Theory of General Conservation: Spherically Symmetric Solutions

Edward Lee Green

University of North Georgia, Dahlonega, GA 30597

E-mail: egreen@ung.edu

Abstract. A theory has been developed which extends the geometrical structure of a real four-dimensional space-time via a field of orthonormal tetrads with an enlarged transformation group. This new transformation group, called the conservation group, contains the group of diffeomorphisms as a proper subgroup and is the foundation of the theory of general conservation. Field equations for the free field have been obtained from an appropriate Lagrangian. In this paper, this theory is further extended by development of a suitable Lagrangian for a field with sources. Spherically symmetric solutions for both the free field and the field with sources are given. An external, free-field model is developed. The theory implies that the external stress-energy tensor has non-compact support and hence may give the geometrical foundation for dark matter. The resulting models are compared to the internal and external Schwarzschild models. The theory may explain the Pioneer anomaly and the corona heating problem.

PACS numbers: 04.50.-h, 12.10.-g,04.40.-b

1. Introduction

Let $\tilde{V}^\alpha$ be a vector density of weight +1. Then a conservation law of the form $\tilde{V}^\alpha{}_{,\alpha} = 0$ is invariant under all transformations satisfying

$$x^\nu{}_{,\overline{\alpha}}(x^{\overline{\alpha}}{}_{,\nu,\mu} - x^{\overline{\alpha}}{}_{,\mu,\nu}) = 0 \quad . \tag{1}$$

This property defines the group of conservative transformations, of which, the group of diffeomorphisms is a proper subgroup [1-3].

In an effort to model dark matter cosmic acceleration, many theorists have simply modified general relativity in some fashion. We claim that our modification which is based on an enlargement of the transformation group is perhaps the only one with a solid guiding principle. Einstein himself felt it was a mistake to simply add a cosmological constant Λ. Recently, theoretical developments of $f(R)$ gravity [8], quintessence [9] and other modifications of general relativity [10] have a similar *ad hoc* flavor.

A suitable Lagrangian for the free field is given by

$$\mathcal{L}_f = \frac{1}{16\pi}\int C^\alpha C_\alpha \, h \, d^4x \tag{2}$$

where $h = \sqrt{-g}$ is the determinant of the tetrad and C_α is the curvature vector defined by $C_\alpha \equiv h_i{}^\nu(h^i{}_{\alpha,\nu} - h^i{}_{\nu,\alpha}) \ = \gamma^\mu{}_{\alpha\mu}$ [2-6].

Additional field variables Λ^i_I are introduced to yield a theory with an extended covariant derivative. The resulting field equations are [6]:

$$C_\mu = 0 \quad . \tag{3}$$

We assume for this talk that we are working with a solution of the field equations for which $\Lambda^I_i = \delta^I_i$. Using (3) we see that the field equations may be also expressed in the form

$$G_{\mu\nu} = \Upsilon_{(\mu}{}^\alpha{}_{\nu);\alpha} + \Upsilon^\alpha{}_{\sigma(\nu}\Upsilon^\sigma{}_{\mu)\alpha} + \frac{1}{2}g_{\mu\nu}\Upsilon^{\alpha\beta\sigma}\Upsilon_{\alpha\sigma\beta} \quad \equiv 8\pi(T_\mathrm{f})_{\mu\nu} \tag{4}$$

with free field stress energy tensor $\mathbf{T}_\mathrm{f}$.

In the presence of sources the Lagrangian is of the form

$$\mathcal{L} = \mathcal{L}_\mathrm{f} + \mathcal{L}_\mathrm{s} = \int\left(\frac{1}{16\pi}C^\alpha C_\alpha + L_s\right) h \, d^4x \tag{5}$$

where $L_s = L_s(x^\alpha)$ is the appropriate Lagrangian density function for the source. Thus we have the following identity for the Einstein tensor,

$$G_{\mu\nu} = \left(\Upsilon_{(\mu}{}^\alpha{}_{\nu);\alpha} + \Upsilon^\alpha{}_{\sigma(\nu}\Upsilon^\sigma{}_{\mu)\alpha} + \frac{1}{2}g_{\mu\nu}\Upsilon^{\alpha\beta\sigma}\Upsilon_{\alpha\sigma\beta}\right) + 8\pi(T_\mathrm{s})_{\mu\nu} \tag{6}$$

or

$$G_{\mu\nu} = 8\pi(T_\mathrm{f})_{\mu\nu} + 8\pi(T_\mathrm{s})_{\mu\nu} \qquad . \tag{7}$$

We call $\mathbf{T}_\mathrm{f}$ the free field stress energy and $\mathbf{T}_\mathrm{s}$ the stress energy for the source.

2. Spherically symmetric solutions.

2.1. Free Fields.

We now exhibit spherically symmetric solutions of the field equations for a free field (5). The tetrad in spherical coordinates may be generally expressed by

$$h^i{}_\mu = \begin{bmatrix} e^\Phi & 0 & 0 & 0 \\ 0 & (1+\frac{1}{2}r\Phi')\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ 0 & (1+\frac{1}{2}r\Phi')\sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ 0 & (1+\frac{1}{2}r\Phi')\cos\theta & -r\sin\theta & 0 \end{bmatrix} \tag{8}$$

where the upper index refers to the row and the prime indicates differentiation with respect to r. One finds that $C_\mu = 0$ for this tetrad. The new metric is

$$ds^2 = -e^{2\Phi(r)}dt^2 + (1+\frac{1}{2}r\Phi'(r))^2dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \quad . \tag{9}$$

One finds that

$$e^{-2\Phi(r)}G_{tt} = \frac{2}{r^2}\cdot\frac{d}{dr}\left(\frac{r}{2} - \frac{r}{2(1+\frac{1}{2}r\Phi')^2}\right) \equiv \frac{2}{r^2}w'(r) \equiv 8\pi\rho_f \quad , \tag{10}$$

where $w(r) \equiv \frac{r}{2} - \frac{r}{2(1+\frac{1}{2}r\Phi')^2}$. Hence

$$\Phi'(r) = \frac{2}{r}\left[(1-\frac{2w(r)}{r})^{-\frac{1}{2}} - 1\right] \quad . \tag{11}$$

Thus

$$g_{rr} = (1+\frac{1}{2}r\Phi')^2 = \left(1-\frac{2w(r)}{r}\right)^{-1} \quad , \tag{12}$$

and

$$g_{tt} = -e^{2\Phi(r)} \quad ,\texttt{where}\ \Phi(r) = \int\frac{2}{r}\left[\left(1-\frac{2w(r)}{r}\right)^{-\frac{1}{2}} - 1\right]dr \tag{13}$$

The radial pressure [12] of the free field is given by

$$8\pi p_r = \frac{4r\sqrt{1-\frac{2w(r)}{r}} - 4r + 6w(r)}{r^3} \tag{14}$$

and the tangential pressure is

$$8\pi p_T = \frac{8r - 9w(r) - 8r\sqrt{1-\frac{2w(r)}{r}} + rw'(r)}{r^3} \quad . \tag{15}$$

Since $p_r \neq p_T$ there are shear stresses and we see that $(T_\mathrm{f})_{\mu\nu}$ does not model a perfect fluid.

We note that $(T_f)^\mu_\nu = diag[-\rho,\, p_r\,,\, p_T\,,\, p_T]$. The conservation of energy condition, $T^\mu{}_{\nu;\mu} = 0$ is vacuous for $\nu = 0, 2$ and 3. The only nontrivial condition is when $\nu = 1$ representing the radial coordinate and in this case yields

$$(\,\rho + p_r)\Phi' + p_r' - \frac{2}{r}\,(p_T - p_r) = 0 \tag{16}$$

3. External Solutions.

In order for the external solution to agree with the weak-field solution as $r \to \infty$, we will identify $w(r) = \frac{1}{2}M$ for $r \geq$ R, the radius of the star. Thus $\frac{1}{2}M = \frac{r}{2} - \frac{r}{2(1+\frac{1}{2}r\Phi')^2}$ and hence

$$\Phi(r) = \int \left[\frac{2}{r\sqrt{1-\frac{M}{r}}} - \frac{2}{r} \right] dr \tag{17}$$

resulting in

$$g_{tt} = \frac{-1}{256}\Big(1+\sqrt{1-\frac{M}{r}}\Big)^8 \quad . \tag{18}$$

We thus obtain the following line element:

$$ds^2 = \frac{-1}{256}\Big(1+\sqrt{1-\frac{M}{r}}\Big)^8 dt^2 + \Big(1-\frac{M}{r}\Big)^{-1} dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \ . \tag{19}$$

Expanding g_{tt} and g_{rr} in powers of $\frac{M}{r}$, we find that asymptotically (for $r >> M$), to second order,

$$ds^2 \approx -\Big(1-\frac{2M}{r}+\frac{5M^2}{4r^2}\Big)dt^2 + \Big(1+\frac{M}{r}+\frac{M^2}{r^2}\Big)dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \ . \tag{20}$$

Using (18-20), the Einstein field equations for the external solution are

$$\begin{aligned}
G_{tt} = 8\pi T_{tt} &= 0 \\
G_{rr} = 8\pi T_{rr} &= \frac{M\Big(3\sqrt{1-\frac{M}{r}}-1\Big)}{r^3\Big(1-\frac{M}{r}\Big)\Big(1+\sqrt{1-\frac{M}{r}}\Big)} \\
\frac{G_{\theta\theta}}{r^2} = \frac{G_{\phi\phi}}{r^2\sin^2\theta} &= \frac{8\pi T_{\theta\theta}}{r^2} = \frac{8\pi T_{\phi\phi}}{r^2\sin^2\phi} = \frac{-M\Big(9\sqrt{1-\frac{M}{r}}-7\Big)}{2r^2\Big(1+\sqrt{1-\frac{M}{r}}\Big)} \quad .
\end{aligned} \tag{21}$$

Or

$$\begin{aligned}
8\pi\rho &= 0 \\
8\pi p_r &= \frac{M\Big(3\sqrt{1-\frac{M}{r}}-1\Big)}{r^3\Big(1+\sqrt{1-\frac{M}{r}}\Big)} \\
8\pi p_T &= \frac{-M\Big(9\sqrt{1-\frac{M}{r}}-7\Big)}{2r^2\Big(1+\sqrt{1-\frac{M}{r}}\Big)} \quad .
\end{aligned} \tag{22}$$

Asymptotically for $r >> M$, we have

$$\begin{aligned}
8\pi p_r &\approx \frac{M}{r^3}\Big(1-\frac{M}{2r}\Big) \\
8\pi p_T &\approx -\frac{M}{2r^3}\Big(1-\frac{2M}{r}\Big) \quad .
\end{aligned} \tag{23}$$

${T^\mu}_{\nu;\mu} = 0$ leads to

$$-\frac{dp_r}{dr} + \frac{2}{r}\Big(p_T - p_r\Big) = \Big(\rho + p_r\Big)\Phi' \quad . \tag{24}$$

For the metric of (19), asymptotically $\Phi' \approx \frac{M}{r^2}$ and thus

$$-\frac{dp_r}{dr} + \frac{2}{r}\Big(p_T(r) - p_r(r)\Big) \approx \frac{M^2}{8\pi r^5} \quad . \tag{25}$$

3.1. A Noncompact External Solution.

One particularly simple noncompact model is given by

$$w(r) = \frac{M}{2} - \frac{M^2}{8r} \quad . \tag{26}$$

With this choice, $1 - \frac{2w(r)}{r} = 1 - \frac{M}{r} + \frac{M^2}{4r^2} = (1 - \frac{M}{2r})^2$. Thus

$$\begin{aligned} ds^2 &= -\Big(1 - \frac{M}{2r}\Big)^4 dt^2 + \Big(1 - \frac{M}{2r}\Big)^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \qquad (27) \\ &\approx -\Big(1 - \frac{2M}{r} + \frac{3M^2}{r^2}\Big)dt^2 + \Big(1 + \frac{M}{r} + \frac{3M^2}{4r^2}\Big)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \end{aligned}$$

and hence

$$\begin{aligned} 8\pi\rho &= \frac{M^2}{4r^4} \\ 8\pi p_r &= \frac{M}{r^3}\Big(1 - \frac{3M}{4r}\Big) \\ 8\pi p_T &= -\frac{M}{2r^3}\Big(1 - \frac{5M}{2r}\Big) \end{aligned} \tag{28}$$

These equations are exact. Also, (25) holds for this metric as well.

4. Motion of a Test Particle and Comparison with Predictions of General Relativity.

We now investigate the motion of a test particle in the external field solution. The Lagrangian with the source term given by

$$L_s = \rho(x) = \mu \int \delta^4_\epsilon(x - \gamma(s))(-u^\mu u_\mu)^{\frac{1}{2}} ds \tag{29}$$

where δ^4_ϵ approximates the Dirac delta function with a space-like volume of ϵ. The condition $T^{\beta\alpha}{}_{;\beta} = 0$ leads to

$$\epsilon\, (T_f)^{\beta\alpha}{}_{;\beta} \; + \; \frac{\mu}{\sqrt{-u^\nu u_\nu}}\, u^\beta u^\alpha_{\;;\beta} = 0 \qquad . \tag{30}$$

The only nonzero component of $(T_f)^{\beta\alpha}{}_{;\beta}$ is the radial component ($\alpha = 1$) as we saw in (25). The $u^\beta u^\alpha_{;\beta}$ term corresponds to the geodesic equation. When $\alpha \neq 1$, (30) is equivalent to the geodesic equation for u^α.

This implies that there is an additional outward force, $F_p(r)$, from the radial and tangential pressures, since

$$\begin{aligned} \epsilon\, (T_f)^{\beta r}{}_{;\beta} \; + \; \frac{\mu}{\sqrt{-u^\nu u_\nu}}\, u^\beta u^r{}_{;\beta} &= 0 \\ \epsilon\left(p_r' - \frac{2}{r}\,(p_T - p_r)\right) \; + \; \frac{\mu}{\sqrt{-u^\nu u_\nu}}\, u^\beta u^r{}_{;\beta} &= 0 \end{aligned} \tag{31}$$

Table 1. Values of F_p, $\frac{M}{r}$ and $\frac{M^2}{r^3}$ for various planets

Planet	$\frac{1}{\mu} F_p$	$\frac{F_p}{F_{\rm grav}}$	$\frac{M}{r}$	$\frac{M^2}{r^3}$
Mercury	$3.31 \times 10^{-28}\mathrm{cm}^{-1}$	7.53×10^{-8}	2.55×10^{-8}	$1.12 \times 10^{-28}\mathrm{cm}^{-1}$
Earth	$2.82 \times 10^{-30}\mathrm{cm}^{-1}$	4.28×10^{-9}	9.86×10^{-9}	$6.51 \times 10^{-30}\mathrm{cm}^{-1}$
Jupiter	$3.08 \times 10^{-33}\mathrm{cm}^{-1}$	1.26×10^{-10}	1.90×10^{-9}	$4.62 \times 10^{-32}\mathrm{cm}^{-1}$
Neptune	$3.60 \times 10^{-37}\mathrm{cm}^{-1}$	4.94×10^{-13}	3.28×10^{-10}	$2.39 \times 10^{-34}\mathrm{cm}^{-1}$

i.e.

$$\frac{\mu}{\sqrt{-u^\nu u_\nu}}\, u^\beta u^r{}_{;\beta} \;=\; \epsilon \left(-p_r{}' + \frac{2}{r}\,(p_T - p_r) \right) \;\equiv\; F_p \quad . \tag{32}$$

Thus

$$\ddot{r} \;+\; \frac{2M\sqrt{1-\frac{M}{r}}}{r^2\left(1+\sqrt{1-\frac{M}{r}}\right)} \;-\; \frac{L^2\sqrt{1-\frac{M}{r}}\left(3\sqrt{1-\frac{M}{r}}-2\right)}{r^3}$$
$$+\; \frac{M\left(3\sqrt{1-\frac{M}{r}}-1\right)}{2r^2\left(1-\frac{M}{r}\right)\left(1+\sqrt{1-\frac{M}{r}}\right)}\dot{r}^2 \;=\; \frac{1}{\mu}F_p \quad . \tag{33}$$

A similar computation with the metric given by (27) results in

$$\ddot{r} \;+\; \frac{M(1-\frac{M}{2r})}{r^2} + \frac{M}{2r^2(1-\frac{M}{2r})}\dot{r}^2 - \frac{L^2(1-\frac{M}{2r})(1-\frac{3M}{2r})}{r^3} \;=\; \frac{1}{\mu}\,F_p \quad . \tag{34}$$

Hence $\frac{1}{\mu}\,F_p \approx \frac{\epsilon}{\mu}\frac{M^2}{8\pi r^5}$. Let the average density of the particle be $\tilde{\rho}$, then $\tilde{\rho} = \frac{\mu}{\epsilon}$ and so

$$\frac{1}{\mu}\,F_p \approx \frac{M^2}{8\pi\tilde{\rho}r^5} \qquad . \tag{35}$$

For earth, $\tilde{\rho} \approx 4.0971 \times 10^{-28}\mathrm{cm}^{-2}$. Typical values of $\tilde{\rho}$ for planets range between $3 \times 10^{-29}\mathrm{cm}^{-2}$ and $5 \times 10^{-28}\mathrm{cm}^{-2}$. Consider the ratio of F_p to the magnitude of the (Newtonian) gravitational force of the sun, $F_{\rm grav} \equiv \frac{\mu M}{r^2}$. This ratio is $\frac{F_p}{F_{\rm grav}} \approx \frac{M}{8\pi\tilde{\rho}r^3}$. For Mercury this ratio is approximately 7.53×10^{-8}. See Table 1.

Kepler's Law. For the metric of (19) we find that

$$\omega^2 r^3 \approx M\Big(1 - \frac{5M}{4} - \frac{M}{8\pi\tilde{\rho}r^3}\Big) \quad . \tag{36}$$

For the motion under the metric (27) one gets

$$\omega^2 r^3 \approx M\Big(1 - \frac{3M}{2r} - \frac{M}{8\pi\tilde{\rho}r^3}\Big) \quad . \tag{37}$$

Thus we see that when $\frac{M}{r}$ and $\frac{F_p}{F_{\rm grav}}$ as very small that we have excellent agreement with Kepler's Law.

Radial Motion. For pure radial motion ($L = 0$), metric (19) asymptotically yields

$$\ddot{r} \approx -\frac{M}{r^2}\Big(1 - \frac{M}{4r}\Big) - \frac{M}{2r^2}\Big(1 + \frac{M}{2r}\Big)\dot{r}^2 + \frac{M^2}{8\pi\tilde{\rho}r^5} \quad , \quad . \tag{38}$$

From the metric (27), one finds

$$\ddot{r} \approx -\frac{M}{r^2}\left(1-\frac{M}{2r}\right) - \frac{M}{2r^2}\left(1+\frac{M}{2r}\right)\dot{r}^2 + \frac{M^2}{8\pi\tilde{\rho}r^5} \quad . \tag{39}$$

The magnitude of the $\dot{r}^2$ terms do not appear to be large enough to explain the Pioneer anomaly. For Pioneer, the magnitude of these terms at planet Pluto is approximately 10^{-15} m s^{-2} which is much less that the anomalous value of about 8.74×10^{-10} m s^{-2}.

However, we do see that there is an explanation of the Pioneer anomaly. These radial equations have additional outward accelerations that are not part of the standard external Schwarzschild solution equations. We see an additional outward acceleration given by

$$a_{\rm out} = \frac{M^2}{4r^3} + \frac{M^2}{8\pi\tilde{\rho}r^5} \quad . \tag{40}$$

For the Pioneer spacecraft, $\tilde{\rho} \approx \frac{1}{3}$g/cm^3. This yields $\frac{M^2}{8\pi\tilde{\rho}r^5} \approx 4.67 \times 10^{-29}$cm^{-1} at 1 A.U. from the sun. At Earth, we see using the value of $\frac{M^2}{r^3}$ from Table 1, that $a_{\rm out} \approx 0.25 \times 6.51 \times 10^{-30} + 4.67 \times 10^{-29}$ cm^{-1}. Thus for the Pioneer spacecraft at Earth's distance from the sun has an outward acceleration of

$$a_{\rm out} \approx 4.83 \times 10^{-29}\text{cm}^{-1} \qquad (\text{at Earth}). \tag{41}$$

This is an extra outward acceleration due to the fact that our theory differs from general relativity and also includes a nonzero stress-energy tensor. At a distance of Jupiter from the Sun, with the same value of $\tilde{\rho}$ yields $\frac{M^2}{8\pi\tilde{\rho}r^5} \approx 1.23 \times 10^{-32}$cm^{-1}. Thus, using the value of $\frac{M^2}{r^3}$ from Table 1, we see that

$$a_{\rm out} \approx 1.39 \times 10^{-32}\text{cm}^{-1} \qquad (\text{at Jupiter}). \tag{42}$$

For distances that are greater than the distance from the Sun to Jupiter we see that $\Delta a_{\rm out} \approx 4.83 \times 10^{-29}$cm^{-1} and under the general relativity model, this would be interpreted as an additional Sun-ward acceleration. Converting this value to standard units yields

$$\Delta a_{\rm out} \approx 4.34 \times 10^{-10}\ \text{m}\,\text{s}^{-2} \tag{43}$$

which is about 50% of the anomalous acceleration. For the metric (27) at earth we have $a_{\rm out} \approx 4.99 \times 10^{-29}$cm^{-1} which yields a value of

$$\Delta a_{\rm out} \approx 4.49 \times 10^{-10}\ \text{m}\,\text{s}^{-2} \tag{44}$$

which is 51% of the anomalous acceleration. The difference may be explained by thermal forces [16].

Redshift. The redshift $z = \frac{\Delta\lambda}{\lambda} = |g_{tt}|^{-\frac{1}{2}} - 1$ for stationary objects. From (19) we find that

$$z = 16\left(1+\sqrt{1-\frac{M}{r}}\right)^{-4} - 1 \ \approx \frac{M}{r} + \frac{7M^2}{8r^2} \quad , \quad r >> M \quad , \tag{45}$$

and from (27) we find

$$z = \left(1-\frac{M}{2r}\right)^{-2} - 1 \ \approx \frac{M}{r} + \frac{3M^2}{4r^2} \quad , \quad r >> M \quad . \tag{46}$$

Asymptotically, these results agree with the value found in the Schwarzschild geometry, i.e. $z \approx \frac{M}{r}$. At the distance of the earth from the sun, the relative differences compared to the standard result are 8.6×10^{-9} and 7.4×10^{-9} for these results.

Precession of Perihelion. We now consider the precession of perihelion problem. The results are as follows. From the metric given in (19), we find that

$$-\frac{f''(u_0)}{2(L^\dagger)^2} \approx 1 - \frac{15}{4}u_0 + \frac{12u_0^2}{16\pi\hat{\rho}L^2} \quad . \tag{47}$$

For Mercury, the u_0^2 term (last term) of this expression is approximately 10^{-4} times the value of the preceding term and thus the perihelion is shifted by

$$\Delta\phi \approx \left(\frac{15\pi}{4}\right)\frac{M}{r_0} \quad . \tag{48}$$

where r_0 is the radius of the near-circular orbit. If the metric (27) is used one finds

$$-\frac{f''(u_0)}{2(L^\dagger)^2} \approx 1 - \frac{7}{2}u_0 + \frac{12u_0^2}{16\pi\hat{\rho}L^2} \quad . \tag{49}$$

and hence

$$\Delta\phi \approx \left(\frac{7\pi}{2}\right)\frac{M}{r_0} \quad . \tag{50}$$

Both of these results are less than the standard result of $\frac{6\pi M}{r_0}$, with (94) being $\frac{5}{8}$ of the standard result and (96) being $\frac{7}{12}$ of the standard result.

In the Newcomb's calculation of the precession of Mercury in 1882, [17], it was stated that "a planet or a group of planets between Mercury and the Sun" could explain the additional $43.03''$ per century. It seems reasonable to define the average pressure by $\bar{p} = \frac{1}{3}(p_r + 2p_T)$ and thus the inertial mass per unit volume of the halo is given by $\rho + \bar{p}$. The inertial mass of the halo between the Sun and Mercury for the metric given by (19) is given by

$$\Delta m \approx \int_{r_{sun} \le r \le r_{merc}} \frac{M^2}{8\pi \cdot 2r^4}\, d^3x \approx 0.0784 \text{ cm} \quad . \tag{51}$$

This is about 17.7% of the mass of Earth. For the metric of (27), the inertial mass is $\frac{5}{3}$ times larger, giving 0.131 cm which represents 29.4% of the Earth's mass. It is possible that these values of Δm may explain the remaining fraction of the anomalous precession. Since it is spherically symmetric, other effects on the orbit of Mercury should be minimal.

Isotropic Form and Temperature of the Corona. The tetrad that produces the isotropic form for the metric given in (27) above, is generated by the transformation, $r \to r + \frac{M}{2}$. The resulting metric is

$$ds^2 = -\left(1 + \frac{M}{2r}\right)^{-4} dt^2 + \left(1 + \frac{M}{2r}\right)^2 \left[\, dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \,\right] \tag{52}$$

and the stress energy tensor yields

$$8\pi\rho = \frac{M^2}{4r^4(1 + \frac{M}{2r})^4}$$

$$8\pi p_r = \frac{M}{r^3(1+\frac{M}{2r})^4}\left(1-\frac{M}{4r}\right) \tag{53}$$
$$8\pi p_T = \frac{-M}{2r^3(1+\frac{M}{2r})^4}\left(1-\frac{2M}{r}\right)$$

To first order, the isotropic metric and its corresponding stress energy tensor are equivalent to the metric of (27).

Suppose we apply the ideal gas law to the halo with a pressure of $\bar{p}=\frac{1}{3}(p_r+2p_T)$ and assume that the halo is comprised of particles with mass $\hat{m}$ ev. The resulting temperature is $T=\frac{7}{3}\hat{m}$ cm $=2.70\times10^4$ K. If the masses of the constituent matter in the halo are approximately 36 ev, the resulting temperature would be approximately 10^6 K and hence would explain the high temperature of the corona. These considerations also suggest that the dark matter may be a mixture of the neutrinos ν_e, ν_μ and ν_τ.

Deflection of Light and Time Delay. For null rays which model photon motion, $ds^2=0$, and we see that

$$\left|\frac{d\mathbf{r}}{dt}\right| = [f(r)]^{\frac{3}{4}} \approx \left(1-\frac{2M}{r}\right)^{\frac{3}{4}} \approx 1-\frac{3M}{2r} \approx \frac{1}{1+\frac{3M}{2r}} \quad . \tag{54}$$

The denominator represents a refraction index of $n_0(r)\equiv 1+\frac{3M}{2r}$. We see that $n_0(r)-1=\frac{3M}{2r}$ is precisely 75% of the value in general relativity (where $n(r)-1=\frac{2M}{r}$). Let $\bar{p}=\frac{1}{3}(p_r+p_T+p_T)$ be the average pressure. We propose that additional refraction occurs due to the medium and the value of $n_1(r)-1$ is proportional to $\rho+\bar{p}$, viz.

$$n_1(r)-1 = 8\pi\alpha(\rho+\bar{p}) \quad , \tag{55}$$

where α is a positive constant and the factor of 8π is included for convenience. This formula may be justified by the Lorentz-Lorenz relation [18].

Hence $n_1(r)\approx 1+\frac{\beta M^2}{r^4}$, with $\beta=\frac{1}{2}\alpha$ for (19) and $\beta=\frac{5}{6}\alpha$ for the stress energy tensor (27). We multiply this by the corresponding refractive index which is calculated from the metric, hence

$$n(r) = n_0(r)\cdot n_1(r) \approx 1+\frac{3M}{2r}+\frac{\beta M^2}{r^4} \quad . \tag{56}$$

From [14] we find that the angle of deflection of light passing near the surface of the sun (i.e. a minimum radius of r_0) is given by

$$\Delta\phi = \int_{1+\frac{3M}{2r_0}+\frac{\beta M^2}{r_0^4}}^{\infty} \frac{2\,dr}{r\sqrt{r^2-1}} - \int_1^{\infty} \frac{2\,dr}{r\sqrt{r^2(1+\frac{3M}{2r}+\frac{\beta M^2}{r^4})^2-1}} \tag{57}$$

Defining $A\equiv\frac{3M}{2r_0}$, $B\equiv\frac{\beta M^2}{r_0^4}$ and changing variables to $w\equiv\frac{r}{r_0}$ we find that

$$\Delta\phi = \int_{1+A+B}^{\infty} \frac{2\,dw}{w\sqrt{w^2-1}} - \int_1^{\infty} \frac{2\,dw}{w\sqrt{w^2(1+\frac{A}{w}+\frac{B}{w^3})^2-1}} \tag{58}$$

Noting that A and B are much less than 1, we find that (104) yields

$$\Delta\phi \approx -2A-\frac{3\pi}{2}B+(2A+8B)\sqrt{2(A+B)} \tag{59}$$

For the time delay we follow MTW [12]. Suppose the photon is moving along a path which is approximated in Cartesian coordinates by $y = b$, $z = 0$ for $-a_T \leq x \leq a_R$. The total time of transit from transmitter to reflector and back is

$$t_{TRT} = 2\int_{-a_T}^{a_T}\left(1 + \frac{3M}{2\sqrt{x^2+b^2}} + \frac{\beta M}{(x^2+b^2)^2}\right)dx \quad . \tag{60}$$

The second and third terms of this integral correspond to the delay effect. We see that the second term (which when integrated will be called $\Delta\tau$) is 75% of the general relativity value. The value of $\Delta\tau$ is

$$\Delta\tau = 3M\ln\left|\frac{(\sqrt{a_R^2+b^2}+a_R)(\sqrt{a_T^2+b^2}+a_T)}{b^2}\right| \quad . \tag{61}$$

The third term which will be called $\Delta(\Delta\tau)$ when integrated has a value

$$\Delta(\Delta\tau) \equiv \frac{\beta M^2}{b^3}\left[\arctan\left(\frac{x}{b}\right) + \frac{bx}{x^2+b^2}\right]\Bigg|_{-a_T}^{a_R} \approx \frac{\beta M^2\pi}{b^3} \tag{62}$$

when a_T and a_R are large compared to b. We assume that $\frac{da_T}{d\tau} \approx \frac{da_R}{d\tau} \approx 0$ and hence the rate of change of the total time delay is

$$\frac{d}{d\tau}\Big(\Delta\tau + \Delta(\Delta\tau)\Big) \approx \frac{-6M}{b}\left(1 + \frac{\beta M\pi}{2b^3}\right)\frac{db}{d\tau} \quad . \tag{63}$$

Thus, agreement with the general relativity value would occur if $\frac{\beta M\pi}{2b^3} = \frac{1}{3}$ and hence if $\beta = \frac{2b^3}{3\pi M}$. If $b = r_0 \approx 6.960 \times 10^{10}$ cm, then we find that $\beta \approx 4.844 \times 10^{26}$ cm^2. For the metric (19) we find $\alpha \approx 9.688 \times 10^{26}$ cm^2 and for (27), $\alpha \approx 5.813 \times 10^{26}$ cm^2.

We note that these values of α are fairly typical. For example, the corresponding value of α for hydrogen (H_2) gas is approximately 7.9×10^{26} cm^2. However, the Lorentz-Lorenz relation in its most basic form [18] relates the number density to the refraction. If the consideration of the corona temperature is correct, the number density of the halo near the sun is approximately 5×10^7 times that of hydrogen gas. Thus, the dark matter is seen to interact weakly.

As already noted, with $b = r_0$ and $\beta = \frac{2b^3}{3\pi M}$, leads to the general relativity result of $\frac{-8M}{b}\frac{db}{d\tau}$ for the time delay. For the deflection of light problem, we find that $B = \frac{2M}{3\pi r_0}$ and hence yields $\Delta\phi \approx \frac{-4M}{r_0} + \left(\frac{3M}{r_0} + \frac{16M}{3\pi r_0}\right)\sqrt{\frac{3M}{r_0} + \frac{4M}{3\pi r_0}}$. For $M \approx 1.477 \times 10^5$ cm and $r_0 \approx 6.960 \times 10^{10}$ cm, we find that $\Delta\phi \approx -8.462 \times 10^{-6}$. We note that this value of $\Delta\phi$ is approximately $1.75''$.

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