RIGID SURFACES IN GENERAL RELATIVITY

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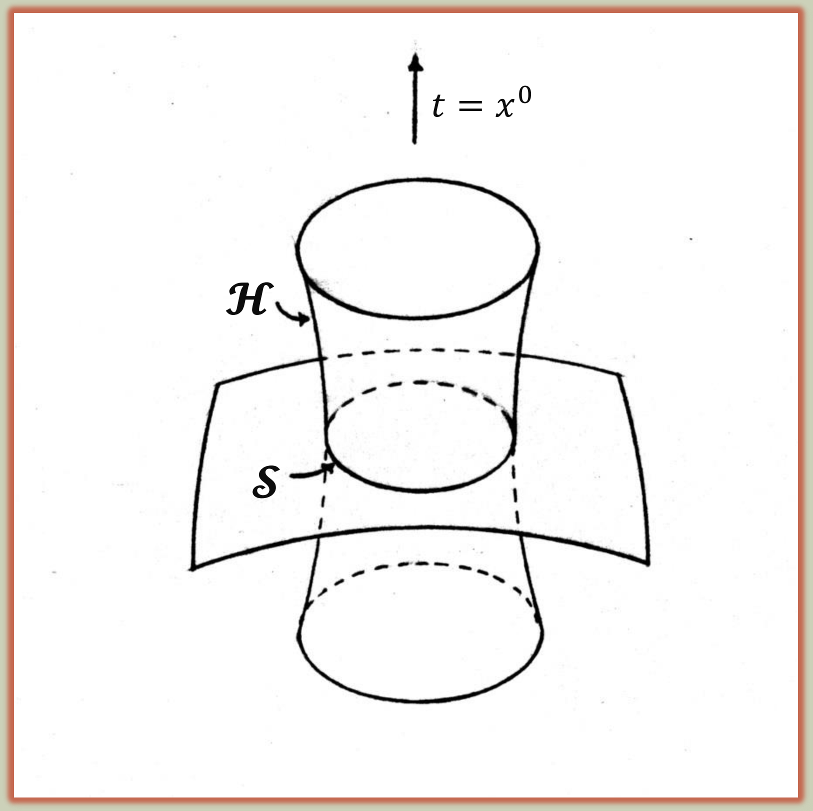
OUTLINE

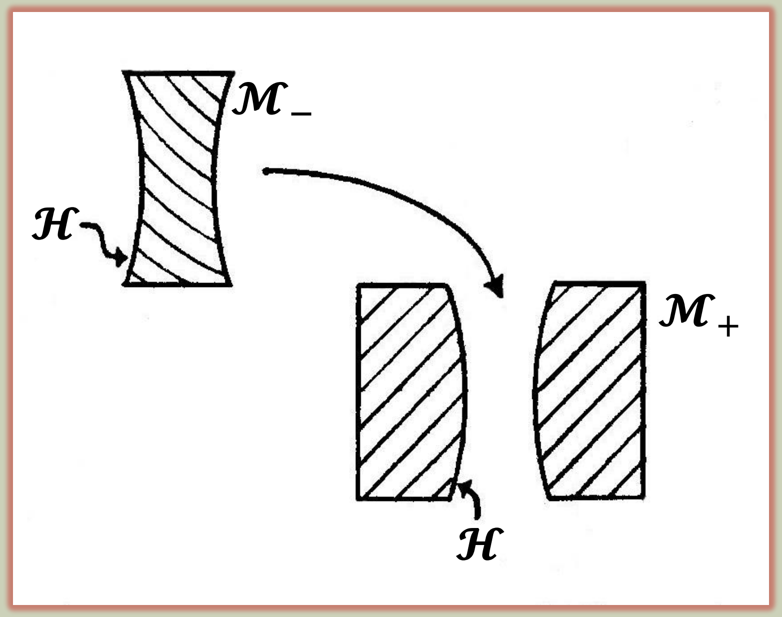
- Motivation: Defining energy with the thin shell formalism
- Rigid Surfaces: Definition, Initial Value Formulation, and Applications

NOTATION



- $\mathcal{M}$ is 4d spacetime with coords x^μ and signature $(-,+,+,+)$
- $\mathcal{H}$ is hypersurface of $\mathcal{M}$ with timelike tangent vector
- Spacelike hypersurfaces of $\mathcal{H}$ will be called $\mathcal{S}$
- σ_{AB} is induced metric on $\mathcal{S}$
  - y^A $\rightarrow$ coordinates on $\mathcal{S}$
  - $y^a = (\tau, y^A)$ $\rightarrow$ coords. on $\mathcal{H}$

GLUING SPACETIMES



- Let $\mathcal{H}$ spatially enclose a region of spacetime (for spatial sections $\mathcal{S}$, $\nexists\ \partial\mathcal{S}$)
- Try to isometrically embed $\mathcal{H}$ in flat spacetime
- If successful, one can "glue" region of curved spacetime $\mathcal{M}_-$ to region of flat spacetime $\mathcal{M}_+$ at boundary $\mathcal{H}$
- $+,-$ subscripts denote qtys defined on different sides of $\mathcal{H}$

THIN SHELL FORMALISM

- Using the thin-shell formalism, one can define a surface stress tensor $\mathcal{T}_{ab}$ on $\boldsymbol{\mathcal{H}}$:

$$\mathcal{T}_{ab} = -\frac{1}{8\pi}\left([\mathcal{K}_{ab}] - q_{ab}[\mathcal{K}]\right)$$

- Where $\mathcal{K}_{ab}$ is the extrinsic curvature of $\boldsymbol{\mathcal{H}}$, $\mathcal{K}$ is the mean curvature, and I use notation from (Poisson 2004) for the bracket (where A is some quantity):

$$[A] \coloneqq A(\boldsymbol{\mathcal{M}}_+)\Big|_{\boldsymbol{\mathcal{H}}} - A(\boldsymbol{\mathcal{M}}_-)\Big|_{\boldsymbol{\mathcal{H}}}$$

- In words: evaluate the quantity A at the hypersurface $\boldsymbol{\mathcal{H}}$ embedded in $\boldsymbol{\mathcal{M}}_+$, do the same for $\boldsymbol{\mathcal{H}}$ embedded in $\boldsymbol{\mathcal{M}}_-$, and take the difference.

DEFINING ENERGY

- If χ^a is a timelike Killing vector on $\boldsymbol{\mathcal{H}}$, and $\boldsymbol{\mathcal{S}}$ is a spacelike hypersurface of $\boldsymbol{\mathcal{H}}$. If $\boldsymbol{\mathcal{M}}_-$ is a vacuum, then the following quantity is independent of $\boldsymbol{\mathcal{S}}$:

$$E_{\mathcal{S}} \coloneqq -\oint_{\mathcal{S}} \mathcal{T}_{ab}\chi^a n^b \sqrt{\sigma}\, d^2 y$$

- One can interpret $E_{\mathcal{S}}$ as the amount of negative mass needed to make spacetime flat outside.
- Equivalent to an intepretation by Booth & Mann; also, if $\boldsymbol{\mathcal{H}}$ is orthogonal to constant time surfaces, this reduces to the Brown-York QLE (Booth & Mann 1999)

SOLVING FOR $\mathcal{H}$

- Given an initial 2 surface $\mathcal{S}_0$, can one construct an initial value problem for $\mathcal{H}$?

- Given a flow in $\mathcal{M}$, defined by a timelike vector field V^μ, and a 2-surface $\mathcal{S}_0$ parametrically defined by $x_0^\mu(y^A)$, one can form $\mathcal{H}$ by evolving $\mathcal{S}_0$ according to the flow:

$$\frac{\partial x^\mu}{\partial \tau} = V^\mu$$

- Can one, in general, define a flow in spacetime such that $\mathcal{H}$ has a timelike Killing vector?

- Function counting implies no; Killing's equations on $\mathcal{H}$ are too restrictive.

RIGID SURFACES

- One can, however, define surfaces (called *Rigid Surfaces*) that have the following properties:

$$\frac{\partial \sigma_{AB}}{\partial \tau} = 0 \quad , \qquad g_{\mu\nu} V^{\mu} V^{\nu} = -1$$

  Where σ_{AB} is the induced metric of $\boldsymbol{\mathcal{S}}_{\tau}$, and V^{μ} defines the flow $\frac{\partial x^{\mu}}{\partial \tau} = V^{\mu}$.

- Unit V^{μ} ensures that V^{μ} is a 4-velocity of some observer

- If $\boldsymbol{\mathcal{H}}$ has a unit timelike Killing vector, it is a rigid surface, but the reverse is not true. Rigid surfaces with a unit timelike Killing vector will be called *Conserving Surfaces*.

RIGID SURFACES: INITIAL VALUE PROBLEM

- If $\mathcal{H}$ is parametrically defined $x^\mu(\tau, y^A)$, induced metric is:

$$\sigma_{AB} = g_{\mu\nu}(x^\alpha)\frac{\partial x^\mu}{\partial y^A}\frac{\partial x^\nu}{\partial y^B}$$

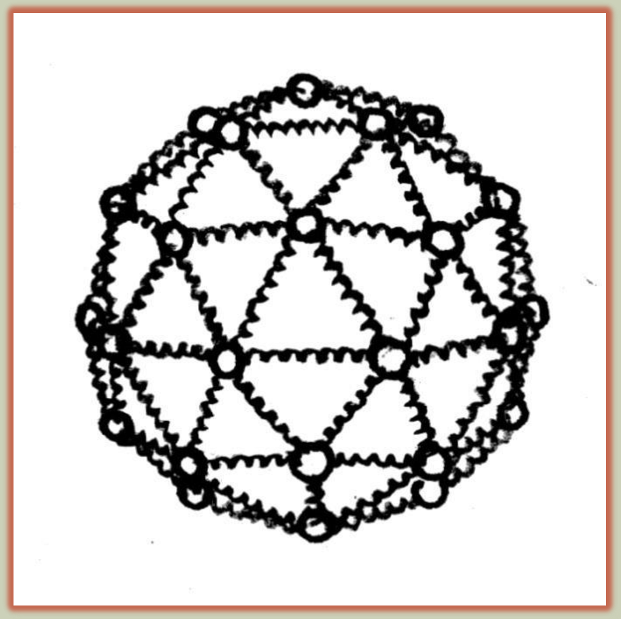
- The defining properties of rigid surfaces form a set of four PDEs for $x^\mu(\tau, y^A)$

- Initial Value Problem: Solve for $x^\mu(\tau, y^A)$, given ICs:

$$x^\mu(\tau_0, y^A) = x_0^\mu(y^A) \quad , \qquad V_0^\mu = \left(\frac{\partial x^\mu}{\partial \tau}\right)_{\tau=\tau_0}$$

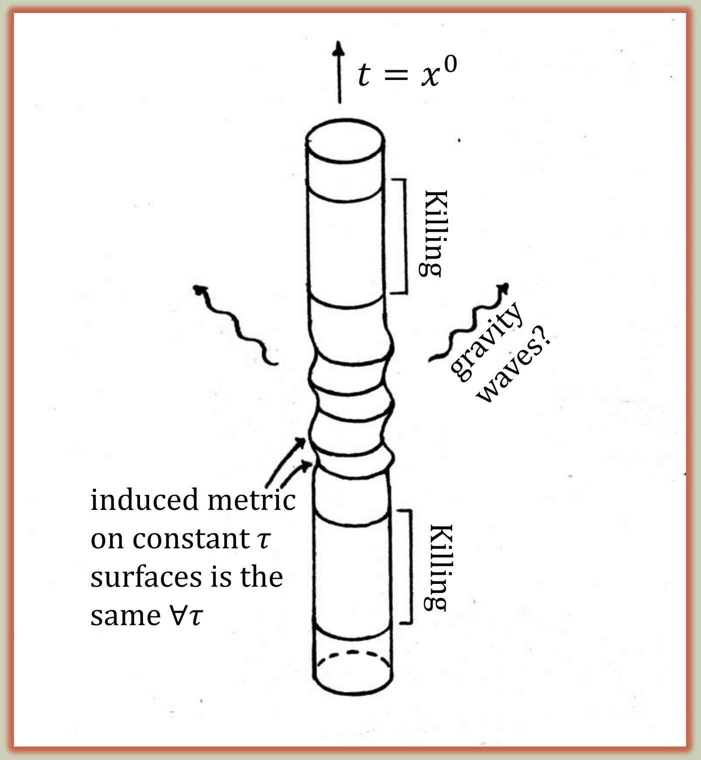
- Problem: Solutions are not unique in general (they are not unique in Minkowski space); extra conditions may be needed (zero net acceleration? Conserving Surfaces?)

RIGIDITY



- Physically, an object is Rigid if it resists deformation.
- If V^{μ} is orthogonal to $\mathcal{S}$, $\exists$ observers moving with rigid surface that do not see change in intrinsic geometry
- Consider a surface formed form identical masses and springs (spring const. k, equil. length l)
- For large k and small l, worldsheet will approximate a Rigid Surface.

RIGID SURFACES AND ENERGY



- For Conserving surfaces, E_S makes sense as an energy, but is non-unique for non-Conserving Rigid Surfaces
- Conserving surfaces may not be embeddable in generic vacuum spacetimes
- If one can find a condition to make the IVP for Rigid Surfaces unique, Rigid Surfaces may become Conserving in different regions of spacetime.
- One can calculate energy flux through the Rigid Surface by comparing E_S between the Conserving regions

SOME UNRESOLVED QUESTIONS

- The initial value problem
  - How far into the future can one extend a Rigid Surface, if at all?
  - What happens to Rigid Surfaces near a singularity?
  - Non-Uniqueness; are there extra conditions that fix this?
    - Zero net acceleration
    - See next bullet point.

- Under what conditions can one embed a Conserving Surface (a Rigid Surface with timelike Killing vector) in spacetime?
  - $B^{\mu} \coloneqq \sigma^{\mu}_{\nu} V^{\nu} \neq 0$. Perhaps one can place conditions on B^{μ}; if one has the condition that $\sigma^{\mu}_{\nu} \mathcal{L}_{V} B^{\nu} = 0$, Rigid Surface is Conserving.

- Would like a way to solve for constraint function:

$$f(x^{\mu}) = 0$$

REFERENCES

1. Poisson, E., *A Relativist's Toolkit* (Cambridge University Press 2004)

2. Booth, I.S., Mann, R.B., *Static and Infalling Quasilocal Energy of Charged and Naked Black Holes*, Phys. Rev. D 60:124009, (1999)

3. Brown, J.D., York, J.W., *Quasilocal energy and conserved charges derived from the gravitational action*, Phys. Rev. D 47:1407 (1993)