Second-Order Self-Force Calculations

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Introduction

- First-Order Self-Force Review
- The Good Stuff: Second-Order Enhancements
- Second-Order Perturbations
- Difficulties and Challenges
Assume a particle mass $\mu$ is in quasi-stationary circular orbit around a Schwarzschild black hole with mass $M$.

- Numerical values for: $h^{1\text{ret}}, h^{1S}, h^{1R}$.

- Self-Force is $\mathcal{O}(\mu)$ and generated by the Regular field:

$$u^b \nabla_b u^a = -(g^{ab}_0 + u^a u^b) u^c u^d (\nabla_c h^{1R}_{bd} - \frac{1}{2} \nabla_b h^{1R}_{cd}).$$

- Physical spacetime local to the particle:

$$g_{ab} = g^{0}_{ab} + h^{1R}.$$

- Worldline of the particle is perturbed away from the background geodesic, $\gamma_0(\tau)$, by an $\mathcal{O}(\mu)$ correction,

$$\gamma_0 \rightarrow \gamma_0 + \gamma^{1R},$$

which is determined by the first-order geodesic equation.
Second-Order Enhancements: Energy

• To zeroth-order, the total energy of this binary system is simply $M$.

• From the perturbed geodesic equation,

$$\frac{du_a}{d\tau} = \frac{1}{2} u^b u^c \frac{\partial}{\partial x^a} (g^0_{bc} + h^{1R}_{bc}),$$

one finds,

$$\mu E = \mu \frac{(r - 2M)}{\sqrt{r(r - 3M)}} \left[1 + O(h^{1R})\right] + O(\mu^2).$$

• For $\ell = 0$ we find the $O(\mu)$ correction to the energy is $\mu E$, which has no contribution from the self-force term $h^{1R}_{ab}$.

• The self-force effects in the energy of the system first appear at $O(\mu^2)$!
Second-Order Enhancements: Phase Evolution

- Define a gauge-invariant measure for the energy,

\[ M_\infty = M + \mu E + O(\mu^2), \]

which is chosen to be the Bondi mass.

- Given the orbital frequency, \( \Omega(E) \), and the rate of energy-loss, \( dE/dt \), the phase of the orbit may be expressed as,

\[ \phi(t) = \int_0^t dt \Omega(E(t)) = \int_0^t dt \left( \Omega_o + t \left[ \frac{d\Omega}{dE} \frac{dE}{dt} \right] + \cdots \right), \]

where \( \Omega_o \) is the frequency of the orbit at time \( t = 0 \).

- Regge & Wheeler (1957) give \( E \approx E_{1st} = -u_t \), and \( \frac{dE}{dt} \) has been known to first-order since the 1970's.
Second-Order Enhancements: Phase Evolution

- Let the second-order corrections be given by $\Delta_{2nd}$, then after integration,

$$
\phi(t) = t\Omega_0 + \frac{1}{2} t^2 \left[ \frac{d\Omega}{dE} \frac{dE}{dt} \right]_{1st} \left[ 1 + \Delta_{2nd} + \cdots \right].
$$

- To estimate the error in one full cycle of the orbit, we find the de-phasing timescale:

$$
\text{Error} \sim \frac{1}{2} t^2 \left[ \mathcal{O} \left( \frac{\mu}{M^3} \right) \right]_{1st} \left( 1 + \left[ \mathcal{O} \left( \frac{\mu}{M} \right) \right]_{2nd} + \cdots \right),
$$

and by ignoring the higher-order terms above second-order, the $\mathcal{O} \left( \mu^2 / M^2 \right)$ contributions result in a de-phasing timescale,

$$
t_{1st} \sim M \sqrt{\frac{M}{\mu}} \quad t_{2nd} \sim \left( \frac{M}{\mu} \right) t_{1st}.
$$
A Second-Order Primer

• We now express the physical spacetime to second-order,

\[ g_{ab} = g_{ab}^0 + h_{ab}^{1\text{ret}} + h_{ab}^{2\text{ret}}, \]

where \( h^{n \text{ret}} = \mathcal{O}(\mu^n) \) and \( G_{ab}(g^0) = 0. \)

• The full second-order problem may be expressed as,

\[ G_{ab}(g^0 + h^{1\text{ret}} + h^{2\text{ret}}) = 8\pi T_{ab}(\gamma_0 + \gamma_{1R}) + \mathcal{O}(\mu^3). \]

• By expanding about \( g^0 \) and defining:

\[ G_{ab}^{(n)}(g, h) \equiv \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} G_{ab}(g + \lambda h) \right]_{\lambda=0}, \]

we recover the second-order perturbation equation:

\[ G_{ab}^{(1)}(g^0, h^{2\text{ret}}) = 8\pi T_{ab}(\gamma_0 + \gamma_{1R}) - 8\pi T_{ab}(\gamma_0) - G_{ab}^{(2)}(g^0, h^{1\text{ret}}) + \mathcal{O}(\mu^3). \]
In general, the stress-energy of a point particle is given by,

\[ T_{ab}(\gamma_0) = \mu \frac{u_a u_b}{\sqrt{-g}} \frac{d\tau}{dT} \delta^3[X^i - \gamma_0^i(T)]. \]

Taking a closer look at the stress-energy in the second-order equation,

\[ T_{ab}(\gamma_0 + \gamma_{1R}) - T_{ab}(\gamma_0) = \mu \Delta \left( \frac{u_a u_b}{\sqrt{-g}} \frac{d\tau}{dT} \right) \delta^3[X^i - \gamma_0^i(T)] \]

\[ -\mu \frac{\bar{u}_a \bar{u}_b}{\sqrt{-g_0}} \frac{d\tau}{dT} \gamma_{1R} \frac{\partial}{\partial X_j} \delta^3[X^i - \gamma_0^i(T)]. \]

The \( \Delta \) operator represents an \( \mathcal{O}(\mu) \) change to the quantities within the parentheses.

This term is \( \mathcal{O}(\mu^2) \) in the perturbation, which is to be expected.

**Question:** Are the integrability conditions for the second-order perturbation equation satisfied?
Integrable?

- **Answer:** Yes and No (and Yes)!

- Moving away from $\gamma_0$, the Einstein equations reduce to the form:

  $$G_{ab}^{(1)}(g^0, h^{2\text{ret}}) = -G_{ab}^{(2)}(g^0, h^{1\text{ret}}) + \mathcal{O}(\mu^3),$$

  and the integrability condition $\nabla^a G_{ab}^{(2)}(g^0, h) = 0$ has been shown by Habisohn (1986) to be satisfied for a vacuum solution.

- On or near $\gamma_0$, the integration becomes tricky, due to the singular nature of $h^{1\text{ret}}$ and $h^{2\text{ret}}$ at the location of the particle.

- The divergence of these singular terms is finite and discontinuous on $\gamma_0$ due to the ignorance in our knowledge of the singular field.

- This ignorance, however, can be shown to only affect the $\mathcal{O}(\mu^3)$ calculations, and thus the problem is well-posed to second-order.
In Conclusion

• Second-order effects are well worth the effort!

• A correct description of the second-order stress-energy requires information obtained from the first-order self-force problem.

• The second-order equations are soluble to $O(\mu^2)$ everywhere.

Thank You
Appendix: Equations

\[ 2G^{(1)}_{ab}(g, h) = -\nabla^c \nabla_c h_{ab} - \nabla_a \nabla_b h + 2\nabla_{(a} \nabla^c h_{b)c} - 2R^{c \, d}_{a \, b} h_{cd} + g_{ab}(\nabla^c \nabla_c h - \nabla^c \nabla^d h_{cd}) \]

\[ 2G^{(2)}_{ab}(g, h) = \frac{1}{2} \nabla_a h^{cd} \nabla_b h_{cd} + h^{cd} \nabla_b \nabla_a h_{cd} + \frac{1}{2} \nabla_a h^c_b \nabla_c h + \nabla^c h_{a} \nabla^d h_{b} - \nabla^c h_{b} \nabla^d h_{a} - \nabla^c h_{a} \nabla^d h_{b} - \nabla^c h_{b} \nabla^d h_{a} + h^{cd} \nabla_d \nabla_{a} h_{bc} - h^{cd} \nabla_d \nabla_{c} h_{ab} - h^{cd} \nabla_d \nabla_{c} h_{cd} + h_{ab} \nabla_d \nabla^{d} h - \nabla_c h_{bd} \nabla^{d} h_{a} + \nabla_d h_{bc} \nabla^{d} h_{c} + g_{ab}(h^{cd} \nabla_d \nabla_e h^e_c - h^{cd} \nabla_d \nabla_c h + \frac{1}{4} \nabla_d h \nabla^d h + \nabla_c h^{cd} \nabla_e h^e_d - \nabla^d h \nabla_e h^e_d + h^{cd} \nabla_e \nabla_d h^e_c - h^{cd} \nabla_e \nabla^e h_{cd} + \frac{1}{2} \nabla_d h_{ce} \nabla^e h^{cd} - \frac{3}{4} \nabla_e h_{cd} \nabla^e h^{cd}) \]