The Schwarzschild metric: It's the coordinates, stupid!

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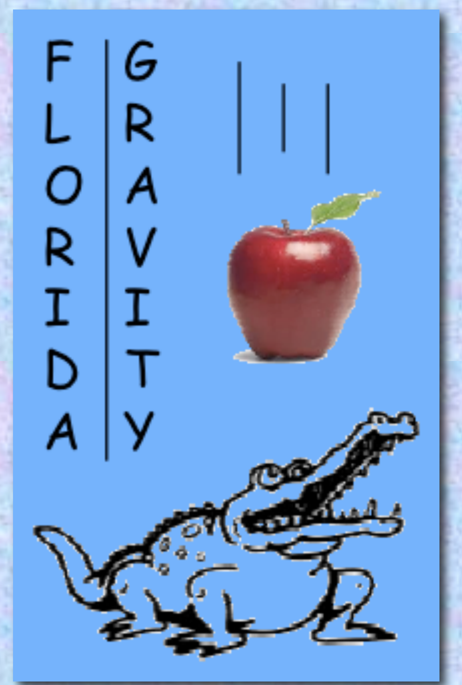
It's the coordinates, stupid!*



*During Bill Clinton's first campaign for the US Presidency, his top advisor James Carville tried to keep him focused on talking about the first Bush administration's economic failures by repeatedly telling him: "It's the economy, stupid!"

Finding the Schwarzschild metric: The normal way

$$ds^2 = -e^{-2\Phi}\,dt^2 + \frac{dr^2}{1-2m(r)/r} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2),$$

Calculate the Christoffel symbols, Riemann, Ricci and Einstein tensors

Vacuum Einstein eq'ns

$$G^0{}_0 = -\frac{2}{r^2}\,\partial_r m = 0$$

$$G^r{}_r = -\frac{2}{r}\left(1-\frac{2m}{r}\right)\partial_r\Phi - \frac{2m}{r^3} = 0.$$

Voilà!

$$m(r) = GM/c^2 = \text{constant}$$

$$\Phi(r) = -\frac{1}{2}\ln\left(1 - 2GM/c^2 r\right)$$

The "relaxed" Einstein Equations

$$\mathfrak{g}^{\alpha\beta} \equiv \sqrt{-g}\, g^{\alpha\beta} \qquad \partial_\beta \mathfrak{g}^{\alpha\beta} = 0$$

Lorenz gauge
Harmonic coordinates

Einstein's equations: $$\Box \mathfrak{g}^{\alpha\beta} = \frac{16\pi G}{c^4}\tau^{\alpha\beta}$$

$$\tau^{\alpha\beta} = (-g)\left(T^{\alpha\beta} + t^{\alpha\beta}_{\mathrm{LL}} + t^{\alpha\beta}_{\mathrm{H}}\right)$$

$$\frac{16\pi G}{c^4}(-g)t^{\alpha\beta}_{\mathrm{H}} = \left(\partial_\mu \mathfrak{g}^{\alpha\nu}\partial_\nu \mathfrak{g}^{\beta\mu} - (\mathfrak{g}^{\mu\nu} - \eta^{\mu\nu})\partial_{\mu\nu}\mathfrak{g}^{\alpha\beta}\right)$$

Finding the Schwarzschild metric: Harmonic coordinates*

Static, spherical symmetry:

$$\mathfrak{g}^{00} = N(r)\,,$$
$$\mathfrak{g}^{0j} = 0\,,$$
$$\mathfrak{g}^{ij} = \alpha(r)P^{ij} + \beta(r)n^i n^j \qquad P^{ij} = \delta^{ij} - n^i n^j$$

Lorenz gauge

$$\frac{d\beta}{dr} = \frac{2}{r}(\alpha - \beta)$$

*Exercise 8.9 of Poisson-Will, *Gravitation: Newtonian, post-Newtonian, Relativistic*

Pieces of the field equations

Left-hand side

$$\Box \mathfrak{g}^{00} = \nabla^2 N$$

$$\Box \mathfrak{g}^{ij} = \nabla^2 \mathfrak{g}^{ij}$$

Right-hand side

$$\frac{16\pi G}{c^4}(-g)t^{00}_{\rm LL} = N\left[\frac{7}{8}\beta\frac{N'^2}{N^2} + \frac{3}{8}\frac{\beta'^2}{\beta} - \frac{1}{2}\frac{\alpha'\beta'}{\alpha} + \frac{1}{2}\beta\frac{N'\alpha'}{N\alpha} + \frac{1}{4}\frac{N'\beta'}{N}\right]$$

$$\frac{16\pi G}{c^4}(-g)t^{ij}_{\rm LL} = P^{ij}\left[\frac{3}{8}\alpha\frac{\beta'^2}{\beta} + \beta\frac{\alpha'^2}{\alpha} - \frac{1}{8}\alpha\beta\frac{N'^2}{N^2} + \frac{1}{2}\beta\frac{N'\alpha'}{N} + \frac{1}{4}\alpha\frac{N'\beta'}{N} + \frac{1}{2}\alpha'\beta'\right]$$
$$+ n^i n^j\left[\frac{1}{8}\beta'^2 + \frac{1}{2}\beta\frac{\alpha'\beta'}{\alpha} + \frac{1}{8}\beta^2\frac{N'^2}{N^2} - \frac{1}{2}\beta^2\frac{N'\alpha'}{N\alpha} - \frac{1}{4}\beta\frac{N'\beta'}{N}\right]$$

$$\frac{16\pi G}{c^4}(-g)t^{00}_{\rm H} = \nabla^2 N - 2\alpha\frac{N'}{r} - \beta N''$$

$$\frac{16\pi G}{c^4}(-g)t^{ij}_{\rm H} = \nabla^2\mathfrak{g}^{ij} - P^{ij}\left[\beta\alpha'' + 2\alpha\frac{\alpha'}{r} + \alpha'\beta' - \alpha\frac{\beta'}{r}\right]$$
$$+ n^i n^j\left[\beta'^2 + 2\beta\frac{\beta' - \alpha'}{r} - 3\alpha\frac{\beta'}{r}\right]$$

3 differential equations

Define: $X \equiv \frac{\alpha'}{\alpha}\,, \quad Y \equiv \frac{\beta'}{\beta}\,, \quad Z \equiv \frac{N'}{N}\,,$

3 equations

$$X' + XY + \frac{1}{r}(2X - Y) = Q \quad (1)$$

$$XY + \frac{1}{r}(2X + Y) = -Q \quad (2)$$

$$Z' + YZ + \frac{2}{r}Z = Q \quad (3)$$

$$Q := \frac{1}{8}\Big(3Y^2 - Z^2 + 2YZ + 4XZ - 4XY\Big)$$

Solutions

Equation (1)+(2): $\frac{X'}{X} + 2\frac{\beta'}{\beta} + \frac{4}{r} = 0$ implies $r^4\beta^2\frac{\alpha'}{\alpha} = c$

Case 1: c=0

$$\alpha = 1 \qquad \beta' = \frac{2}{r}(1-\beta) \qquad \boxed{\beta = 1 - \frac{b}{r^2}}$$

Subst X=0 in Eq. (2): $\left(\frac{N'}{N} - \frac{\beta'}{\beta}\right)^2 = 4\frac{\beta'^2}{\beta^2} + 8\frac{\beta'}{r\beta}$ $\quad \boxed{N = \left(1 - \frac{b}{r^2}\right)\left(\frac{r-\sqrt{b}}{r+\sqrt{b}}\right)^{\pm 2}}$

Newtonian limit $\quad g_{00} = \sqrt{-g}\,g_{00} \to -1 \mp \frac{2\sqrt{b}}{r} + O(r^{-2}) \qquad \sqrt{b} = \mp GM/c^2$

$$\boxed{\begin{aligned} N &= -\frac{(1+GM/c^2r)^3}{1-GM/c^2r} \\ \alpha &= 1 \\ \beta &= 1 - \left(\frac{GM}{c^2r}\right)^2 \end{aligned}}$$

The metric in harmonic coordinates

$$g_{00} = -\frac{1 - GM/c^2r}{1 + GM/c^2r}$$

$$g_{ij} = (1 + GM/c^2r)^2 P^{ij} + \frac{1 + GM/c^2r}{1 - GM/c^2r} n^i n^j$$

Obtain from Schwarzschild coordinates by substituting

$$r_S = r + GM/c^2$$

The other solution

Equation (1)+(2): $\frac{X'}{X} + 2\frac{\beta'}{\beta} + \frac{4}{r} = 0$ implies $r^4\beta^2\frac{\alpha'}{\alpha} = c$

Case 2: c ≠0

$$W(r) \equiv r^2\beta$$

$$W'' - \frac{W'}{r} = c\frac{W'}{W^2}$$

Pierre has tried:

- Change of variables

$$W(r) = G(\rho) \qquad \rho = a\ln r$$

$$G(\rho) = e^{d\rho}H(\rho)$$

- Abel's resolution

$$p(G) = G'$$

$$\omega'(z)\omega(z) - \omega(z) = \frac{3}{8}\left(z - \sqrt{z^2 + 4c}\right)$$

- Inversion $r(W)$:

$$r(W)r(W)'' + r(W)'^2 = -c\frac{r(W)r(W)'^2}{W^2}$$

What does the other solution mean?

$$\partial_\beta \mathfrak{g}^{\alpha\beta} = \partial_\beta \left(\sqrt{-g} g^{\gamma\beta} \partial_\gamma X_H^{(\alpha)} \right) = 0 \qquad \Box_g X_H^{(\alpha)} = 0$$

$$X_H = r(r_S) \sin\theta \cos\phi$$
$$Y_H = r(r_S) \sin\theta \sin\phi$$
$$Z_H = r(r_S) \cos\theta \qquad \tilde{M} = GM/c^2$$

$$(r_S^2 - 2\tilde{M} r_S) r'' + 2(r_S - \tilde{M}) r' - 2r = 0$$

Legendre's equation (L=1), with solution:

$$r = a P_1(r_S/\tilde{M} - 1) + b Q_1(r_S/\tilde{M} - 1)$$
$$= a'(r_S - \tilde{M}) + b' \left[(r_S - \tilde{M}) \ln(1 - 2\tilde{M}/r_S) + 2\tilde{M} \right]$$

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