Lorentzian manifolds and causal sets as partially ordered measure spaces

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What is a causal set?

A partially ordered set, a set C with an order relation $\prec$, satisfying

$$x \prec y \,,\;\; y \prec z \quad \Rightarrow \quad x \prec z$$

and

$$x \prec y \,,\;\; y \prec x \quad \Rightarrow \quad x = y \,,$$

which is locally finite, in the sense that $\forall\; x$, y the interval (or Alexandrov set) $A(x, y)$ is finite,

$$\text{cardinality}\{z \mid x \prec z \prec y\} < \infty \,.$$

Causet elements are interpreted as events in a discretized spacetime, and the order relation as a causal relation.

Most recent review paper: S. Surya, "Directions in causal set quantum gravity" arXiv:1103.6272.

For example:

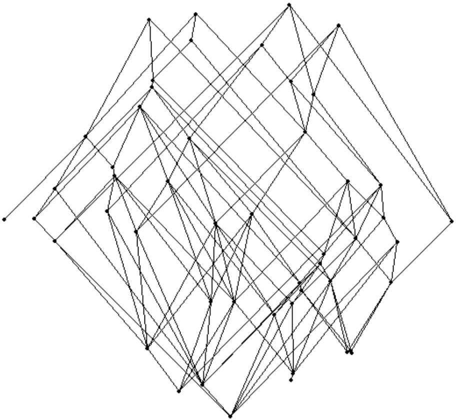


Figure: A 50-element causal set [this one was generated by a uniform random sprinkling in a diamond-shaped region of 2D Minkowski space]. Notice the types of links (only “nearest neighbors” are shown).

Spacetime as a causal set

The proposal that spacetime can be modeled as a causal set, proposed by Sorkin and collaborators [Bombelli et al. 1987, with precursor proposals by 't Hooft 1978, Myrheim 1978], combines:

- Spacetime metric = conformal structure + volume element (motivated by results by Hawking et al. and Malament).
- Spacetime discreteness or "atomicity" (motivated by various general issues, and supported by results in other approaches).

The two ideas work particularly well together. A metric can be discretized (Regge calculus), but the conformal or causal structure and the volume element naturally become combinatorial variables.

How does the correspondence with the continuum work?

Causal sets and Lorentzian manifolds

When does a causal set C correspond to some (M, g)?
Idea: when there is an embedding $i : C \to M$ such that:

- Partial order = Causal relations: $x \prec y \Leftrightarrow i(x) \in J^-(i(y))$,
- The embedded points are uniformly distributed in (M, g) (in a covariant sense; in particular, randomly distributed, which implies an effective local Lorentz invariance),
- (M, g) has no length scales smaller than the embedding one.

Assumption: with these conditions (M, g) is "unique", since M has no structure at small scales, while at larger scales light cones are determined by $\prec$, and volumes by counting.

Can we prove this? Does dynamics lead to this kind of C?

Classical Sequential Growth Dynamics

Idea: A framework which describes dynamics as a stochastic sequential growth process. If one imposes discrete versions of

- *General covariance:* The probability of growing a poset C is independent of the order in which elements are added, and
- *Bell causality:* Relative probabilities for transitions are not influenced by what happens in unrelated parts of the poset,

then the possible stochastic evolutions are the Rideout-Sorkin models, each characterized by non-negative real numbers t_0, t_1, t_2, ..., where t_k is related to the probability that a new "birth" in the causet has some k-element subset to its past [Rideout & Sorkin 1999].

What spacetimes does this produce?
Is there a GR-based formulation?

The action in terms of causal relations and volumes

Instead of a first-principles approach, find a way to write down the Lagrangian for gravity and matter fields on a manifold in terms of causal relations and the volume element. For the scalar curvature,

$$\mathcal{L}_{\rm EH} = \frac{1}{D_d} \left[\frac{E_d}{\tau^2} \left(\frac{V(\tau)}{k_d\, \tau^d} - 1 \right) \right. \\ \left. - \frac{1}{\tau^{d+2}} \int_{A(p,q)} \left(\frac{V(p,x)}{k_d\, \tau^d(p,x)} - 1 \right) {\rm d}^d x \right],$$

and there are similar expressions for the Lagrangian density of scalar and other fields. Rewrite these in terms of causal sets and use them in a sum over histories [R Sverdlov & LB, `arXiv:0801`].

Quantum measures...

One can also study the dynamics of a particle on a causal set.

Some examples of kinematical results

• *Dimension:* Find subsets that force a certain dimensionality, or use statistical methods. There are good dimension estimators for posets embedded in Minkowski space [Meyer 1993; Reid 2002].

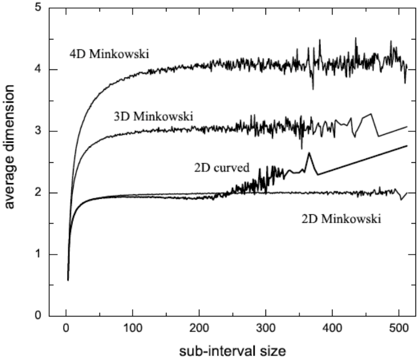
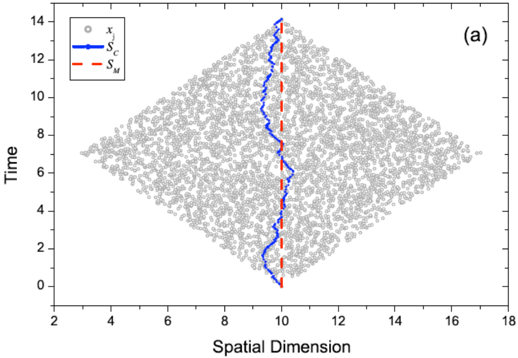


Figure: Dimension estimate, midpoint method [Reid 2002].

- *Timelike distances:* For two related points, the length ℓ of the longest chain in the poset (geodesic) is a good estimator of the timelike separation τ between the corresponding manifold points, $\tau = c_d\,\ell$ [Brightwell & Gregory 1991; Ilie et al. 2005].

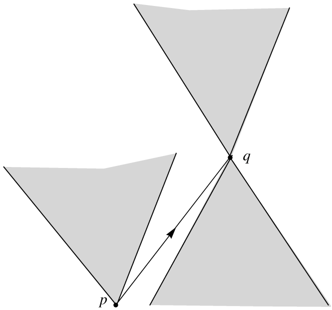


- *Spatial topology:* Spatial hypersurfaces correspond to sets of unrelated points and have no structure, but the topology can be obtained by thickening the hypersurface [Major et al. 2006].

Partially ordered measure spaces (causal measure spaces)

Spaces with a measure $\mu(B)$ for each measurable subset B and a partial order that is compatible, in the sense that all pasts and futures $I^{\pm}(x)$ are measurable.

All chronological Lorentzian manifolds (M, g) are spaces of this type. So are all causal sets, but there are also many other types.



Lorentzian distance and the closeness of causal measure spaces

(with Johan Noldus and Julio Tafoya, arXiv:1212.0601)

A Lorentzian distance can be defined on any causal measure space,

$$d(x, y) := \mu(A(x, y)) \quad (\text{if } x \prec y,\ 0 \text{ otherwise})\ .$$

In turn, this can be used to define a (regular) distance for pairs of causal measure spaces,

$$\begin{aligned} d_{\mathrm{GH}}(X, Y) := \inf\{\epsilon \mid \text{there exist } \xi : X \to Y,\ \zeta : Y \to X \\ \text{with } |d_X(x_1, x_2) - d_Y(\xi(x_1), \xi(x_2))| < \epsilon \\ \text{and } |d_X(x_1, x_2) - d_Y(\xi(x_1), \xi(x_2))| < \epsilon\}\ . \end{aligned}$$

- When restricted to compact Lorentzian manifolds satisfying the (future + past) distinguishing condition, d_{GH} is a distance.

Conclusion

• Formally, the fact that d_{GH} is positive-definite goes a long way towards proving the main assumption underlying this approach. We now need to understand physically the type of uncertainty involved in associating a continuum geometry to a causal set.

• The use of d_{GH} as a measure of the closeness between causal sets and manifolds means we can replace causal embeddings as a basis for the correspondence by a more flexible notion, applicable when no causal embedding may be possible.

• In the continuum, causal measure spaces are more general than Lorentzian manifolds. This opens the possibility that the continuum limit of causal set theory may go beyond "fluctuating Lorentzian geometries".