# Stability of approximate Killing fields and black hole spin

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# Introduction

- The gravitational field does not admit a local stress-energy tensor.
- Example quasi-local integrals for angular momentum

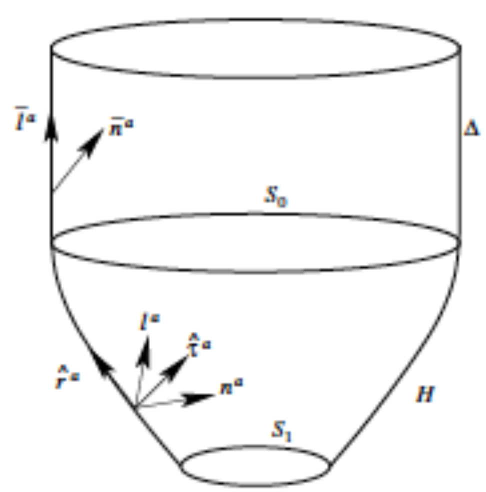
$$J_{\mathrm{Gen}}[S,\phi] := \frac{1}{8\pi G}\oint_S \mathrm{dA}\ \hat{r}^a \phi^b K_{ab}$$

$$J_{\mathrm{BY}}[S,\phi] := -\oint_S \mathrm{dA}\ \hat{r}^a \phi^b \tau_{ab}$$

  - Unique formulae do not exist without a preferred vector field.
- Rotational symmetry generating vector fields, when admitted, provide a natural way of specifying angular momentum.

$$\mathcal{L}_\xi g^{ab} = -2\nabla^{(a}\xi^{b)} \equiv 0$$

- Dynamic horizons provide a rich environment to study quasi-local integrals.



# The Most Metric Preserving Vector Fields

- Minimize this integral to construct approximate Killing fields:

$$I[\Xi] = \frac{1}{2}\int_M \mathrm{d^n x}[\frac{1}{2}(\mathcal{L}_\Xi g_{ab})(\mathcal{L}_\Xi g^{ab}) + k_\Xi(\Xi_a \Xi^a - V^{-1})]$$

  - The constant Lagrange multiplier $k_\Xi$ avoids trivial solutions $\Xi^a = 0$.
  - This action leads to the following eigenproblem

$$\Delta_K \Xi^a := -2\nabla_b \nabla^{(b}\Xi^{a)} = k_\Xi \Xi^a$$

- The Killing Laplacian is Hermitian, (strongly) elliptic, positive and has a complete and discrete spectrum with solutions accumulating at infinity.
- Approximate isometries correspond to the lowest eigenmodes.
- I will argue that eigenfields of the Killing Laplacian are stable under small geometric changes to $M$.

Matzner R A 1968 *Journal of Mathematical Physics* 9
G Lovelace, R Owen, H Pfeiffer, T Chu; Physical Review D 78 (8), 084017
G Cook and B Whiting; Phys. Rev. D 76, pp. 041501(R) (2007)
Beetle, C; arXiv:0808.1745 [gr-qc]

# Defining AKVF Stability

- The approximate Killing field changes under a perturbation of the geometry. The magnitude of the change is given by

$$||\dot{\xi}|| = ||[\boldsymbol{\Delta}_{\mathbf{K}} - \boldsymbol{\kappa}]^{-1}_{\xi^{\perp}}[\dot{\boldsymbol{\Delta}}_{\mathbf{K}} - \dot{\boldsymbol{\kappa}}]\xi|| \sim ||\dot{\boldsymbol{\Delta}}_{\mathbf{K}}\xi||$$

- Strategy for bounding the perturbed Killing Laplacian:

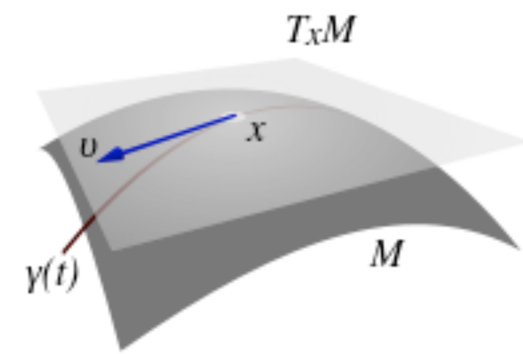
  1. Demonstrate that norms of $1^{st}$ derivatives of an arbitrary vector field are bound by the Killing Laplacian.
  2. Bound the perturbed Killing Laplacian by a Sobolev inequality.
     1. One can estimate the constant appearing in this bound.
  3. Bound $2^{nd}$ order Sobolev norms by the Killing Laplacian

$$||\dot{\boldsymbol{\Delta}}_{K}\phi|| \leq C_1||\phi||_{W^2_2(\nabla)} \leq C_2||\boldsymbol{\Delta}_{K}\phi|| + ||\mathbf{l.o.t.}||$$

# First-Order Terms

- Focus on the 1st order terms of the Sobolev norm

$$||\nabla\phi||^2 = \int_M \mathrm{d^n x}\ \nabla_a\phi_b\nabla^a\phi^b$$
$$\leq \langle\phi, [\Delta_K + R]\phi\rangle - ||\nabla\cdot\phi||^2$$
$$\leq \langle\phi, \Delta_K\phi\rangle + r_{\max}||\phi||^2$$




  - The Ricci tensor operator is bounded.
- Calculating the bound of the Ricci operator.
  - The following inequality holds for (positive) symmetric bi-linear forms:

$$\mathrm{Tr}(BC) \leq \mathrm{B_{max}}\mathrm{Tr}(C) \leq \mathrm{Tr}(B)\mathrm{Tr}(C)$$

  - Apply this inequality in a point-wise sense to the Ricci term.

$$\langle\phi, R\phi\rangle = (R, \phi\phi) \leq \max_{TM} \sup_{T_xM} \{r\} \int_M \mathrm{d^n x}\ \phi_a\phi^a = r_{\max}||\phi||^2$$

- Result: The Killing Laplacian bounds the 1st order Sobolev.

$$||\phi||^2_{W_1^2(\nabla)} \leq \langle\phi, \Delta_K\phi\rangle + (1 + r_{\max})||\phi||^2$$

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# Bounding the Perturbed Killing Laplacian

- The previous methods can be used to manage 2nd order terms of the perturbed Killing Laplacian.

$$\dot{\boldsymbol{\Delta}}_K \phi^a = Z^a{}_B (\nabla\nabla\phi)^B + Y^a{}_M (\nabla\phi)^M + X^a{}_b \phi^b$$

  - The coefficients are built from the perturbed metric and its derivatives.

- Management is easier in terms of norms instead of tensors.

$$\begin{aligned} ||\dot{\boldsymbol{\Delta}}_K \phi|| &\leq ||Z(\nabla\nabla\phi)|| + ||Y(\nabla\phi)|| + ||X\phi|| \\ &= \sqrt{\langle \nabla\nabla\phi, \mathcal{Z}(\nabla\nabla\phi)\rangle} + \sqrt{\langle \nabla\phi, \mathcal{Y}(\nabla\phi)\rangle} + \sqrt{\langle \phi, \mathcal{X}\phi\rangle} \end{aligned}$$

  - The coefficient operators are bounded "multiplication" operators.

$$||\dot{\boldsymbol{\Delta}}_K \phi|| \leq C(\dot{g}, \partial\dot{g}, \partial\partial\dot{g})||\phi||_{W_2^2(\nabla)}$$

  - Since the coefficient operators are positive, one can simply use traces maximized over the tangent bundle to determine their bounds.

- Result: Sobolev norms bound smoothly varying elliptic operators.

# The Hessian is Bounded by the Killing Laplacian

- Douglis-Nirenberg demonstrated that under specific conditions
$$||\phi||_{W_2^2(\partial)} \leq C_L(||L\phi|| + ||\phi||_{W_1^2(\partial)})$$
for a $2^{nd}$ order elliptic operator $L$, but the constant is unknown!

- Only the highest order terms need to be controlled.
  - After some manipulation, one finds that
$$||H\phi||^2 \leq ||\Delta_K\phi||^2 + \langle\nabla\phi, \mathcal{U}(\nabla\phi)\rangle + \langle\mathcal{V}\phi, \nabla\phi\rangle - \langle\phi, R\phi\rangle$$
or $$||H\phi|| \leq ||\Delta_K\phi|| + C'(\partial g, \partial\partial g, \partial\partial\partial g)||\phi||_{W_1^2(\nabla)}$$

- Result: Norms of $2^{nd}$ order elliptic operators acting on arbitrary vector fields are bound in a ***known*** way by the Killing Laplacian.

# Conclusions

- The perturbed Killing Laplacian is relatively bounded by the Killing Laplacian
  $||\dot{\xi}|| \sim ||\dot{\boldsymbol{\Delta}}_{\mathbf{K}}\xi|| \leq \kappa C(\dot{g}, \partial\dot{g}, \partial g, \partial\partial g) + C''(\dot{g}, \partial\dot{g}, \partial\partial\dot{g}, \partial g, \partial\partial g, \partial\partial\partial g)$
  - This result also provides a bound for the change of the eigenvalue.
- Generally - Eigenfields of smoothly varying elliptic operators on compact manifolds are stable.

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## Thank You


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