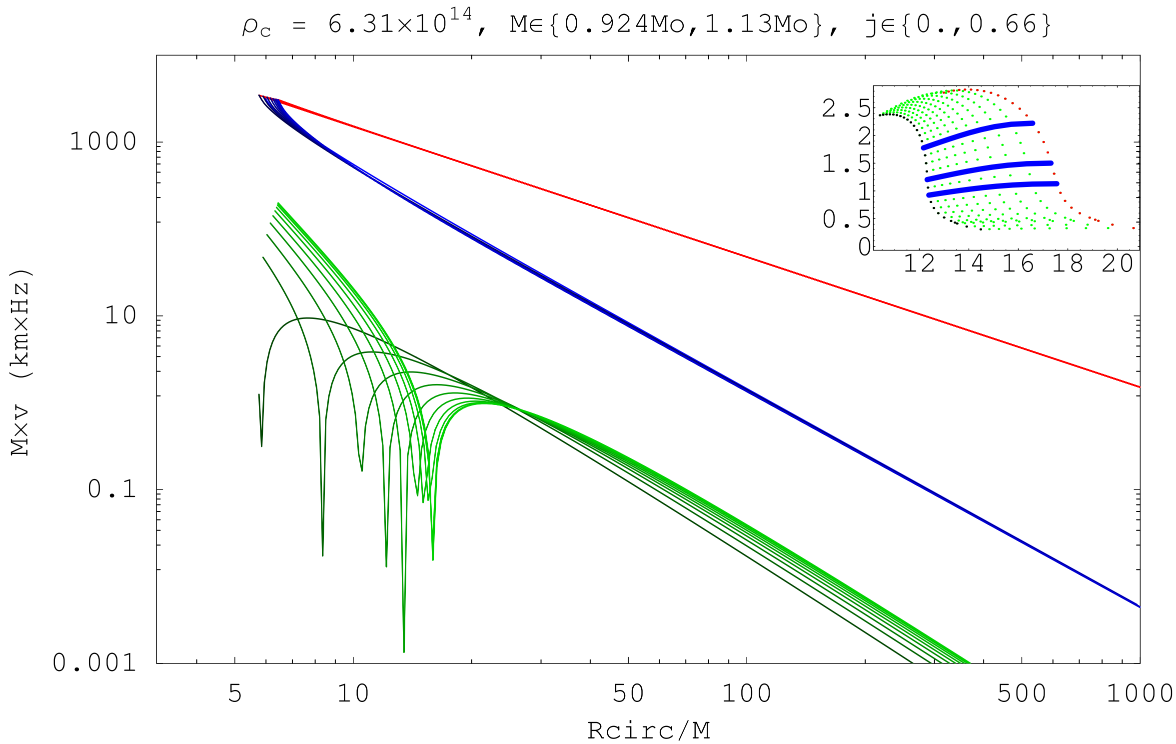
Plots of the orbital, periastron and nodal precession frequencies.



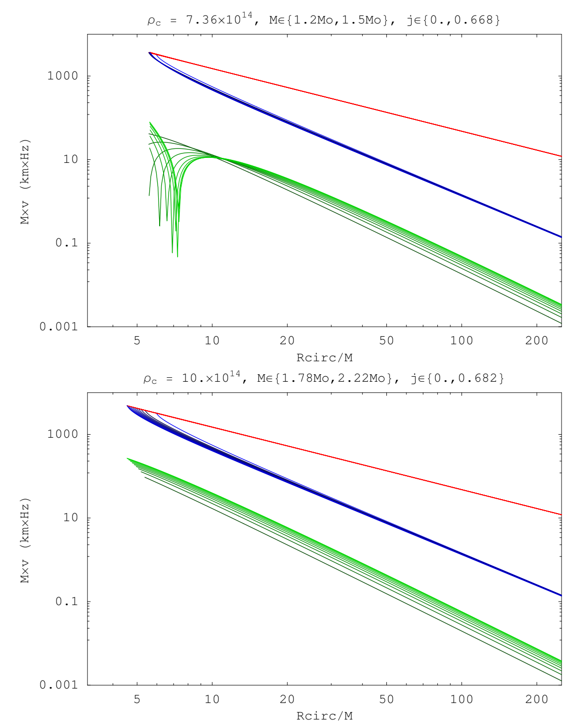
Plots of the periastron and the nodal precession frequencies against the orbital frequency.

Frequencies for the spacetime around neutron stars: The effect of rotation

Orbital frequency, periastron precession frequency and nodal precession frequency:
Orbital, $M\nu_\phi$, and precession, $M\nu_a$, "scaled frequencies" for neutron star models constructed with the APR EOS for various j up to the Kepler limit(0-0.7) and the same central density.



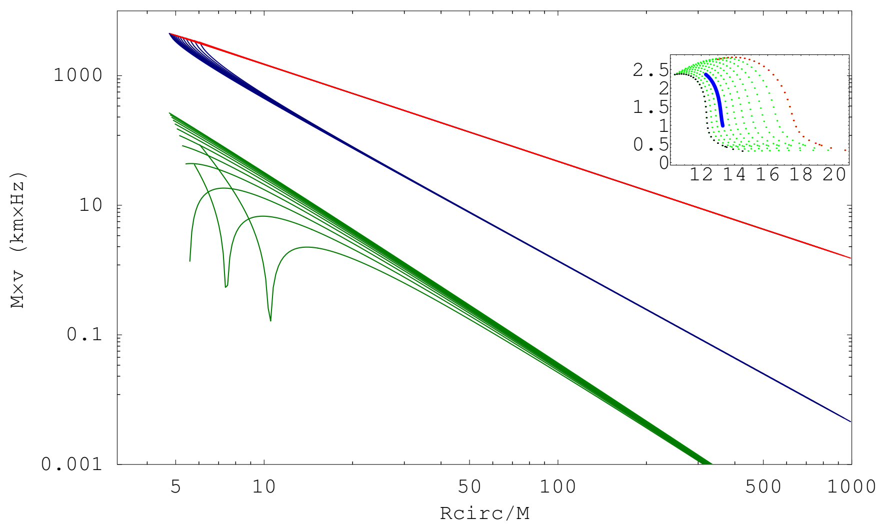
Plots of the orbital, periastron and nodal precession frequencies for different rotations for models with $\rho_c = 6.3 \times 10^{14} g/cm^3$. Rotation in the range, $f \sim 0.3 - 0.9$kHz.

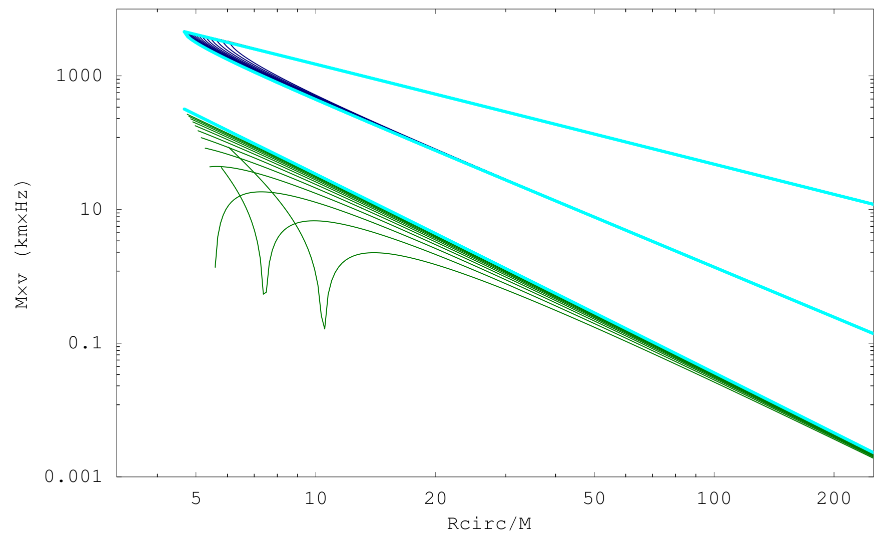


Same plots for $\rho_c = 7.3 \times 10^{14} g/cm^3$ (upper) and $\rho_c = 10^{15} g/cm^3$ (lower).

Frequencies for the spacetime around neutron stars: Effect of higher moments

Orbital frequency, periastron precession frequency and nodal precession frequency:
Orbital, $M\nu_\phi$, and precession, $M\nu_a$, "scaled frequencies" for neutron star models constructed with the APR EOS for approximately the same $j = 0.39$ and different higher moments.



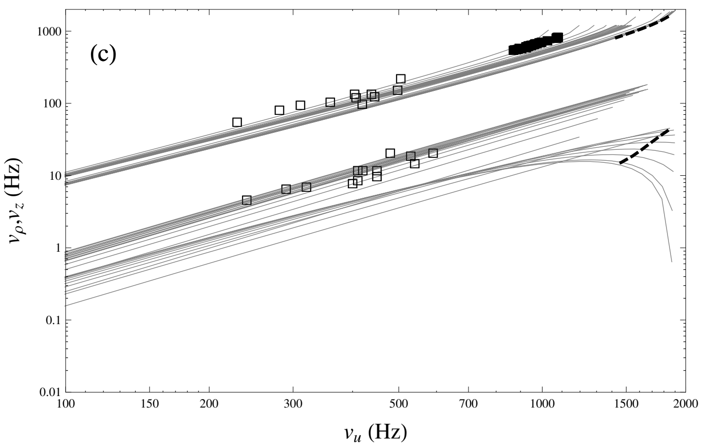


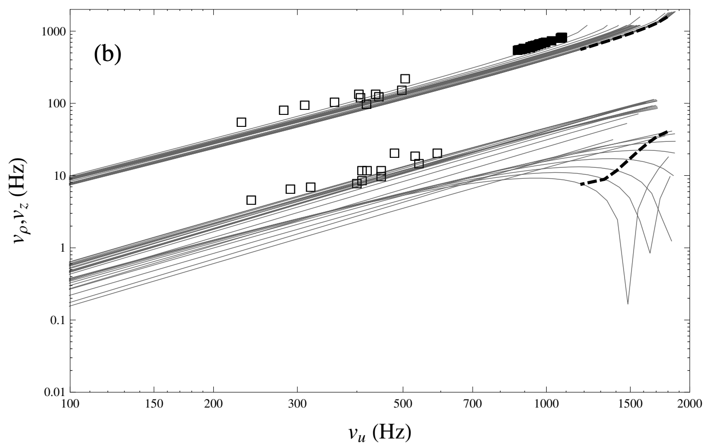
M	0.986	1.133	1.288	1.447	1.604	1.755	1.896	2.021	2.368
j	0.3837	0.382	0.381	0.3809	0.3817	0.3834	0.3861	0.3894	0.4069
q	-1.427	-1.132	-0.905	-0.7334	-0.6044	-0.508	-0.4362	-0.3836	-0.2781
j^2	-0.1472	-0.1459	-0.1452	-0.1451	-0.1457	-0.147	-0.149	-0.1516	-0.1655
f	$\sim 0.5 - 1.2$kHz with the rotation frequency increasing with the increasing mass								

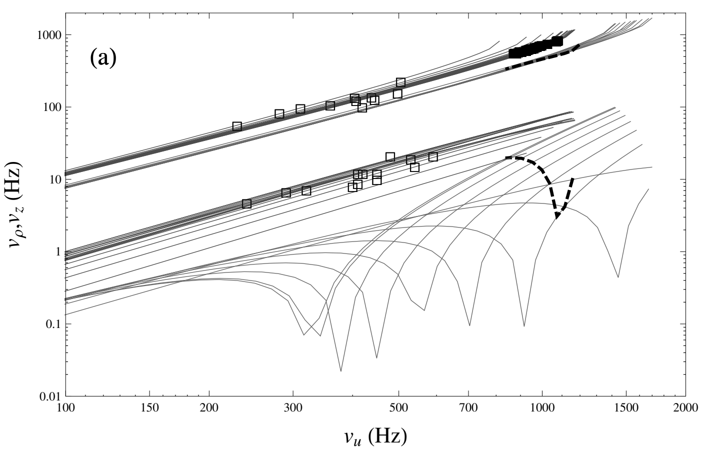
The general effect of a quadrupole being higher than that of Kerr is to make κ_z higher than Ω in the region near the ISCO. The size of that region increases with q.

Frequencies for the spacetime around neutron stars: Effect of the EOS [14]

Plots of the precession frequencies against the orbital frequency for 3 NS sequences of various rotations for AU, FPS and L EOS. The models of the sequences have masses equal to $1.4M_{\odot}$, the mass of the maximum mass non-rotating model that the EOS admits and the mass of a hyper-massive model without non-rotating limit.







The general effect of the EOS is that stiffer EOSs produce higher quadrupole. Kerr black holes behave like a super-soft star.

[14]G.P., 2012 MNRAS, 422, 2581-2589.

The characteristics of the orbits in a spacetime can be associated to its multipole moments.

The precession frequencies are related to the multipole moments,

$$\frac{\Omega_\rho}{\Omega} = 3\upsilon^2 - 4\frac{S_1}{M^2}\upsilon^3 + \left(\frac{9}{2} - \frac{3M_2}{2M^3}\right)\upsilon^4 - 10\frac{S_1}{M^2}\upsilon^5 + \left(\frac{27}{2} - 2\frac{S_1^2}{M^4} - \frac{21M_2}{2M^3}\right)\upsilon^6 + ...$$

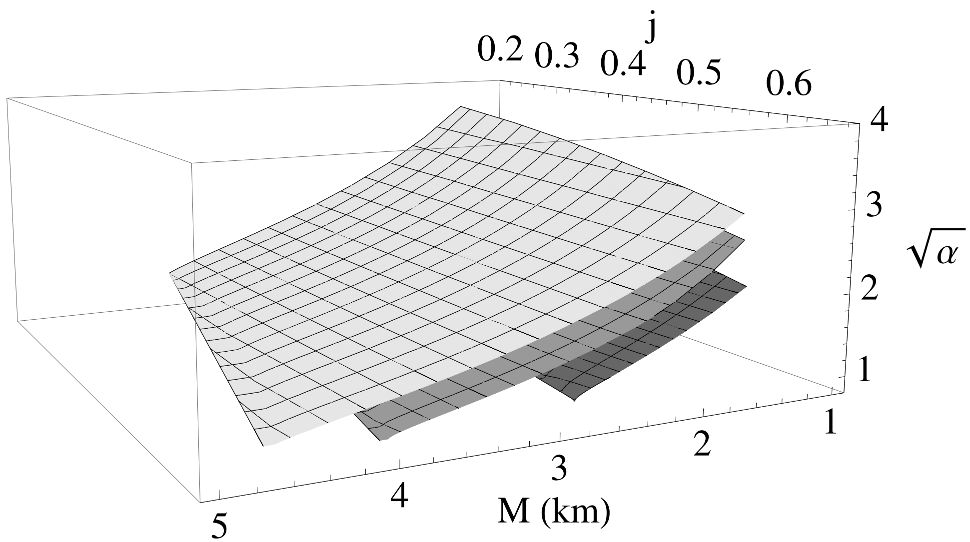
$$\frac{\Omega_z}{\Omega} = 2\frac{S_1}{M^2}\upsilon^3 + \frac{3M_2}{2M^3}\upsilon^4 + \left(7\frac{S_1^2}{M^4} + 3\frac{M_2}{M^3}\right)\upsilon^6 + \left(11\frac{S_1M_2}{M^5} - 6\frac{S_3}{M^4}\right)\upsilon^7 + ...$$

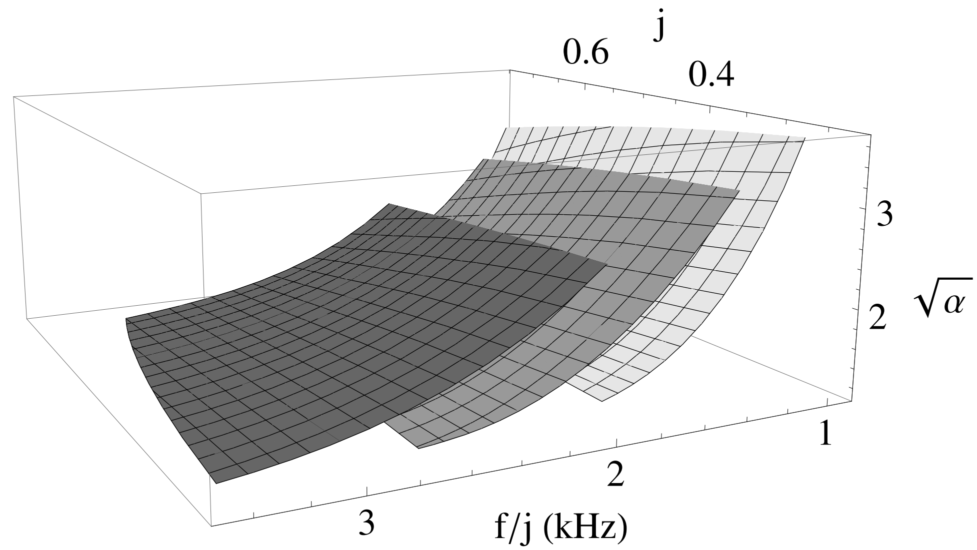
where $\upsilon = (M\Omega)^{1/3}$ and $S_1 = J$.

If specific QPOs are associated to the orbital frequencies, they could then be used to measure the multipole moments of the central object[15] and from the moments, probe its structure.

[15] G.P., 2012 MNRAS, 422, 2581-2589.

An application example: A "measurement" of the first 3 moments (M, J, Q) could select an EoS[16] out of the realistic EOSs candidates.





[16] G.P. and T. A. Apostolatos, Phys.Rev.Lett. **112** 121101 (2014)

Thank You.