Second-Order Self-Force Calculations

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Introduction

- First-Order Self-Force Review
- The Good Stuff: Second-Order Enhancements
- Second-Order Perturbations
- Difficulties and Challenges

First-Order Self-Force in Review: Results

Assume a particle mass μ is in quasi-stationary circular orbit around a Schwarzschild black hole with mass M.

- Numerical values for:

$$h^{1\mathrm{ret}}, h^{1\mathrm{S}}, h^{1\mathrm{R}}.$$

- Self-Force is $\mathcal{O}(\mu)$ and generated by the Regular field:

$$u^b \nabla_b u^a = -(g_0^{ab} + u^a u^b) u^c u^d (\nabla_c h^{1\mathrm{R}}_{db} - \tfrac{1}{2} \nabla_b h^{1\mathrm{R}}_{cd}).$$

- Physical spacetime local to the particle:

$$g_{ab} = g^0_{ab} + h^{1\mathrm{R}}.$$

- Worldline of the particle is perturbed away from the background geodesic, $\gamma_0(\tau)$, by an $\mathcal{O}(\mu)$ correction,

$$\gamma_0 \to \gamma_0 + \gamma_{1\mathrm{R}},$$

  which is determined by the first-order geodesic equation.

Second-Order Enhancements: Energy

- To zeroth-order, the total energy of this binary system is simply M.
- From the perturbed geodesic equation,

$$\frac{\mathrm{d}u_a}{\mathrm{d}\tau} = \frac{1}{2} u^b u^c \frac{\partial}{\partial x^a}(g^0_{bc} + h^{1\mathrm{R}}_{bc}),$$

one finds,

$$\mu E = \mu \frac{(r - 2M)}{\sqrt{r(r-3M)}} \left[1 + \mathcal{O}(h^{1\mathrm{R}})\right] + \mathcal{O}(\mu^2).$$

- For $\ell = 0$ we find the $\mathcal{O}(\mu)$ correction to the energy is μE, which has no contribution from the self-force term $h^{1\mathrm{R}}_{ab}$.
- The self-force effects in the energy of the system first appear at $\mathcal{O}(\mu^2)$!

Second-Order Enhancements: Phase Evolution

- Define a gauge-invariant measure for the energy,

$$M_\infty = M + \mu E + \mathcal{O}(\mu^2),$$

which is chosen to be the Bondi mass.

- Given the orbital frequency, $\Omega(E)$, and the rate of energy-loss, dE/dt, the phase of the orbit may be expressed as,

$$\begin{aligned} \phi(t) &= \int_0^t \mathrm{d}t\, \Omega(E(t)) \\ &= \int_0^t \mathrm{d}t\, \left(\Omega_\mathrm{o} + t\left[\frac{\mathrm{d}\Omega}{\mathrm{d}E}\frac{\mathrm{d}E}{\mathrm{d}t}\right] + \cdots\right), \end{aligned}$$

where Ω_o is the frequency of the orbit at time $t = 0$.

- Regge &Wheeler (1957) give $E \approx E_{1\mathrm{st}} = -u_t$, and $\frac{\mathrm{d}E}{\mathrm{d}t}$ has been known to first-order since the 1970's.

Second-Order Enhancements: Phase Evolution

- Let the second-order corrections be given by Δ_{2nd}, then after integration,

$$\phi(t) = t\Omega_{\mathrm{o}} + \frac{1}{2}t^2\left[\frac{\mathrm{d}\Omega}{\mathrm{d}E}\frac{\mathrm{d}E}{\mathrm{d}t}\right]_{\mathrm{1st}}[1 + \Delta_{\mathrm{2nd}} + \cdots].$$

- To estimate the error in one full cycle of the orbit, we find the de-phasing timescale:

$$\mathrm{Error} \sim \frac{1}{2}t^2\left[\mathcal{O}\left(\frac{\mu}{M^3}\right)\right]_{\mathrm{1st}}\left(1 + \left[\mathcal{O}\left(\frac{\mu}{M}\right)\right]_{\mathrm{2nd}} + \cdots\right),$$

and by ignoring the higher-order terms above second-order, the $\mathcal{O}\left(\mu^2/M^2\right)$ contributions result in a de-phasing timescale,

$$t_{\mathrm{1st}} \sim M\sqrt{\frac{M}{\mu}} \qquad t_{\mathrm{2nd}} \sim \left(\frac{M}{\mu}\right)t_{\mathrm{1st}}.$$

A Second-Order Primer

- We now express the physical spacetime to second-order,

$$g_{ab} = g^0_{ab} + h^{1\mathrm{ret}}_{ab} + h^{2\mathrm{ret}}_{ab},$$

where $h^{n\,\mathrm{ret}} = \mathcal{O}(\mu^n)$ and $G_{ab}(g^0) = 0$.

- The full second-order problem may be expressed as,

$$G_{ab}(g^0 + h^{1\mathrm{ret}} + h^{2\mathrm{ret}}) = 8\pi T_{ab}(\gamma_0 + \gamma_{1\mathrm{R}}) + \mathcal{O}(\mu^3).$$

- By expanding about g^0 and defining:

$$G^{(n)}_{ab}(g, h) \equiv \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\lambda} G_{ab}(g + \lambda h)\right]_{\lambda=0},$$

we recover the second-order perturbation equation:

$$\begin{aligned} G^{(1)}_{ab}(g^0, h^{2\mathrm{ret}}) &= 8\pi T_{ab}(\gamma_0 + \gamma_{1\mathrm{R}}) - 8\pi T_{ab}(\gamma_0) \\ &\quad - G^{(2)}_{ab}(g^0, h^{1\mathrm{ret}}) + \mathcal{O}(\mu^3). \end{aligned}$$

A Second-Order Primer

- In general, the stress-energy of a point particle is given by,

$$T_{ab}(\gamma_0) = \mu \frac{u_a u_b}{\sqrt{-g}} \frac{\mathrm{d}\tau}{\mathrm{d}T} \delta^3[X^i - \gamma_0^i(T)].$$

- Taking a closer look at the stress-energy in the second-order equation,

$$\begin{aligned} T_{ab}(\gamma_0 + \gamma_{1\mathrm{R}}) - T_{ab}(\gamma_0) &= \mu\Delta\left(\frac{u_a u_b}{\sqrt{-g}} \frac{\mathrm{d}\tau}{\mathrm{d}T}\right)\delta^3[X^i - \gamma_0^i(T)] \\ &\quad -\mu \frac{\bar{u}_a \bar{u}_b}{\sqrt{-g^0}} \frac{\mathrm{d}\tau}{\mathrm{d}T} \gamma_{1\mathrm{R}}^j \frac{\partial}{\partial X^j} \delta^3[X^i - \gamma_0^i(T)]. \end{aligned}$$

  The Δ operator represents an $\mathcal{O}(\mu)$ change to the quantities within the parentheses.

- This term is $\mathcal{O}(\mu^2)$ in the perturbation, which is to be expected.

- Question: Are the integrability conditions for the second-order perturbation equation satisfied?

Integrable?

- Answer: Yes and No (and Yes)!

- Moving away from γ_0, the Einstein equations reduce to the form:

$$G^{(1)}_{ab}(g^0, h^{2\mathrm{ret}}) = -G^{(2)}_{ab}(g^0, h^{1\mathrm{ret}}) + \mathcal{O}(\mu^3),$$

  and the integrability condition $\nabla^a G^{(2)}_{ab}(g^0, h) = 0$ has been shown by Habisohn (1986) to be satisfied for a vacuum solution.

- On or near γ_0, the integration becomes tricky, due to the singular nature of $h^{1\mathrm{ret}}$ and $h^{2\mathrm{ret}}$ at the location of the particle.

- The divergence of these singular terms is finite and discontinuous on γ_0 due to the ignorance in our knowledge of the singular field.

- This ignorance, however, can be shown to only affect the $\mathcal{O}(\mu^3)$ calculations, and thus the problem is well-posed to second-order.

In Conclusion

- Second-order effects are well worth the effort!
- A correct description of the second-order stress-energy requires information obtained from the first-order self-force problem.
- The second-order equations are soluble to $\mathcal{O}(\mu^2)$ everywhere.

Thank You

Appendix: Equations

$$
\begin{aligned}
2G^{(1)}_{ab}(g,h) \;&=\; -\nabla^c\nabla_c h_{ab} - \nabla_a\nabla_b h + 2\nabla_{(a}\nabla^c h_{b)c} \\
&\quad -2R_a{}^c{}_b{}^d h_{cd} + g_{ab}(\nabla^c\nabla_c h - \nabla^c\nabla^d h_{cd})
\end{aligned}
$$

$$
\begin{aligned}
2G^{(2)}_{ab}(g,h) \;&=\; \tfrac{1}{2}\nabla_a h^{cd}\nabla_b h_{cd} + h^{cd}\nabla_b\nabla_a h_{cd} + \tfrac{1}{2}\nabla_a h_b^{\ c}\nabla_c h \\
&\quad +\tfrac{1}{2}\nabla_b h_a^{\ c}\nabla_c h - \tfrac{1}{2}\nabla_c h\nabla^c h_{ab} - \nabla_a h_b^{\ c}\nabla_d h_c^{\ d} \\
&\quad -\nabla_b h_a^{\ c}\nabla_d h_c^{\ d} + \nabla^c h_{ab}\nabla_d h_c^{\ d} - h^{cd}\nabla_d\nabla_a h_{bc} \\
&\quad -h^{cd}\nabla_d\nabla_b h_{ac} + h^{cd}\nabla_d\nabla_c h_{ab} - h_{ab}\nabla_d\nabla_c h^{cd} \\
&\quad +h_{ab}\nabla_d\nabla^d h - \nabla_c h_{bd}\nabla^d h_a^{\ c} + \nabla_d h_{bc}\nabla^d h_a^{\ c} \\
&\quad +g_{ab}(h^{cd}\nabla_d\nabla_e h_c^{\ e} - h^{cd}\nabla_d\nabla_c h + \tfrac{1}{4}\nabla_d h\nabla^d h \\
&\quad +\nabla_c h^{cd}\nabla_e h_d^{\ e} - \nabla^d h\nabla_e h_d^{\ e} + h^{cd}\nabla_e\nabla_d h_c^{\ e} \\
&\quad -h^{cd}\nabla_e\nabla^e h_{cd} + \tfrac{1}{2}\nabla_d h_{ce}\nabla^e h^{cd} - \tfrac{3}{4}\nabla_e h_{cd}\nabla^e h^{cd})
\end{aligned}
$$