Mathematical Methods 1/6/2016

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Introduction

This course is designed to introduce the basic techniques needed to study the differential equations which arise in acoustics and vibration. There are no generally applicable techniques for solving differential equations. There are specific techniques which work in special cases, but in general solving an arbitrary system of differential equations is considered a hopeless task. Luckily, some of the special cases are of physical interest.

There are several ways that differential equations are classified. The *order* of a differential equation is the highest derivative appearing in the equation. A differential equation is said to be *linear* if the unknown functions appear linearly, otherwise the equation is said to be *nonlinear*. A differential equation in which only one independent variable is differentiated is said to be an ordinary differential equation. If several independent variables are differentiated then these derivatives are necessarily partial derivatives and the equation is said to be a partial differential equation. A differential equation is said to be *homogeneous* if multiplying the unknown function by a constant has the effect of multiplying the entire equation by a constant.

Example 1.1: The n-dimensional wave equation,

$$\Big(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)p = 0,$$

is a second order linear partial differential equation.

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Example 1.2: The n-dimensional plate equation,

$$\Big(\Delta^2 + K\frac{\partial^2}{\partial t^2}\Big)u = 0,$$

is a fourth order linear partial differential equation.

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Example 1.3: Assuming a sinusoidal time dependence $p(\mathbf{x},t) = f(\mathbf{x})\sin(\omega t)$ in the wave equation one obtains the n-dimensional Helmholtz equation,

$$\Big(\Delta + \frac{\omega^2}{c^2}\Big) f = 0,$$

a second order linear partial differential equation if $n > 1$. If $n = 1$ it is a second order linear ordinary differential equation.

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Example 1.4: Newton's law,

$$m\frac{d^2x}{dt^2} = F(x)$$

is a second order nonlinear ordinary differential equation except in the special case $F = -kx$ when it becomes a 1-dimensional Helmholtz equation.

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Example 1.5: The equations of lossless fluid mechanics; the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$

and the Euler equation,

$$\rho\Big(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\Big)\mathbf{v} = -\nabla P$$

along with an isentropic equation of state, $\rho = f(P)$; are a system of nonlinear first order partial differential equations. Here ρ is the density, p the pressure and $\mathbf{v}$ the velocity of the fluid at a given point in time and space.

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In the above ∇ is the n-dimensional gradient operator

$$\nabla \cdot \mathbf{v} = \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}$$

and Δ is the n-dimensional Laplace operator

$$\begin{aligned}\Delta &= \nabla \cdot \nabla \\ &= \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.\end{aligned}$$

Most of our time will be spent on linear problems. The solutions to a homogeneous linear differential equation satisfy the *principle of superposition.* Let L be a linear differential operator so that $\mathrm{L}f = 0$ is a homogeneous linear differential equation. Then for any functions f_1 and f_2 and constants a and b

$$\mathrm{L}\big(af_1 + bf_2\big) = a\mathrm{L}f_1 + b\mathrm{L}f_2.$$

In particular, if both f_1 and f_2 are solutions, $\mathrm{L}f_1 = 0$ and $\mathrm{L}f_2 = 0$, then $af_1 + bf_2$ is also a solution. This is never the case for nonlinear equations.

Inhomogeneous linear differential equations are of the form

$$\mathrm{L}f = g$$

where g is known. If f_1 and f_2 are solutions of this inhomogeneous problem then

$$\mathrm{L}(f_1 - f_2) = g - g = 0$$

so that f_1 and f_2 differ by a solution of the homogeneous problem $\mathrm{L}f = 0$.

If the solution to a differential equation is to represent a physical quantity it must be real valued. However it is sometimes more convenient to find complex valued solutions. For linear equations $\mathrm{L}f = 0$ with L and f both real, both the real and imaginary parts of f are also solutions, and are real valued.

A large class of solvable (in the sense that they can be reduced to algebraic equations) linear equations are those with constant coefficients. The solutions to such equations are superpositions of exponential functions. That is because

$$\frac{d}{dx}e^{kx} = ke^{kx}$$

so that differentiation can be replaced by multiplication by a number, k.

Example 1.6: An example of a linear system of ordinary differential equations with constant coefficients is

$$\begin{aligned}
\big(2\frac{d^2}{dx^2} - \frac{d}{dx} + 1\big)u + 3\frac{d}{dx}v - \frac{d^2}{dx^2}w &= 0\\
-2\frac{d}{dx}u + \big(\frac{d^2}{dx^2} - 2\big)w &= 0\\
\big(-\frac{d^2}{dx^2} + 2\frac{d}{dx}\big)v + \frac{d}{dx}w &= 0.
\end{aligned}$$

The general solution will be superpositions of solutions of the form

$$\begin{pmatrix} u(x) \\ v(x) \\ w(x) \end{pmatrix} = \begin{pmatrix} A \\ B \\ C \end{pmatrix} e^{kx}.$$

To see what restrictions are placed on the constants A, B, C and k substitute into the differential equation. One obtains the algebraic equation

$$\begin{pmatrix} 2k^2 - k + 1 & 3k & -k^2 \\ -2k & 0 & k^2 - 2 \\ 0 & -k^2 + 2k & k \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus, k must be chosen so that the above matrix has determinant 0 and then, for such k, the vector $\begin{pmatrix} A \\ B \\ C \end{pmatrix}$ must be an eigenvector of eigenvalue 0.

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Example 1.7: If c is a constant then the n-dimensional wave equation has constant coefficients. The exponential solutions are the so-called plane wave solutions

$$p(\mathbf{x}, t) = Ae^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}.$$

Substituting into the wave equation one finds that

$$\mathbf{k}\cdot\mathbf{k} = \frac{\omega^2}{c^2}.$$

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Example 1.8: Acoustics is typically based on a linear approximation to fluid dynamics. Consider an undisturbed fluid at rest with constant density ρ_0 and pressure P_0 and no velocity. Let primed variables denote small disturbances from this quiescent state:

$$\rho = \rho_0 + \rho',$$

$$P = P_0 + P'$$

$$\mathbf{v} = \mathbf{v}'.$$

Expanding the equations of fluid dynamics in the primed variables and keeping only linear terms one obtains, identifying $\rho_0 = f(P_0)$,

$$\rho' = f'(P_0)P',$$

$$f'(P_0)\frac{\partial P'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}' = 0$$

and

$$\rho_0 \frac{\partial \mathbf{v}'}{\partial t} + \nabla P' = 0.$$

One finds plane wave solutions

$$\begin{pmatrix} P \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} A \\ \mathbf{B} \end{pmatrix} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega t}$$

with

$$-f'(P_0)\omega A + \rho_0 \mathbf{k} \cdot \mathbf{B} = 0$$

and

$$-\rho_0 \omega \mathbf{B} + \mathbf{k} A = 0.$$

Note first that if $\mathbf{k} \cdot \mathbf{B} = 0$ then $A = 0$ and the solution is trivial. Assuming that $\mathbf{k} \cdot \mathbf{B} \neq 0$ the second equation implies that

$$\mathbf{k} \cdot \mathbf{k} A = \rho_0 \omega \mathbf{k} \cdot \mathbf{B}.$$

Substituting into the first equation one finds

$$\omega^2 = \frac{\mathbf{k} \cdot \mathbf{k}}{f'(P_0)}.$$

Any $\mathbf{k}$ and ω satisfying this relation and any $\mathbf{B}$ with $\mathbf{k} \cdot \mathbf{B} \neq 0$ leads to a solution with

$$A = \frac{\rho_0 \omega \mathbf{k} \cdot \mathbf{B}}{\mathbf{k} \cdot \mathbf{k}}.$$

The factor $\frac{1}{f'(P_0)}$ is identified with the speed of sound squared.

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Review of Linear Algebra

In this chapter the basic concepts of linear algebra are reviewed. Most often our field of scalar quantities will be the field of complex numbers $\mathbf{C}$, however, we will sometimes have occasion to restrict our scalars to being real. Thus, when generality seems desirable, the symbol $\mathbf{K}$ will be used to denote either the set of real numbers $\mathbf{R}$ or the set of complex numbers $\mathbf{C}$.

Vector Spaces:

Crudely put, a *vector space* is a set which is closed under linear superposition: for vectors v and w, $av + bw$ is also a vector. Here a and b are scalars (ordinary numbers). Most often our field of scalar quantities will be the field of complex numbers $\mathbf{C}$, however, we will sometimes have occasion to restrict our scalars to being real.

The precise definition is: V is *vector space* over $\mathbf{K}$ if the elements of V can be multiplied by elements of $\mathbf{K}$ and added to each other so that given any $a, b \in \mathbf{K}$ and $v, w \in V$

$$av + bw \in V.$$

Example 2.1: $\mathbf{R}$ is a vector space over itself. $\mathbf{C}$ is both a vector space over itself and over $\mathbf{R}$.

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Example 2.2: The set of n-vectors

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

where $x_j \in \mathbf{R}$ is a vector space over $\mathbf{R}$. This space is called n-dimensional Euclidean space and is denoted $\mathbf{R}^n$.

Similarly, if $x_j \in \mathbf{C}$ the space is called n-dimensional complex space and is denoted $\mathbf{C}^n$. $\mathbf{C}^n$ is a vector space over both $\mathbf{R}$ and $\mathbf{C}$.

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Example 2.3: The set of all complex valued functions of a real variable is a vector space over both **R** and **C** since if f and g are any functions then $af + bg$ is also a function. Note that a function f is said to be 0, $f = 0$, only if $f(x) = 0$ regardless of x.

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Given a vector space V, N elements of V, $v_1, v_2, \ldots, v_N$, are said to be *linearly independent* if the only linear combination of the vectors to add up to 0 is the trivial one in which all the coefficients are 0: linear independent if

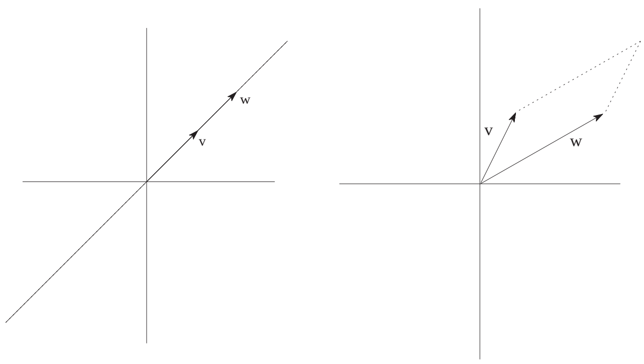
$$a_1 v_1 + a_2 v_2 + \ldots + a_N v_N = 0$$

only for $a_1 = a_2 = \ldots = a_N = 0$.

If $v_1, v_2, \ldots, v_N$ are not linearly independent then there is a *linear dependence relation* between them. That is, there are coefficients $a_1, a_2, \ldots, a_N$ not all of which are 0, for which

$$a_1 v_1 + a_2 v_2 + \ldots + a_N v_N = 0.$$

This means that these vectors are not independent in the sense that any one of them can be expressed as a linear combination of the rest.



For $N = 1$ the situation is trivial: one vector is always linearly dependent on itself. For $N = 2$ the vectors v_1 and v_2 are linearly dependent if they are proportional to each other:

$a_1v_1 + a_2v_2 = 0 \Longrightarrow v_2 = -\frac{a_2}{a_1}v_1$. Geometrically, representing vectors in the usual way as arrows pointing out from the origin in an n-dimensional Euclidean space, two vectors are linearly dependent if they are parallel. Conversely, while one vector determines a line, two linearly independent vectors determine a plane. Similarly, given two linearly independent vectors v_1 and v_2 and a third vector v_3, there is a linear dependence relation between v_1, v_2 and v_3 only if v_3 lies in the plane determined by v_1 and v_2. Otherwise v_1, v_2 and v_3 determine a 3-dimensional subspace of V.

In general, given $v_1, v_2, \ldots, v_N \in V$, the subspace of V given by the set of all linear combinations

$$a_1v_1 + a_2v_2 + \ldots + a_Nv_N$$

is itself a vector space and is called the vector space *spanned* by $v_1, v_2, \ldots, v_N$. The largest number of linearly independent vectors among the $v_1, v_2, \ldots, v_N$ is called the *dimension* of this subspace. It can be shown that in an n-dimensional vector space V

i) Any n linearly independent vectors in V span V.

ii) No set of m vectors can span V if $m < n$.

iii) No set of m vectors is linearly independent if $m > n$.

It follows that, given any n linearly independent vectors $b_1, b_2, \ldots, b_n$ in V, any vector v can be written as a linear combination of the b_j,

$$v = c_1b_1 + c_2b_2 + \ldots + c_nb_n.$$

Here the c_j are in $\mathbf{K}$. This follows since V is n-dimensional and thus there must be a linear dependence relation between any vector v and the set of the $b_j's$. A set of n linearly independent vectors in an n-dimensional vector space is called a *basis* for the vector space.

Example 2.4: $\mathbf{R}^n$ is an n-dimensional vector space over $\mathbf{R}$ since the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

are linearly independent. These n vectors form a basis for $\mathbf{R}^n$ known as the standard basis.

A non-standard basis for $\mathbf{R}^3$ is

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Similarly $\mathbf{C}^n$ is an n-dimensional vector space over $\mathbf{C}$. However, $\mathbf{C}^n$ is also a 2n-dimensional vector space over $\mathbf{R}$ since the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} i \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ i \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ i \end{pmatrix}$$

are linearly independent.

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Example 2.5: The set of n^{th} degree polynomials with complex coefficients,

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n,$$

is an (n+1)-dimensional vector space over $\mathbf{C}$ since the monomials

$$1,\ x,\ x^2,\ \ldots,\ x^n$$

are a basis for this set. Note that the set of all polynomials with complex coefficients, without restriction on degree, is infinite dimensional over $\mathbf{C}$.

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An *inner product* on a vector space V is a function of two vectors, say v and w, usually denoted by $\langle v, w\rangle$, which satisfies

i) $\langle v, w\rangle \in \mathbf{K}$

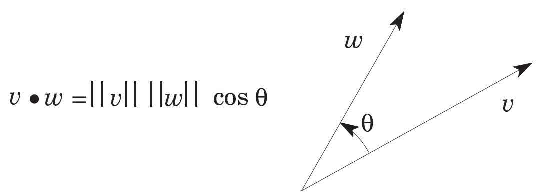
ii) $\langle v, w\rangle = \overline{\langle w, v\rangle}$

iii) $\langle v, aw_1 + bw_2\rangle = a\langle v, w_1\rangle + b\langle v, w_2\rangle$.

iv) $\langle v, v\rangle > 0$ if $v \neq 0$.

Here $\overline{z}$ is the complex conjugate of z, a and b are in $\mathbf{K}$ and w_1 and w_2 are in V.

Inner products are generalizations of the dot product in $\mathbf{R}^n$. In fact, for $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$, $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x} \cdot \mathbf{y}$ is an inner product. In $\mathbf{C}^n$ the standard inner product is $\langle \mathbf{x}, \mathbf{y}\rangle = \overline{\mathbf{x}} \cdot \mathbf{y}$. Once one has an inner product one can speak of orthogonality. Two vectors, v, w, are *orthogonal* if $\langle v, w\rangle = 0$. Similarly, the *norm* of a vector is $\|v\| = \sqrt{\langle v, v\rangle}$. An *orthonormal basis* for a vector space is a basis whose elements are all mutually orthogonal and whose norms are all 1. Recall that in the case of the standard inner product in $\mathbf{R}^n$ orthogonality means geometrically perpendicular and norm means geometric length.



If $b_1, \ldots, b_n$ is an orthonormal basis then

$$\langle b_j, b_k\rangle = \delta_{j,k}.$$

The nice thing about an orthonormal basis is that if one wishes to express any vector v as a linear superposition of the b_j,

$$v = c_1 b_1 + c_2 b_2 + \ldots + c_n b_n,$$

then the coefficients c_j are easy to find:

$$c_j = \langle b_j, v\rangle.$$

Example 2.6: Let f and g be functions on $[-1, 1]$. Define

$$\langle f, g\rangle = \int_{-1}^{1} \overline{f(x)}\, g(x)\, dx.$$

It's easy to check that this is an inner product on the vector space of functions on $[-1,1]$.

Note that with respect to this inner product the functions

$$s_n(x) = \sin(n\pi x)$$

$$c_n(x) = \cos(n\pi x),$$

for $n = 1, 2, 3, \ldots$, and $c_0 = \frac{1}{\sqrt{2}}$ are all mutually orthogonal. Further, they all have norm 1. The basic fact of Fourier analysis is that the span of these functions is large enough function space to satisfy all of our needs. Thus, they can be used as if they were an ordinary orthonormal basis. The precise statement is that if $\|f\|$ is finite then, given the Fourier coefficients

$$a_n = \int_{-1}^{1} c_n(x)\, f(x)\, dx$$

$$b_n = \int_{-1}^{1} s_n(x)\, f(x)\, dx,$$

one has

$$\lim_{N\to\infty} \left\| f - \sum_{n=0}^{N} \left(a_n c_n + b_n s_n \right) \right\| = 0.$$

The way to read this expression is that any f with finite norm can be well approximated, with respect to this norm, by superpositions of sine and cosine functions.

The difference between equality with respect to this norm and actual equality is that the Fourier series can differ from the function but only on a set which is so small that it doesn't effect the integral giving the norm (most often an isolated point). This is the origin of the Gibbs phenomenon.

It turns out to be true in a similar sense that any function with finite norm can be well approximated by polynomials. However, the monomials $f_n(x) = x^n$ are neither mutually orthogonal nor of norm 1. They can be orthogonalized by the Gram-Schmidt procedure, and then normalized (by dividing an unnormalized function by the square root of its norm). The standard choice of orthogonal polynomials with respect to this norm are known as the Legendre polynomials. The first few are

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

and so on. These functions are not normalized,

$$\int_{-1}^{1} P_j(x)^2\, dx = \frac{2}{2j+1},$$

but can be used as if they formed an orthogonal basis.

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Matrices and Linear Operators:

Matrices arise in several ways. The most straightforward way is from systems of linear equations: a system of m linear equations for n unknowns $x_1, x_2, \ldots, x_n$,

$$\sum_{k=1}^{n} a_{jk} x_k = b_j$$

for $j = 1, 2, \ldots, m$, can be written as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

or

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

where $\mathbf{A}$ is the obvious $m \times n$ matrix, $\mathbf{x} \in \mathbf{R}^n$ and $\mathbf{b} \in \mathbf{R}^m$. This looks very much like the 1-dimensional linear equation $ax = b$ and should be thought of (as much as possible) in the same way.

Given an $m \times n$ matrix $\mathbf{A}$ the *transpose* of $\mathbf{A}$, $\mathbf{A}^t$, is the $n \times m$ matrix obtained by exchanging rows for columns,

$$\mathbf{A}^t = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}.$$

The *adjoint* of $\mathbf{A}$, $\mathbf{A}^*$, is the complex conjugate of the transpose,

$$\mathbf{A}^* = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{21} & \cdots & \bar{a}_{m1} \\ \bar{a}_{12} & \bar{a}_{22} & \cdots & \bar{a}_{m2} \\ \vdots & \vdots & \cdots & \vdots \\ \bar{a}_{1n} & \bar{a}_{2n} & \cdots & \bar{a}_{mn} \end{pmatrix}.$$

We will restrict out attention to the case $m = n$ in which there are as many equations as unknowns. By analogy with the one dimensional case we would like to express the solution of $\mathbf{Ax} = \mathbf{b}$ as $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$, if this can be done. Clearly this procedure works if $\mathbf{A}$ is invertible in the sense that there is a matrix $\mathbf{A}^{-1}$ with $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ (here $\mathbf{I}$ is the $n \times n$ identity matrix, with 1 on the diagonal and 0 elsewhere).

If $\mathbf{A}$ is invertible then the only solution to $\mathbf{Ax} = 0$ is the trivial solution $\mathbf{x} = 0$. The converse turns out to be true as well: if the only solution to $\mathbf{Ax} = 0$ is the trivial solution $\mathbf{x} = 0$ then $\mathbf{A}$ is invertible. Also, it can be shown that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ implies that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$ from which it follows that $(\mathbf{A}^t)^{-1} = (\mathbf{A}^{-1})^t$.

If one writes the columns of $\mathbf{A}$ as vectors

$$\mathbf{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}$$

then

$$\mathbf{Ax} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots x_n\mathbf{a}_n.$$

Thus $\mathbf{Ax} = 0$ for $\mathbf{x} \neq 0$ is a linear dependence relation between the columns of $\mathbf{A}$. Similarly $\mathbf{A}^t\mathbf{x} = 0$ for $\mathbf{x} \neq 0$ is a linear dependence relation between the rows of $\mathbf{A}$ (note that if the rows of $\mathbf{A}$ are not linearly independent then the n equations in the linear system $\mathbf{Ax} = \mathbf{b}$ are not independent). It follows that $\mathbf{A}$ is invertible if either its columns or rows are linearly independent.

A final criteria for the invertibility of $\mathbf{A}$ is based on the *determinant*. The determinant of an $n \times n$ matrix, $\det \mathbf{A}$, may be defined inductively as follows. If $n = 1$, so that $\mathbf{A}$ is just a number, let $\det \mathbf{A} = \mathbf{A}$. If $n > 1$ let $\mathbf{A}^{(jk)}$ be the $n-1 \times n-1$ matrix obtained by removing the j^{th} row and k^{th} column from $\mathbf{A}$. Then define

$$\det \mathbf{A} = \sum_{k=1}^{n} (-1)^k a_{1k} \det \mathbf{A}^{(1k)}.$$

Properties of the determinant are

$$\det(\mathbf{AB}) = \det \mathbf{A} \det \mathbf{B}.$$

If $\mathbf{A}'$ is obtained from $\mathbf{A}$ by exchanging neighboring rows then $\det \mathbf{A}' = -\det \mathbf{A}$. Similarly if $\mathbf{A}'$ is obtained from $\mathbf{A}$ by exchanging neighboring columns then $\det \mathbf{A}' = -\det \mathbf{A}$.

The determinant is a bit abstract. Unfortunately it arises often. For example, it turns out that $\mathbf{A}$ is invertible if and only if $\det \mathbf{A} \neq 0$. In fact, if $\det \mathbf{A} \neq 0$, there is an explicit formula for $\mathbf{A}^{-1}$. If

$$\mathbf{A}^{-1} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix}$$

then

$$\alpha_{jk} = \frac{(-1)^{j+k}}{\det \mathbf{A}} \det \mathbf{A}^{(kj)}.$$

To summarize: the following statements are equivalent.

i) For any $\mathbf{b} \in \mathbf{C}^n$ the equation $\mathbf{Ax} = \mathbf{b}$ has a solution.

ii) $\mathbf{A}$ is invertible.

iii) The columns of $\mathbf{A}$ are linearly independent.

iv) The rows of $\mathbf{A}$ are linearly independent.

v) $\det \mathbf{A} \neq 0$.

vi) The only solution of $\mathbf{Ax} = 0$ is the trivial solution $\mathbf{x} = 0$.

Example 2.7: The system of equations

$$2x - y + z = 1$$

$$-x + 2y = -1$$

$$y - z = 2$$

is equivalent to

$$\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

Using the formula one finds that

$$\det \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix} = -4 + 1 - 1 = -4$$

and

$$\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & -1 \end{pmatrix}^{-1} = -\frac{1}{4} \begin{pmatrix} -2 & 0 & -2 \\ -1 & -2 & -1 \\ -1 & -2 & 3 \end{pmatrix}$$

so that

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \frac{1}{4} \begin{pmatrix} 2 & 0 & 2 \\ 1 & 2 & 1 \\ 1 & 2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{2} \\ \\ \frac{1}{4} \\ \\ -\frac{7}{4} \end{pmatrix}. \end{aligned}$$

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Example 2.8: If $b_1, \ldots, b_n$ is a basis for a vector space V, with inner product $\langle \cdot, \cdot \rangle$, to find the coefficients c_j in the expansion $v = c_1 b_1 + c_2 b_2 + \ldots + c_n b_n$ one must solve the n equations

$$\langle b_j, v \rangle = \sum_{k=0}^{n} \langle b_j, b_k \rangle c_k.$$

It follows that

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \langle b_1, b_1 \rangle & \cdots & \langle b_1, b_n \rangle \\ \vdots & \ldots & \vdots \\ \langle b_n, b_1 \rangle & \cdots & \langle b_n, b_n \rangle \end{pmatrix}^{-1} \begin{pmatrix} \langle b_1, v \rangle \\ \vdots \\ \langle b_n, v \rangle \end{pmatrix}.$$

Note that the condition that the matrix in this last expression be invertible is equivalent to the linear independence of the basis vectors.

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A linear transformation T on a vector space V is a function for which $Tv \in V$ for all $v \in V$ and which is linear, $T(av + bw) = aTv + bTw$. Given a basis for V, $b_1, b_2, \ldots, b_n$, any $v \in V$ can be written

$$v = c_1 b_1 + c_2 b_2 + \ldots + c_n b_n.$$

If T is linear one has

$$Tv = c_1 T b_1 + c_2 T b_2 + \ldots + c_n T b_n.$$

But $Tb_j \in V$ can be written as a superposition of the b_j,

$$Tb_j = a_{1j} b_1 + a_{2j} b_2 + \ldots + a_{nj} b_n,$$

so that

$$Tv = \sum_{jk} c_k a_{jk} b_j.$$

It follows that

$$Tv = \sum_j \alpha_j b_j$$

with

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

The matrix in this last expression is the matrix representing T in the basis $b_1, b_2, \ldots, b_n$.

Example 2.9: Let V be the vector space of polynomials of degree 3 or less. Then the derivative operator is a linear transformation from V into itself. Explicitly, if

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

is an element of V then the derivative of p is

$$\frac{d}{dx} p(x) = a_1 + 2a_2 x + 3a_3 x^2.$$

With respect to the basis $\{1, x, x^2, x^3\}$ one has

$$\frac{d}{dx} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

in the sense that if one specifies a third degree polynomial by listing the coefficients of the various powers in a 4 vector, arranged in increasing powers beginning with the constant term,

$$p(x) = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix},$$

then one has

$$\begin{aligned} \frac{d}{dx} p(x) &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \\ &= \begin{pmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ 0 \end{pmatrix}. \end{aligned}$$

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A vector v spans a 1-dimensional subspace. Eigenvectors of a linear transformation T are vectors which, under the action of T, are not moved out of the subspace they span. That means that there is some number λ for which

$$Tv = \lambda v.$$

λ is an *eigenvalue* of T and v is an *eigenvector* corresponding to λ.

Now consider matrices $\mathbf{A}$ with complex matrix elements a_{jk}. Recall the *adjoint* of $\mathbf{A}$, $\mathbf{A}^*$, is the matrix with matrix elements $\bar{a_{kj}}$, that is, the complex conjugate of the transpose of $\mathbf{A}$. There is a class of matrices called *normal* which satisfy $\mathbf{A}^*\mathbf{A} = \mathbf{A}\mathbf{A}^*$. The importance of normal matrices is that their eigenvectors can be chosen to give an orthonormal basis. While the concept of a normal matrix is a bit abstract, for us it will be sufficient to note that self-adjoint matrices, those for which $\mathbf{A}^* = \mathbf{A}$, are normal. Thus any self-adjoint $n \times n$ matrix has n orthonormal eigenvectors which can be used as a basis.

Note that given a self-adjoint matrix $\mathbf{A}$ and eigenvectors $b_1, b_2, \ldots, b_n$ chosen to be orthonormal one can form the matrix U whose columns are the b_j. Then one finds that

$$U^*\mathbf{A}U = \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_n \end{pmatrix}$$

where λ_j is the eigenvalue corresponding to b_j. Further, $U^*U = \mathbf{I}$ so that $\langle Uv, Uw \rangle = \langle v, w \rangle$. Given any vector $\mathbf{v}$ one has

$$\mathbf{v} = \sum_{j=1}^{n} \langle b_j, \mathbf{v} \rangle \, b_j.$$

But

$$U^*v = \begin{pmatrix} \langle b_1, \mathbf{v} \rangle \\ \vdots \\ \langle b_n, \mathbf{v} \rangle \end{pmatrix}$$

so that $U^*\mathbf{v}$ is the vector of the coefficents of the expansion of $\mathbf{v}$ with respect to the basis of the b_j. Noting that $U^*(\mathbf{A}\mathbf{v}) = (U^*\mathbf{A}U)(U^*\mathbf{v})$ one sees that $U^*\mathbf{A}U$ is the matrix representation of $\mathbf{A}$ with respect to the basis of eigenvectors b_j.

Note that finding eigenvalues is equivalent to asking if there is any λ with

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0.$$

In turn, this is possible only if the *secular equation*,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0,$$

is satisfied. In general this is an n^{th} degree polynomial equation for λ which has at most n roots.

Example 2.10: Consider the 2×2 matrix

$$\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

the secular equation is

$$(1-\lambda)(2-\lambda) - 1 = \lambda^2 - 3\lambda + 1 = 0$$

which has roots

$$\lambda_{\pm} = \frac{3}{2} \pm \frac{\sqrt{5}}{2}.$$

To find eigenvectors we need to solve

$$\begin{aligned}\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \left[\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} \frac{3}{2} \pm \frac{\sqrt{5}}{2} & 0 \\ 0 & \frac{3}{2} \pm \frac{\sqrt{5}}{2} \end{pmatrix} \right] \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2} \mp \frac{\sqrt{5}}{2} & -1 \\ -1 & \frac{1}{2} \mp \frac{\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.\end{aligned}$$

Although this is two equations they are not independent (since the determinant of the matrix is zero). It follows that

$$(-\frac{1}{2} \mp \frac{\sqrt{5}}{2})x - y = 0.$$

The eigenvectors can be chosen to be

$$\begin{pmatrix} 1 \\ -\frac{1}{2} - \frac{\sqrt{5}}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{5}}{2} \\ 1 \end{pmatrix},$$

the first corresponding to λ_+, the second to λ_-.

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Example 2.11: The matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is a simple example of a 2×2 matrix which only has one linearly independent eigenvector. The secular equation is

$$(1 - \lambda)^2 = 0$$

so that there is only one eigenvalue, $\lambda = 1$. The eigenvectors must satisfy

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

so that $b = 0$. Thus all eigenvectors are of the form

$$a \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

which is, up to linear lindependence, one vector.

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Review of Calculus of Several Variables

Differentiation:

Derivatives of functions of one variable are straightforward to define:

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Other notation for the derivative is $f'(x)$. Note that the equation for the line tangent to the graph $y = f(x)$ at the point $(a, f(a))$ is

$$y = f(a) + f'(a)(x - a).$$

For functions of several variables this definition won't work since h would have to be a vector and one can't divide by vectors. What is straightforward to define are the partial derivatives

$$\frac{\partial f}{\partial x_j} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{j-1}, x_j + h, x_{j+1}, \ldots, x_n) - f(x_1, \ldots, x_n)}{h}.$$

Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be an m-dimensional vector. The 1-dimensional definition is usually generalized by introducing an $m \times n$ matrix $Df(\mathbf{x})$, depending on $\mathbf{x}$, for which

$$\lim_{\mathbf{h} \to 0} \frac{1}{\|\mathbf{h}\|} \big\| f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) - Df(\mathbf{x})\mathbf{h} \big\| = 0.$$

If such a matrix exists, it turns out there can be only one, and $Df(\mathbf{x})$ is called the derivative of f. Note that the derivative matrix gives the best linear approximation to f about $\mathbf{x}$. Luckily, if the derivative of f exists at a point $\mathbf{x}$,

$$Df(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \ldots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ldots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \ldots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

The partial derivative matrix might exist even if the derivative does not, but we will not have occasion to deal with such pathologies. Further, if they are continuous, then mixed partial derivatives are equal:

$$\frac{\partial^2 f}{\partial x_j \partial x_k} = \frac{\partial^2 f}{\partial x_k \partial x_j}.$$

Note that if $f : \mathbf{R}^n \to \mathbf{R}$ then

$$Df(\mathbf{x})^t = \nabla f.$$

Let $f : \mathbf{R}^n \to \mathbf{R}^m$ and $g : \mathbf{R}^k \to \mathbf{R}^n$ be functions. Recall the chain rule,

$$\frac{\partial}{\partial x_j} f\big(g(\mathbf{x})\big) = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}\Big|_{g(\mathbf{x})} \frac{\partial g_k}{\partial x_j}.$$

In this matrix notation the chain rule has an appealing form. Recall the notation $(f \circ g)(\mathbf{x}) = f\big(g(\mathbf{x})\big)$. The chain rule becomes matrix multiplication,

$$D(f \circ g)(\mathbf{x}) = Df\big(g(\mathbf{x})\big)\, Dg(\mathbf{x}).$$

Example 3.1: Consider a partical moving under the influence of a potential energy $V(\mathbf{x})$. Let $\mathbf{x}(t)$ be the curve describing the partical's trajectory; t is time. Recall that the force felt by the partical is the gradient of V, $\mathbf{F} = \nabla V$ while the velocity with which the partical moves is $\mathbf{v}(t) = \frac{d\mathbf{x}(t)}{dt}$. The rate of change of potential energy with time is

$$\begin{aligned} \frac{\partial V\big(\mathbf{x}(\mathbf{t})\big)}{\partial t} &= \nabla V(\mathbf{x}) \cdot \frac{d\mathbf{x}(t)}{dt} \\ &= \mathbf{F} \cdot \mathbf{v}. \end{aligned}$$

This last expression is known as the mechanical power gained by the partical.

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Example 3.2: Let $f : \mathbf{R}^n \to \mathbf{R}$ be a function of n variables. Let $\mathbf{x}_0 \in \mathbf{R}^n$. The direction of $\nabla f(\mathbf{x}_0)$ is the direction of greatest increase of f at $\mathbf{x}_0$. The level surface of f at $\mathbf{x}_0$ is orthogonal to $\nabla f(\mathbf{x}_0)$. In particular, $\nabla f(\mathbf{x})$ is normal to the surface given by $f(\mathbf{x}) = 0$.

To see this one need only consider the curve $\mathbf{x}_0 = \mathbf{x}_0 + \hat{\mathbf{d}}s$. Then

$$\frac{df(\mathbf{x}_0 + \hat{\mathbf{d}}s)}{ds}\Big|_{s=0} = \nabla f(\mathbf{x}_0) \cdot \hat{\mathbf{d}}.$$

Clearly this is maximal when $\hat{\mathbf{d}}$ is in the same direction as $\nabla f(\mathbf{x})$, and is zero when $\hat{\mathbf{d}} \perp \nabla f(\mathbf{x})$.

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Example 3.3: The formalism of Thermodynamics consists of arcane notation for variants of the chain rule. In the simplest situations three macroscopic quantities are used to specify the state of a system: pressure P, temperature T and density ρ. In general an equation connecting these three variables, called an equation of state, is either derived or inferred from the microscopic properties of the system. The equation of state can generally be written

$$F(P, \rho, T) = 0$$

for some function F. In the case of an ideal gas $F(P, \rho, T) = P - \rho RT$ where R is the gas constant (Boltzman's constant divided by particle mass) and one has the familiar

$$P = \rho RT.$$

In particular, only two of the variables P, ρ, T are independent, however, one is free to decide which. For example, in an ideal gas, if one decides that P and T should be the independent variables then ρ is given by $\frac{P}{RT}$.

It is common in thermodynamics to use a differential relation to express the equation of state. Imagine that the state of the system varies with some parameter t. Then through the equation of state the derivatives of P, T and ρ are related,

$$\begin{aligned} \frac{d}{dt}F(P, \rho, T) &= \frac{\partial F}{\partial P}\frac{dP}{dt} + \frac{\partial F}{\partial T}\frac{dT}{dt} + \frac{\partial F}{\partial \rho}\frac{d\rho}{dt} \\ &= 0. \end{aligned}$$

If, as an example, one decides that P and T should be the independent variables then ρ becomes a function of P and T for which

$$\frac{d\rho}{dt} = -\frac{\frac{\partial F}{\partial P}}{\frac{\partial F}{\partial \rho}}\frac{dP}{dt} - \frac{\frac{\partial F}{\partial T}}{\frac{\partial F}{\partial \rho}}\frac{dT}{dt}.$$

By the chain rule

$$-\frac{\frac{\partial F}{\partial P}}{\frac{\partial F}{\partial \rho}} = \frac{\partial \rho}{\partial P}$$

and

$$-\frac{\frac{\partial F}{\partial T}}{\frac{\partial F}{\partial \rho}} = \frac{\partial \rho}{\partial T}.$$

In thermodynamics these equation are given the shorthand notation

$$d\rho = \frac{\partial \rho}{\partial P}\Big)_T dP + \frac{\partial \rho}{\partial T}\Big)_P dT$$

with

$$\frac{\partial \rho}{\partial P}\Big)_T = -\frac{\frac{\partial F}{\partial P}}{\frac{\partial F}{\partial \rho}}$$

and

$$\frac{\partial \rho}{\partial T}\Big)_P = -\frac{\frac{\partial F}{\partial T}}{\frac{\partial F}{\partial \rho}}.$$

In general, the notation $\frac{\partial f}{\partial x}\Big)_y$ means choose x and y as the independent variables, solve the equation of state to express the remaining variable as a function of x and y, substitute into f and then differentiate with respect to x. Of course, if one can explicitly solve the equation of state there is no need to use these implicit relations. For the case of the ideal gas law one finds

$$\frac{d\rho}{\rho} = \frac{dP}{P} - \frac{dT}{T}.$$

Now consider some quantity $G(P, \rho, T)$, G might be the entropy or the total energy. Then

$$\frac{d}{dt}G(P, \rho, T) = \frac{\partial G}{\partial P}\frac{dP}{dt} + \frac{\partial G}{\partial T}\frac{dT}{dt} + \frac{\partial G}{\partial \rho}\frac{d\rho}{dt}.$$

But $\frac{d\rho}{dt}$ can be expressed in terms of $\frac{dP}{dt}$ and $\frac{dT}{dt}$ as above, so that one has

$$\frac{d}{dt}G(P, \rho, T) = \Big(\frac{\partial G}{\partial P} + \frac{\partial G}{\partial \rho}\frac{\partial \rho}{\partial P}\Big)\frac{dP}{dt} + \Big(\frac{\partial G}{\partial T} + \frac{\partial G}{\partial \rho}\frac{\partial \rho}{\partial T}\Big)\frac{dT}{dt},$$

generally written in the shorthand notation

$$dG = \frac{\partial G}{\partial P}\Big)_T dP + \frac{\partial G}{\partial T}\Big)_P dT$$

with

$$\frac{\partial G}{\partial P}\Big)_T = \frac{\partial G}{\partial P} + \frac{\partial G}{\partial \rho}\frac{\partial \rho}{\partial P}$$

and

$$\left.\frac{\partial G}{\partial T}\right)_P = \frac{\partial G}{\partial T} + \frac{\partial G}{\partial \rho}\frac{\partial \rho}{\partial T}.$$

In practice, one often solves for the dependent variable and substitutes into G, obtaining a new function of 2 variables, for example $G(P, \frac{P}{RT}, T)$, which can be differentiated without the chain rule.

A function which arises in Acoustics is the entropy

$$S = S(P, \rho, T).$$

Choosing P and ρ for the independent variables one has

$$S = S(P, \rho, \frac{P}{R\rho}).$$

Making the approximation that acoustic processes are isentropic (that is, occur without a change in entropy) one has

$$S(P, \rho, \frac{P}{R\rho}) = \text{const}\ .$$

This is now a relation between ρ and P. Solving, if one can, gives

$$\rho = f(P)$$

for some function f, or, if one prefers,

$$P = g(\rho)$$

for some function g.

The quantity

$$g'(\rho_0) = \left.\frac{\partial P_0}{\partial \rho_0}\right)_S$$

arises in linear acoustics (Example 1.8) as the speed of sound squared. Note that

$$\begin{aligned} f'(P_0) &= \left.\frac{\partial \rho_0}{\partial P_0}\right)_S \\ &= \frac{1}{g'(\rho_0)}. \end{aligned}$$

This quantity can be obtained without producing explicit functions $g(\rho)$ or $f(P)$. Note that the ideal gas law implies that

$$\frac{1}{P}dP = \frac{1}{\rho}d\rho + \frac{1}{T}dT.$$

The chain rule gives

$$dS = \left.\frac{\partial S}{\partial P}\right)_\rho dP + \left.\frac{\partial S}{\partial \rho}\right)_P d\rho.$$

If the entropy is constant then any derivative of S is 0. In particular,

$$\left.\frac{\partial S}{\partial \rho}\right)_S = 0 = \left.\frac{\partial S}{\partial P}\right)_\rho \left.\frac{\partial P}{\partial \rho}\right)_S + \left.\frac{\partial S}{\partial \rho}\right)_P$$

so that

$$\left.\frac{\partial P}{\partial \rho}\right)_S = -\frac{\left.\frac{\partial S}{\partial \rho}\right)_P}{\left.\frac{\partial S}{\partial P}\right)_\rho}.$$

At this point one introduces the specific heats

$$c_V = T\left.\frac{\partial S}{\partial T}\right)_\rho = T\left.\frac{\partial S}{\partial P}\right)_\rho \left.\frac{\partial P}{\partial T}\right)_\rho$$

and

$$c_p = T\left.\frac{\partial S}{\partial T}\right)_P = T\left.\frac{\partial S}{\partial \rho}\right)_P \left.\frac{\partial \rho}{\partial T}\right)_P.$$

Using the ideal gas law

$$c_V = P\left.\frac{\partial S}{\partial P}\right)_\rho$$

and

$$c_p = -\rho\left.\frac{\partial S}{\partial \rho}\right)_P$$

so that

$$\left.\frac{\partial P}{\partial \rho}\right)_S = \frac{P}{\rho}\frac{c_p}{c_V}.$$

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Change of variables:

An important class of functions are the *change of variables*. These have $n = m$ and the determinant of their derivative matrix is non-zero. Let g be a change of variables. Then it can be shown that g is invertible. That is, for every $\mathbf{x}$ there is one $\mathbf{y}$ with $x = g(\mathbf{y})$. $\mathbf{y}$ is called $g^{-1}(\mathbf{x})$ and

$$D(g^{-1})(\mathbf{x}) = \Big(Dg(\mathbf{y})\Big)^{-1}.$$

Thus, if we are concerned with some function $f(\mathbf{x})$ for which $(f \circ g)(\mathbf{y})$ is somehow simpler, we can study $f \circ g$ and invert back to f later. If $m = 1$, that is $f : \mathbf{R}^n \to \mathbf{R}$, then the chain rule gives

$$\begin{aligned} D(f \circ g)(\mathbf{y}) &= Df(\mathbf{x})\, Dg(\mathbf{y}) \\ &= \begin{pmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{pmatrix} \begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial g_n}{\partial y_1} & \cdots & \frac{\partial g_n}{\partial y_n} \end{pmatrix} \end{aligned}$$

or, taking transposes,

$$\begin{pmatrix} \frac{\partial}{\partial y_1} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{pmatrix} (f \circ g) = \begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_n}{\partial y_1} \\ \vdots & \cdots & \vdots \\ \frac{\partial g_1}{\partial y_n} & \cdots & \frac{\partial g_n}{\partial y_n} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} f.$$

Often one sees this formula written in shorthand, setting $\mathbf{x} = g(\mathbf{y})$ and writing

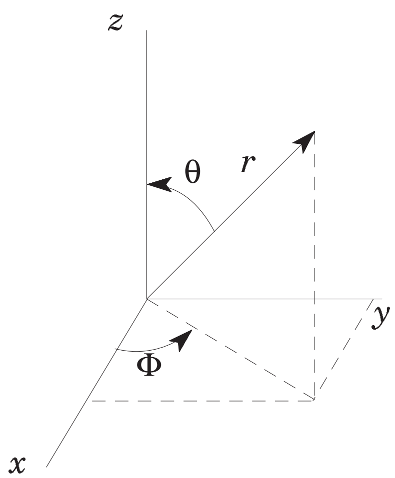
$$\begin{pmatrix} \frac{\partial}{\partial y_1} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_1} \\ \vdots & \cdots & \vdots \\ \frac{\partial x_1}{\partial y_n} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}.$$

Further, since the matrix on the right is a function of $\mathbf{y}$, one obtains a formula for the change of variables in the gradient operator,

$$\begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_1} \\ \vdots & \cdots & \vdots \\ \frac{\partial x_1}{\partial y_n} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial}{\partial y_1} \\ \vdots \\ \frac{\partial}{\partial y_n} \end{pmatrix}.$$

Example 3.4: Spherical coordinates in 3-dimensions can be given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix}.$$



The matrix of partial derivatives for this transformation is

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

and has determinant $r^2 \sin\theta$. Note that this change of variables is singular at $r = 0$ and $\sin\theta = 0$. The singularity at $\sin\theta = 0$ is fairly benign, however, this change of variables does introduce complications at $r = 0$.

To express the Cartesian gradient in spherical coordinates use the formula above:

$$\begin{aligned} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} &= \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{pmatrix} \\ &= \begin{pmatrix} \sin\theta\cos\phi & \frac{1}{r}\cos\theta\cos\phi & -\frac{\sin\phi}{r\sin\theta} \\ \sin\theta\sin\phi & \frac{1}{r}\cos\theta\sin\phi & \frac{\cos\phi}{r\sin\theta} \\ \cos\theta & -\frac{1}{r}\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \phi} \end{pmatrix} \end{aligned}$$

or

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi},$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

and

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial \theta}.$$

To compute the Laplace operator in spherical coordinates, substitute the above expressions into $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. After a somewhat long calculation, making sure to use the product rule for differentiation ($\frac{\partial}{\partial x} f(x)\frac{\partial}{\partial x} = \frac{\partial f}{\partial x}\frac{\partial}{\partial x} + f(x)\frac{\partial^2}{\partial x^2}$), one finds

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Big(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\Big). \tag{3.1}$$

Two other important changes of variables are *polar coordinates* in $\mathbf{R}^2$,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

and cylindrical coordinates in $\mathbf{R}^3$,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}.$$

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Taylor expansions:

Taylor's formula is one of the more useful tools around. In one dimension it reads

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \ldots + \frac{1}{n!}f^{(n)}(a)(x-a)^n + \ldots .$$

The question of how many terms to keep in this sum depends on the application, as does the choice of a. If the infinite series converges then f is said to be *analytic at a*. If f is not analytic at a the series might still be useful. It often happens that, even though the entire series diverges, the first few terms in the series are a reasonable approximation to f near a.

Example 3.5: Consider $\sin x$. It turns out that sin is analytic for all x. Even though $\sin x$ is analytic about $x = 0$ so that the Taylor series for $\sin x$ about $x = 0$,

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\ x^{2n+1},$$

converges, if x is not very close to 0 one needs to keep a prohibitively large number of terms to get a good approximation. In order to uniformly approximate $\sin x$ for some range of x using relatively low order polynomials one needs to piece together different expansions about different points.

Consider $x \in [0, \frac{\pi}{2}]$. About $x = \frac{\pi}{2}$ it is

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\ (x - \frac{\pi}{2})^{2n}.$$

To get a good uniform approximation over this interval it suffices to use these two expansions up to order 5. Between 0 and $\frac{\pi}{4}$ use the expansion about 0. Between $\frac{\pi}{4}$ and $\frac{\pi}{2}$ use the expansion about $\frac{\pi}{2}$.

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A convenient formula for the error in Taylor series is available. Write

$$f(x) = \sum_{n=1}^{N} \frac{1}{n!} f^{(n)}(a)(x-a)^n + \mathcal{E}_N(f, a, x).$$

The error $\mathcal{E}$ is given by

$$\mathcal{E}_N(f, a, x) = \frac{1}{(N+1)!} f^{(N+1)}(c)\ (x-a)^{N+1}$$

where c is unknown, but is between a and x. It is often possible to bound the derivatives of a function over some interval, giving a bound on the error. For example, for all n,

$$|\frac{d^n \sin x}{dx^n}| \leq 1$$

so that approximating $\sin x$ about $x = 0$ by an n^{th} order Taylor polynomial incurs an error no larger than

$$\frac{1}{(N+1)!}\ |x|^{N+1}.$$

To obtain a Taylor expansion for functions of several variables one needs only apply the one variable formula several times. In, say, 3 dimensions, first do a Taylor expansion in x. The coefficients depend on y and z. Expand them in y, and then again in z. One obtains

$$f(x, y, z) = \sum_{j+k+n=0}^{N} \frac{1}{j!k!n!}\ \frac{\partial^{j+k+n} f}{\partial x^j \partial y^k \partial z^n}\Big|_{x=a, y=b, z=c}\ (x-a)^j (y-b)^k (z-c)^n + \dots.$$

This formula gives us a way to look for local max and min for functions of several variables. Again, at a local max or min the graph of the function must flatten out (think of the top of a hill, at the very top you're neither climbing up nor down). Thus one must have $\nabla f = \mathbf{0}$ (a max or min in any direction). To second order the Taylor expansion becomes (setting $a = b = c = 0$)

$$f(x, y, z) = f(\mathbf{0}) + \nabla f(\mathbf{0}) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}\Big|_{\mathbf{0}} & \frac{\partial^2 f}{\partial x \partial y}\Big|_{\mathbf{0}} & \frac{\partial^2 f}{\partial x \partial z}\Big|_{\mathbf{0}} \\ \frac{\partial^2 f}{\partial x \partial y}\Big|_{\mathbf{0}} & \frac{\partial^2 f}{\partial y^2}\Big|_{\mathbf{0}} & \frac{\partial^2 f}{\partial y \partial z}\Big|_{\mathbf{0}} \\ \frac{\partial^2 f}{\partial x \partial z}\Big|_{\mathbf{0}} & \frac{\partial^2 f}{\partial y \partial z}\Big|_{\mathbf{0}} & \frac{\partial^2 f}{\partial z^2}\Big|_{\mathbf{0}} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \dots$$

The second derivative matrix

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2}\Big|_{\mathbf{0}} & \frac{\partial^2 f}{\partial x \partial y}\Big|_{\mathbf{0}} & \frac{\partial^2 f}{\partial x \partial z}\Big|_{\mathbf{0}} \\ \frac{\partial^2 f}{\partial x \partial y}\Big|_{\mathbf{0}} & \frac{\partial^2 f}{\partial y^2}\Big|_{\mathbf{0}} & \frac{\partial^2 f}{\partial y \partial z}\Big|_{\mathbf{0}} \\ \frac{\partial^2 f}{\partial x \partial z}\Big|_{\mathbf{0}} & \frac{\partial^2 f}{\partial y \partial z}\Big|_{\mathbf{0}} & \frac{\partial^2 f}{\partial z^2}\Big|_{\mathbf{0}} \end{pmatrix}$$

is symmetric and thus has three orthogonal eigenvectors with real eigenvalues. Let these eigenvalues be λ_1, λ_2 and λ_3 and let the corresponding eigenvectors be $\mathbf{v}^{(1)}$, $\mathbf{v}^{(2)}$ and $\mathbf{v}^{(3)}$. Assume they are normalized. Then one can write

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = t_1\mathbf{v}^{(1)} + t_2\mathbf{v}^{(2)} + t_3\mathbf{v}^{(3)}$$

where

$$t_j = \mathbf{v}^{(j)} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Thus, if $\nabla f(\mathbf{0}) = \mathbf{0}$, then

$$f(x, y, z) = f(\mathbf{0}) + \frac{1}{2}\Big(\lambda_1 t_1^2 + \lambda_2 t_2^2 + \lambda_3 t_3^2\Big).$$

If all the $\lambda_j > 0$ one has a min. If all the $\lambda_j < 0$ one has a max. If some $\lambda_j < 0$ while others are > 0 then one has a saddle, neither max nor min (like a mountain pass between two peaks). If any of the $\lambda_j = 0$ then it's not clear what's happening and one must do more.

Integration:

Integration of functions of one variable is relatively straightforward. The integral of a function of one variable is defined geometrically to be the area between the graph of the function and the x-axis, with a positive contribution from the part above the x-axis and a negative contribution from the part below. Analytically, one has the *Fundamental Theorem of Calculus* which says that derivatives and integrals undo each other,

$$\int_a^x f'(t)\,dt = f(x) - f(a)$$

or

$$\frac{d}{dx}\int_a^x g(t)\,dt = g(x).$$

Thus, integration of functions of one variable is reduced to guessing at anti-derivatives. Recall that inverse to the chain rule one has the technique of substitution (one dimensional change of variables),

$$\int_a^b f(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f\big(g(y)\big)\,g'(y)\,dy$$

and inverse to the product rule one has integration by parts

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\big|_a^b - \int_a^b f'(x)g(x)\,dx.$$

Numerically, efficient algorithms can be devised to estimate areas under curves.

Example 3.6: Integration by parts is sometimes usefull in generating asymptotic expansions to integrals. A classic example is the *Riemann-Lebesgue lemma* which is the $J = 1$ version of the following.

Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be J times differentiable with the n^{th} derivatives $|f^{(n)}|$ integrable over $\mathbf{R}$ for $n = 0, 1, 2, \ldots, J$. Then necessarily $f^{(n)}(\pm\infty) = 0$. Consider the large ω behavior of the Fourier transform of f,

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt.$$

Noting that

$$e^{i\omega t} = \frac{1}{i\omega}\frac{de^{i\omega t}}{dt}$$

and integrating by parts J times one finds that

$$\hat{f}(\omega) = \frac{(-1)^J}{(i\omega)^J}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^{(J)}(t)e^{i\omega t}\,dt.$$

It follows that

$$|\hat{f}(\omega)| \le \frac{1}{\omega^J}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f^{(J)}(t)|\,dt$$

so that the Fourier transform goes to zero as $\omega \to \infty$ at least as fast as $\frac{1}{\omega^J}$. If $J = \infty$ then the Fourier transform is said to go to zero "exponentially fact". If J is finite and f does not have an intrgrable $J + 1$ derivative then its Fourier transform decreases precisely as $\frac{1}{\omega^J}$ as $\omega \to \infty$.

Another example is provided by the large x asymptotic expansion for the integral (related to the Error function)

$$\int_x^{\infty} e^{-t^2}\,dt.$$

Using

$$e^{-t^2} = -\frac{1}{2t}\frac{de^{-t^2}}{dt}$$

and integrating by parts repeatedly one finds

$$\int_x^\infty e^{-t^2}\,dt = \frac{1}{2x} - \frac{1}{4x^3} + \frac{3}{8x^5} + \dots.$$

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Integration of functions of several variables is a problem. For one thing, the possible regions one can integrate over increases in complexity as the number of dimensions increases. Further, the only analytical tool available is to transform an integral over several variables into several integrals over one variable, and this can be done only for relatively simple functions integrated over extremely simple domains. In addition, while numerical methods exist to estimate integrals over several variables they are by no means as efficient as the one variable algorithms.

To attempt to simplify integrals over several variables there are essentially only two tricks available. The first is to use a change of variables, if one can be found, which either simplifies the function which is being integrated or the domain over which one is integrating. If g is an n-dimensional change of variables then

$$\int_D f(\mathbf{x})\,d^n x = \int_{g^{-1}(D)} f\big(g(\mathbf{y})\big)\,|\det\big(Dg(\mathbf{y})\big)|\,d^n y.$$

Here $g^{-1}(D)$ is the set of points that gets mapped to D by g. The factor $\det\big(Dg(\mathbf{y})\big)$ is known as the *Jacobian determinant* for the change of variables. Note the absolute value. (Why is there no absolute value in the 1-dimensional case?)

Example 3.7: Let D be the 3-dimensional ball of radius R centered at the origin. Let $\mathbf{q}$ be a vector in $\mathbf{R}^3$ and consider

$$\int_D e^{\mathbf{q}\cdot\mathbf{x}}\,dx\,dy\,dz.$$

First, choose the coordinate system so that $\mathbf{q}$ is parallel to the z axis and let

$$\mathbf{q} = \begin{pmatrix} 0 \\ 0 \\ q \end{pmatrix}.$$

Thus $\mathbf{q} \cdot \mathbf{x} = qz$. Then change variables to spherical coordinates so that D becomes simple. The points that get mapped to D in spherical coordinates are $r \in [0, R]$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$. Thus

$$\int_D e^{\mathbf{q}\cdot\mathbf{x}}\, dx\, dy\, dz = \int_0^{2\pi} \int_0^{\pi} \int_0^R e^{qr\cos\theta}\, r^2 \sin\theta\, dr\, d\theta\, d\phi.$$

The integral over ϕ can be done immediately. To do the integral over θ make a one dimensional change of variables $\eta = cos\theta$. Then $d\eta = -\sin\theta\, d\theta$ and

$$\begin{aligned}\int_0^{2\pi} \int_0^{\pi} \int_0^R e^{qr\cos\theta}\, r^2 \sin\theta\, dr\, d\theta\, d\phi &= 2\pi \int_{-1}^{1} \int_0^R e^{q\eta r} r^2\, dr\, d\eta \\ &= \frac{2\pi}{q} \int_0^R \Big(e^{qr} - e^{-qr}\Big)\, r\, dr \\ &= 2\pi \Big[\Big(\frac{R}{q^2} - \frac{1}{q^3}\Big) e^{qR} + \Big(\frac{R}{q^2} + \frac{1}{q^3}\Big) e^{-qR}\Big].\end{aligned}$$

....

The second tool is the n-dimensional generalization of the fundamental theorem of calculus called Stoke's theorem. In it's general form it is deceptively complex and requires a lot of mathematical machinery just to write down. This complexity arises both from the variety of surfaces over which one can integrate in n dimensions as well as from the variety of ways one can take a first derivative.

For our purposes it is sufficient to restrict our attention to two simple low dimensional cases known as Stoke's and Gauss' theorem (classically, Stoke's theorem was the low dimensional case, when the generalization was found it was also called Stoke's theorem).

For functions $\mathbf{f} : \mathbf{R}^2 \to \mathbf{R}^2$ there are two types of first derivatives of interest. The *divergence* of $\mathbf{f}$ is a scalar given by

$$\nabla \cdot \mathbf{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}.$$

Similarly the *curl* is also a scalar given by

$$\nabla \times \mathbf{f} = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}.$$

For functions $\mathbf{f} : \mathbf{R}^3 \to \mathbf{R}^3$ there are also two types of first derivatives of interest. The *divergence* of $\mathbf{f}$ is a scalar given by

$$\nabla \cdot \mathbf{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}.$$

However the *curl* is a vector given by

$$\nabla \times \mathbf{f} = \begin{pmatrix} \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \\ \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \\ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{pmatrix}.$$

Both Stoke's and Gauss' theorem deal with functions of two or three variables. Gauss' theorem for functions of three variables is as follows: let V be a volume in $\mathbf{R}^3$ denote by ∂V the boundary of V and by $\mathbf{n}$ the outward pointing unit normal vector field to ∂V. Then if $\mathbf{u} : \mathbf{R}^3 \to \mathbf{R}^3$ is a vector field

$$\int_V \nabla \cdot \mathbf{u} \, d^3x = \int_{\partial V} \mathbf{n} \cdot \mathbf{u} \, d\sigma.$$

Here $d\sigma$ is the surface element on ∂V. A general discussion of surface elements is beyond the scope of this course. In general it is required to have a parameterization of the surface, $\mathbf{x}(v, w)$, and a change of variables in a neighborhood of the surface from $x,\ y,\ z$ to $v,\ w,\ t$ where t is the distance along the normal vector from the surface.

Finding a surface element is simplified if a change of variables can be found so that the surface in question is obtained by holding one of the new variables constant. For example, consider a sphere of radius R. In spherical coordinates this is a surface of constant r, $r = R$. The volume element in spherical coordinates is

$$r^2 \sin\theta \, d\theta \, d\phi \, dr.$$

Thus, the surface element to a sphere of radius R can be taken to be $R^2 \sin\theta \, d\theta \, d\phi$.

Normal vectors are easier. Consider a surface given by an equation

$$f(\mathbf{x}) = 0.$$

Recall that ∇f is a vector pointing in the direction in which f changes most rapidly. This implies that ∇f is normal to the surface.

Example 3.8: Consider a fluid. As before let $\rho(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ be the density and velocity at point $\mathbf{x}$. Then $\rho\mathbf{v}$ is the mass flux in the system. It gives the mass per unit area and time transported in the direction of $\mathbf{v}$. Given a volume V the change in mass in V is given by

$$\frac{\partial}{\partial t}\int_V \rho\, d^3x = -\int_{\partial V} \rho\mathbf{v}\cdot\mathbf{n}\, d\sigma.$$

That is, the change of mass in V is negative of the rate at which mass enters or leaves V. By Gauss' theorem

$$\int_{\partial V} \rho\mathbf{v}\cdot\mathbf{n}\, d\sigma = -\int_V \nabla\cdot(\rho\mathbf{v})\, d^3x.$$

Since this must be true for any volume V one can conclude that the equation of continuity,

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\mathbf{v})$$

must be true.

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Stokes theorem relates the integral of the curl of a function over a surface to the integral of the function over the boundary of the surface. Let S be a surface in $\mathbf{R}^3$. Then

$$\int_S (\nabla\times\mathbf{u})\cdot\mathbf{n}\, d\sigma = \int_{\partial S} \mathbf{u}\cdot d\mathbf{l}.$$

Here $d\mathbf{l}$ is the line element over the boundary ∂S of S. Line elements are easier to handle than surface elements. If $\mathbf{x}(t)$ is a parameterization of a curve then the line element along this curve is

$$d\mathbf{l} = \mathbf{x}'(t)\, dt.$$

An identity which follows from Gauss' theorem and which is used often in Acoustics is Green's theorem. It is a 3-dimensional version of integration by parts. Let f and g be functions on $\mathbf{R}^3$. Green's theorem states that

$$\int_v \Big(f\Delta g - g\Delta f\Big)\, d^3x = \int_{\partial V} \Big(f\nabla g - g\nabla f\Big)\cdot\mathbf{n}\, d\sigma.$$

It follows from Gauss' theorem on noting that

$$f\Delta g - g\Delta f = \nabla \cdot \Big(f\nabla g - g\nabla f \Big).$$

Dirac delta function:

The Dirac delta function is not a function at all, but is useful notation for the linear transformation which takes a function into it's value at a given point, say w. The rule is

$$\int_V f(\mathbf{x})\delta(\mathbf{x}-\mathbf{w})\, d^n x = \begin{cases} f(\mathbf{w}) & \text{if } \mathbf{w} \in V \\ 0 & \text{if } \mathbf{w} \notin V \end{cases}.$$

Note that this is a linear operation. An integral with a delta function in it is easy, it's not an integral at all.

Example 3.9: Let f be continuous at y and let g_ϵ be any family of functions satisfying

$$\lim_{\epsilon \downarrow 0} g_\epsilon(x) = 0$$

if $x \neq 0$ and

$$\int_{-\infty}^{\infty} g_\epsilon(x)\, dx = 1$$

for all $\epsilon > 0$. Then

$$\lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} f(x) g_\epsilon(x-y)\, dx = f(y).$$

In this sense

$$\lim_{\epsilon \downarrow 0} g_\epsilon(x-y) = \delta(x-y).$$

Examples of such families are

$$g_\epsilon(x) = \begin{cases} 0 & \text{if } |x| > \epsilon \\ \frac{1}{2\epsilon} & \text{if } |x| < \epsilon \end{cases}$$

or

$$g_\epsilon(x) = \frac{1}{\sqrt{\pi\epsilon}} e^{-\frac{1}{\epsilon}x^2}.$$

Example 3.10: If f is continuous at y note that, ignoring the singularity at $y = 0$ and using formal integration by parts,

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\frac{d^2}{dx^2}|x-y|\,dx &= \lim_{\epsilon\downarrow 0}\int_{y-\epsilon}^{y+\epsilon} f(x)\frac{d^2}{dx^2}|x-y|\,dx \\
&= f(y)\lim_{\epsilon\downarrow 0}\frac{d}{dx}|x-y|\big|_{x=y-\epsilon} x = y+\epsilon \\
&= 2f(y).
\end{aligned}$$

In this sense

$$\frac{d^2}{dx^2}|x-y| = 2\,\delta(x-y).$$

....

There is a subtlety under change of variables. Let g be a change of variables. Then, taking the integral notation above seriously (even though $\delta(\mathbf{x})$ is really nonsense) one has, assuming that $\mathbf{0} \in V$,

$$\begin{aligned}
\int_V f(\mathbf{x})\delta(\mathbf{x})\,d^n x &= \int_{g^{-1}(V)} f\big(g(\mathbf{y})\big)\delta\big(g(\mathbf{y})\big)|\det Dg(\mathbf{y})|\,d^n y \\
&= f(\mathbf{0}).
\end{aligned}$$

Thus we find that in order for this notation to be self-consistent we must have

$$\delta\big(g(\mathbf{y})\big)|\det Dg(\mathbf{y})| = \delta\big(y - g^{-1}(\mathbf{0})\big).$$

One often sees this formula expressed by saying that under the change of variables g, $\delta(x)$ becomes

$$\frac{1}{|\det Dg(\mathbf{y})|}\delta\big(\mathbf{y} - g^{-1}(\mathbf{0})\big)$$

or even

$$\frac{1}{|\det Dg\big(g^{-1}(\mathbf{0})\big)|}\delta\big(\mathbf{y} - g^{-1}(\mathbf{0})\big).$$

Unfortunately this doesn't work if g is singular where it's 0, that is, if

$$\det Dg\big(g^{-1}(\mathbf{0})\big) = 0.$$

This is the case in polar coordinates where $\det Dg(r,\theta,\phi) = r^2 \sin\theta$, and $g^{-1}(\mathbf{0})$ means $r = 0$. In polar coordinates there are many ways to see what's going on, but the simplest is to guess. One wants

$$\begin{aligned} \int f(\mathbf{x})\delta(\mathbf{x})\, d^n x &= \int f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\, r^2 \sin\theta \;\, \delta(\mathbf{x})\, dr\, d\theta\, d\phi \\ &= f(\mathbf{0}). \end{aligned}$$

In $\delta(\mathbf{x})$ one expects a factor of $\frac{1}{r^2}$, by analogy with the regular case, and clearly a factor of $\delta(r)$ is needed, but what does one do with θ and ϕ? The problem is that, at $r = 0$, θ and ϕ make no difference. However, formally putting $r = 0$ in f and integrating over θ and ϕ gives the correct answer up to a factor of 4π. Thus, choosing

$$\delta(\mathbf{x}) = \frac{1}{4\pi r^2}\delta(r)$$

seems to be a good guess. It turns out to be right.

Example 3.11: Here we show that in three dimensions

$$\Delta \frac{1}{\|\mathbf{x}-\mathbf{y}\|} = -4\pi\delta(\mathbf{x}-\mathbf{y})$$

in the sense that if f is continuous at $\mathbf{0}$ then

$$\int f(\mathbf{x})\Delta \frac{1}{\|\mathbf{x}-\mathbf{y}\|}\; d^3x = -4\pi f(\mathbf{y}).$$

First note that the shift of variables $\mathbf{x} \mapsto \mathbf{x}-\mathbf{y}$ is a change of variables whose partial derivative matrix is the identity matrix

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, under a shift of variables, partial derivatives don't change. It follows that the formula is true in general if it's true for $\mathbf{y} = \mathbf{0}$.

The representation of Δ in spherical coordinates developed in Example 3.4 is valid everywhere except at $r = 0$. For $r \neq 0$ it shows that

$$\Delta \frac{1}{\|\mathbf{x}\|} = 0.$$

Thus only $r = 0$ can contribute to this integral. To see what's happening at $r = 0$ integrate over a small ball of radius ϵ centered at $\mathbf{x} = \mathbf{0}$, B, and then let $\epsilon \downarrow \mathbf{0}$. Since f is continuous at $\mathbf{0}$ it approaches $f(\mathbf{0})$ and can be replaced by $f(\mathbf{0})$ in the integral. One has from Gauss' theorem

$$\begin{aligned}\int_B \Delta \frac{1}{\|\mathbf{x}\|}\, d^3x &= \int_{\partial B} \mathbf{n} \cdot \big(\nabla \frac{1}{\|\mathbf{x}\|}\big)\, d\sigma \\ &= -\int_0^{2\pi} \int_0^{\pi} \sin\theta\, d\theta\, d\phi \\ &= -4\pi.\end{aligned}$$

Here, that

$$\mathbf{n} \cdot \nabla = \frac{\partial}{\partial r}$$

on the surface of a sphere, is used.

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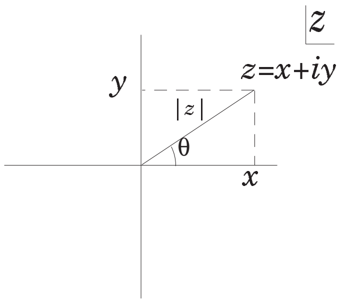
Review of Complex Analysis

The set of complex numbers, **C**, is a 2-dimensional vector space over **R** with, in addition to the vector space structure, a multiplication defined. The standard basis for **C** is the set $\{1, i\}$ where 1 is the real number and i is anything whose square is -1, $i^2 = -1$. Any complex number can be written as a linear combination $x + iy$ where x and y are real. Given $z = x + iy$, x is called the real part of z, $x = \operatorname{Re} z$, and y the imaginary part, $y = \operatorname{Im} z$. The *complex conjugate* of z is $\bar{z} = x - iy$.



Taking seriously the representation of complex numbers as 2-dimensional vectors over **R** and representing them graphically using the real part as the x component and the imaginary part as the y component one sees that $|z|$ defined by

$$|z|^2 = \bar{z}z$$

is the Euclidean norm $\sqrt{x^2 + y^2}$. In polar coordinates $z = x + iy$ becomes

$$z = |z|(\cos\theta + i\sin\theta)$$

where θ is the angle between the x-axis and the vector (x, y). It is a fact, known as *DeMoivre's Theorem* that

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{4.1}$$

This can be seen as a definition of the complex exponential once one has checked that it has all the desired properties. There are many ways to do this. The easiest is to note that

$$\frac{d}{d\theta}(\cos\theta + i\sin\theta) = i(\cos\theta + i\sin\theta),$$

as one expects from $e^{i\theta}$, and, setting $\theta = 0$ one obtains 1, also as expected.

Example 4.1: The set of 2×2 antisymmetric matrices with real matrix elements and equal diagonal elements is algebraically identical to the set of complex numbers. Such a matrix is of the form

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

for real x and y. This is a 2-dimensional vector space over $\mathbf{R}$ with a basis given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so that $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ acts like i.

Many interesting conclusions can be drawn from this representation. For example, note that multiplying a complex number z by $e^{i\theta}$ rotates the vector representing z through an angle θ. It follows that

$$e^{i\theta} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is the matrix which rotates 2-dimensional vectors through an angle θ.

....

Functions, $f : \mathbf{C} \to \mathbf{C}$, given by

$$f(x + iy) = u(x, y) + iv(x, y)$$

are really functions from $\mathbf{R}^2 \to \mathbf{R}^2$ given by

$$F(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}.$$

This leads to some subtleties when one tries to speak of differentiation. On the one hand, one would like the derivative of f to be a complex valued function f'. On the other, the derivative of F is a 2×2 matrix given by the partial derivative matrix. The condition that these two notions be identical leads to the notion of an analytic function as follows.

Note that multiplication by a complex number $a + ib$ is linear and induces a linear transformation of $\mathbf{R}^2$. Explicitly

$$(a + ib)(x + iy) = ax - by + i(ay + bx)$$

is the same as

$$\begin{pmatrix} ax - by \\ ay + bx \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

A function $f = u + iv : \mathbf{C} \to \mathbf{C}$ is said to be *analytic* if the derivative matrix of the associated function from $\mathbf{R}^2 \to \mathbf{R}^2$ corresponds, as a linear transformation, to multiplication by a complex number. Explicitly, if

$$\begin{aligned} DF &= \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \\ &= \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \end{aligned}$$

for some numbers a and b then $f = u + iv$ is analytic. The *Cauchy-Riemann Equations* for an analytic function follow immediately:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = a$$

and

$$-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = b.$$

The derivative of f is then defined to be

$$f' = a + ib.$$

Analyticity (or complex differentiability) turns out to be a very strong condition, much more restrictive than ordinary differentiability. Analytic functions are extremely regular in their behavior. It turns out that if a function f is analytic at some point, say $w \in \mathbf{C}$, then f is infinitely differentiable at w and the Taylor series for f about w converges (this is often taken as an alternative definition of analyticity). In fact the radius of convergence of the Taylor expansion is the distance from w to the closest point of non-analyticity. For example, the expansion

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$$

converges for all z with $|z| < 1$.

Example 4.2: A function f is analytic if it is complex differentiable. Checking for complex differentiability amounts to checking that the Cauchy-Riemann equations hold. This is often straightforward, although sometimes checking that the Taylor series converges is easier. That

$$e^{x+iy} = e^x \big(\cos y + i \sin y \big)$$

is analytic everywhere follows easily from the Cauchy-Riemann equations. That any polynomial

$$p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

is analytic everywhere follows since $p(z)$ is its own Taylor series and, since it is a finite sum, converges everywhere. If $j < 0$ the function z^j is analytic everywhere except at $z = 0$ where it's not differentiable and if $j > 0$ but is not an integer then z^j is also analytic everywhere except at $z = 0$, although if $j > 1$ the reason is subtle (see below).

The Cauchy-Riemann equations show that the function $f(z) = \bar{z}$, complex conjugation, is not analytic anywhere. Neither is any polynomial in $\bar{z}$.

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Complex integration is also more involved than real integration since it is an essentially 2-dimensional concept and necessarily involves integration over curves. Let Γ be some curve in $\mathbf{C}$. In order to integrate a function f over Γ one needs a parameterization of Γ, that is a function $w : [a, b] \to \mathbf{C}$ whose image is Γ. Then

$$\int_\Gamma f(z)\, dz = \int_a^b f\big(w(t)\big)\, w'(t)\, dt.$$

Clearly, given a geometric curve Γ there are many ways to parameterize it. For example, the circle of radius r centered at 0 can be parameterized by $w(t) = re^{it}$ for $t \in [0, 2\pi]$ as well as by $w(t) = t + i\sqrt{r^2 - t^2}$ for $t \in [0, r]$, or even $w(t) = rt^i$ for $t \in [1, e^{2\pi}]$. However, one can show that the integral over Γ is independent of what function $w(t)$ is used to parameterize it. Note as well that there are no subtleties involved with w' since w is a function (or in fact two real valued functions) of one variable.

A great aid in integrating analytic functions is *Cauchy's Theorem* which is a direct application of Gauss' theorem to the real and imaginary parts of fw'. Consider a region S. Then Cauchy's theorem can be stated as

$$\int_{\partial S} f(z)\, dz = 0 \tag{4.2}$$

if f is analytic in S. The reason is simple. Let $f = u + iv$ and $w(t) = x(t) + iy(t)$ be a parameterization of ∂S. Then

$$fw' = ux' - vy' + i(uy' + vx')$$

so that

$$\begin{aligned}\int_{\partial S} f(z)\, dz &= \int \begin{pmatrix} u\big(x(t), y(t)\big) \\ -v\big(x(t), y(t)\big) \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} dt + i \int \begin{pmatrix} v\big(x(t), y(t)\big) \\ u\big(x(t), y(t)\big) \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} dt \\ &= \int_{\partial S} \begin{pmatrix} u \\ -v \end{pmatrix} \cdot d\mathbf{l} + i \int_{\partial S} \begin{pmatrix} v \\ u \end{pmatrix} \cdot d\mathbf{l}\end{aligned}$$

where

$$d\mathbf{l} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} dt$$

is a line element along ∂S. Gauss' law implies that

$$\begin{aligned}\int_{\partial S} \begin{pmatrix} u \\ -v \end{pmatrix} \cdot d\mathbf{l} &= \int_S \nabla \cdot \begin{pmatrix} u \\ -v \end{pmatrix} d^2x \\ &= \int_S \Big(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \Big) d^2x\end{aligned}$$

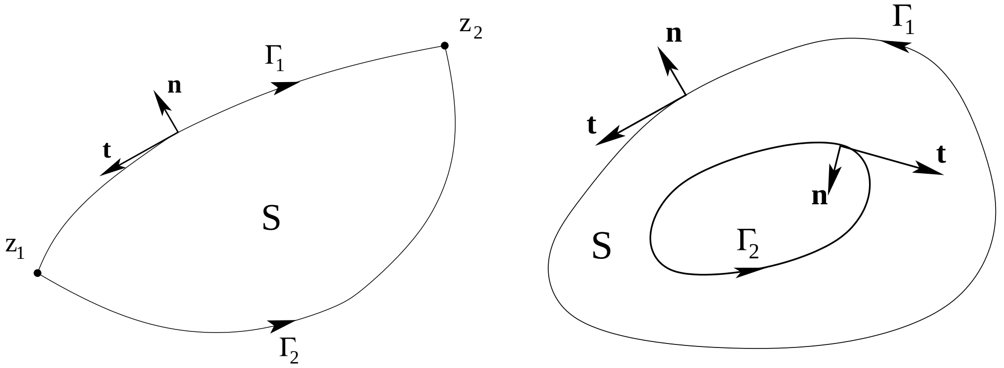
and that

$$\begin{aligned}\int_{\partial S} \begin{pmatrix} v \\ u \end{pmatrix} \cdot d\mathbf{l} &= \int_S \nabla \cdot \begin{pmatrix} v \\ u \end{pmatrix} d^2x \\ &= \int_S \Big(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \Big) d^2x.\end{aligned}$$

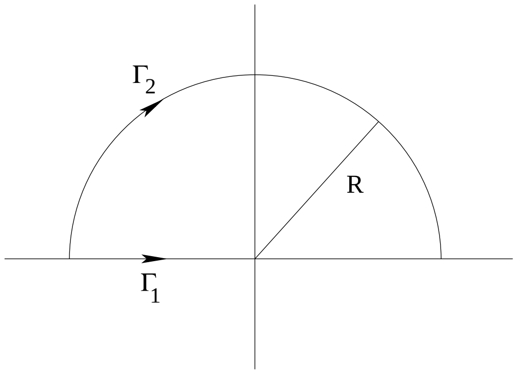
However, if f is analytic then u and v satisfy the Cauchy-Riemann equations which state that the integrands in the two expressions above are 0.

An immediate consequence of Cauchy's theorem is the notion of *path independence*. Let $f : \mathbf{C} \to \mathbf{C}$ and let Γ_1 and Γ_2 be different curves in $\mathbf{C}$, open or closed, for which either curve may be continuously deformed into the other without encountering any point at which f is not analytic. In particular, Γ_1 and Γ_2 enclose a region in which f is analytic. Then

$$\int_{\Gamma_1} f(z)\, dz = \int_{\Gamma_2} f(z)\, dz.$$



In the figures above two situations are depicted. In one the points z_1 and z_2 are connected by two dinstinct paths. The region S is the interior of the closed curve formed by the union of Γ_1 and Γ_2. In the other both curves Γ_j are closed, but one is contained in the interior of the other. The region S is the annular region between the two curves. The orientation of the boundary ∂S is determined by the convention that the outward pointing normal $\mathbf{n}$ and the tangent $\mathbf{t}$ have the orientation of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ respectivlely. The orientations of the curves Γ_j are indicated with arrow heads. Note that the orientations of the Γ_j and ∂S need not coincide.



Example 4.3: Since $\frac{e^{iz}}{z+i}$ is analytic in the upper half-plane one may, for any $R > 0$, replace the integral over $[-R, R]$ (the contour Γ_1 in the figure above) by the integral over the semi-circle of radius R (the contour Γ_2 in the figure above). Parameterizing the semi-circle with $Re^{i\theta}$ and then letting $R \to \infty$ one has

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{ix}}{x+i}\, dx &= \lim_{R\to\infty} \Big[\int_{-\infty}^{-R} \frac{e^{ix}}{x+i}\, dx + \int_{\pi}^{0} \frac{e^{iRe^{i\theta}}}{Re^{i\theta}+i}\, iRe^{i\theta}\, d\theta + \int_{R}^{\infty} \frac{e^{ix}}{x+i}\, dx \Big] \\ &= 0. \end{aligned}$$

Note that this argument requires that the integrand decrease more rapidly with increasing R than R^{-1}.

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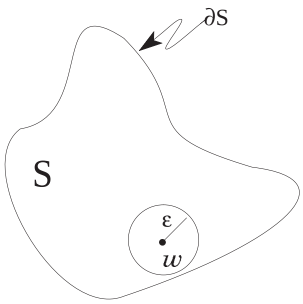

An immediate consequence of path independence is the *Calculus of Residues*: Let S be a region in which f is analytic everywhere except at an isolated point w in the interior of S. Let $C_\epsilon(w)$ be the circle of radius ϵ centered at w. Then

$$\int_{\partial S} f(z)\,dz = \lim_{\epsilon\downarrow 0}\int_{C_\epsilon(w)} f(z)\,dz.$$

The quantity

$$\frac{1}{2\pi i}\lim_{\epsilon\downarrow 0}\int_{C_\epsilon(w)} f(z)\,dz = \lim_{\epsilon\downarrow 0}\frac{\epsilon}{2\pi}\int_0^{2\pi} f(w+\epsilon e^{it})\,e^{it}\,dt,$$

if it exists, is called the residue of f at w and will be denoted by $\mathrm{Res}(f, w)$.

Example 4.4: Using the calculus of residues the integral

$$\int_{-\infty}^{\infty}\frac{e^{ix}}{x^2+1}\,dx$$

can be calculated with very little effort. Let S_R be that part of the disk of radius R which is in the upper half plane. Since $\frac{e^{iz}}{z^2+1}$ has isolated singularities at $z = \pm i$

$$\int_{S_R}\frac{e^{iz}}{z^2+1}\,dz = 2\pi i\,\mathrm{Res}(\frac{e^{iz}}{z^2+1}, i).$$

But

$$\lim_{R\to\infty}\int_{S_R}\frac{e^{iz}}{z^2+1}\,dz = \int_{-\infty}^{\infty}\frac{e^{ix}}{x^2+1}\,dx$$

so that

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{e^{ix}}{x^2+1}\,dx &= 2\pi i\,\mathrm{Res}(\frac{e^{iz}}{z^2+1}, i)\\ &= \lim_{\epsilon\downarrow 0}\epsilon i\int_0^{2\pi}\frac{e^{i(i+\epsilon e^{it})}}{(i+\epsilon e^{it})^2+1}\,e^{it}\,dt\\ &= \frac{1}{2e}\int_0^{2\pi} dt\\ &= \frac{\pi}{e}.\end{aligned}$$

To get from the second to the third lines in this last expression the integrand was expanded in series in ϵ using

$$e^{i(i+\epsilon e^{it})} = e\frac{1}{e}\Big(1 + \epsilon e^{it} + \dots\Big)$$

and

$$\begin{aligned}\frac{1}{(i+\epsilon e^{it})^2+1} &= \frac{1}{2i\epsilon e^{it} + \epsilon^2 e^{2it}} \\ &= \frac{1}{2i\epsilon e^{it}}\Big(1 - \frac{1}{2}\epsilon e^{it} + \dots\Big)\end{aligned}$$

and noting that all but the first terms vanish in the limit $\epsilon \downarrow 0$.

....

Consider a function f which has non-analytic (to be called *singular*) behavior at an isolated point w. There are three possibilities. w is either a *pole*, a *branch point* or an *essential singularity.*

A function f has a pole at $z = w$ if there is some $m \in \{1, 2, 3, \dots\}$ for which $(x - w)^m f(z)$ is analytic at w. The smallest such m is called the order of the pole. A pole is thus a singularity of the form

$$f(z) \sim \frac{\text{const}}{(z-w)^m}$$

for some positive integer n. The integer m is the order of the pole. If $m = 1$ the pole is said to be *simple*. Functions which have poles in some region S but are otherwise analytic in S are said to be *meromorphic* in S. The function

$$\frac{1 + z - 3z^2}{(z-1)^2}$$

is meromorphic in $\mathbf{C}$ with a pole of order 2 at 1. The function

$$\frac{1}{\sin z}$$

is meromorphic in $\mathbf{C}$ with simple poles at $m\pi$ for any integer m. Similarly $\tan z$ has simple poles at $(m + \frac{1}{2})\pi$ for any integer m. The function

$$\frac{e^{iz}}{z^2+1}$$

has simple poles at $\pm i$.

A branch point is a point around which a function is multivalued. These typically arise with inverse functions when there isn't a unique inverse. The simplest example is the square root. Any number $w \neq 0$ has two square roots. Explicitly, given z with $z^2 = w$ the number $-z$ also satisfies $(-z)^2 = w$. Thus, in defining a square root a choice must be made. In $\mathbf{R}$ one generally chooses $\sqrt{\ }$ to be the positive square root. In $\mathbf{C}$ things are more subtle and there are many more possible choices. Note that this structure collapses at 0 since $-0 = 0$, so $\sqrt{0}$ is unique. If $z = |z|e^{i\phi}$ then the two possible square roots of z are $|z|^{\frac{1}{2}}e^{\frac{i}{2}\phi}$ and $|z|^{\frac{1}{2}}e^{i\pi+\frac{i}{2}\phi}$. Here $|z|^{\frac{1}{2}}$ is the positive root of $|z|$. Choose the first and follow a path which surrounds 0 and comes back to z. The simplest is a circle of radius $|z|$. After going around once ϕ has increased by 2π which means the first square root has changed to the second.

The canonical example of a multivalued function is the natural logarithm, $\ln z$. The defining relation for the natural logarithm is

$$e^{\ln z} = z.$$

But this equation does not have a unique solution: any integer multiple of $2\pi i$ can be added to $\ln z$ since

$$e^{2m\pi i+\ln z} = e^{2m\pi i}e^{\ln z} = z.$$

Thus, there are an infinite number of natural logarithms, all differing by integer multiples of $2\pi i$. Again, the point at which this degenerates is 0. Note that if

$$\ln(|z|e^{i\phi}) = \ln|z| + i\phi$$

then traversing a circle around 0 adds $2\pi i$ to the logarithm.

Essential singularities are isolated singularities which are neither poles nor branch points. An example is $e^{\frac{1}{z}}$ at $z = 0$. Essential singularities are as messy they get. We won't encounter them often.

For functions with poles there is an extension of the notion of a Taylor expansion called a *Laurent expansion.* Explicitly, given a function f with a pole of order m at w one can write

$$\begin{aligned} f(z) &= \sum_{n=-m}^{\infty} a_n (z-w)^n \\ &= \frac{a_{-m}}{(z-w)^m} + \frac{a_{-m+1}}{(z-w)^{m-1}} + \ldots + \frac{a_{-1}}{z-w} + a_0 + a_1(z-w) + a_2(z-w)^2 + \ldots \end{aligned} \tag{4.3}$$

for z sufficiently close to w. A Laurent expansion gives complete information about the behavior of f near w. In particular, for z very close to w one can approximate $f(z)$ by

$$f(z) \approx \frac{a_{-m}}{(z-w)^m}.$$

The coefficients a_n can be determined using the formula

$$\begin{aligned} \int_{C_\epsilon(w)} (z-w)^n dz &= i\epsilon^{n+1} \int_0^{2\pi} e^{i(n+1)t}\, dt \\ &= 2\pi i \delta_{n,-1} \end{aligned}$$

where $\delta_{j,k}$ is 1 if $j = k$ and 0 if $j \neq k$. Thus

$$a_n = \frac{1}{2\pi i} \int_{C_\epsilon(w)} (w-z)^{-n-1} f(z)\, dz. \tag{4.4}$$

Note that

$$a_{-1} = \mathrm{Res}(f, w).$$

Although formula (4.4) for a_n works fine, in practice it is often easier to note that $(z - w)^m f(z)$ is analytic at w and Taylor expand it. Then a_n is the $(n+m)^{\text{th}}$ Taylor coefficient of $(z-w)^m f(z)$,

$$a_n = \frac{1}{(n+m)!} \frac{d^{n+m}}{dz^{n+m}} (z-w)^m f(z)\Big|_{z=w}.$$

Example 4.5: Rather than using either of the above formulae to find the coefficients a_n it's often easier to just expand. The first few terms of the Laurent series about 0 for

$$\begin{aligned} \frac{1}{e^z - 1} &= \frac{1}{z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \ldots} \\ &= \frac{1}{z} \frac{1}{1 + \frac{1}{2}z + \frac{1}{6}z^2 + \ldots} \\ &= \frac{1}{z}\Big(1 - \frac{1}{2}z + \frac{1}{12}z^2 + \ldots\Big) \\ &= \frac{1}{z} - \frac{1}{2} + \frac{1}{12}z + \ldots. \end{aligned}$$

If $\ln z$ is chosen so that $\ln 1 = 0$ the first few terms of the Laurent series about 1 for

$$\begin{aligned}
\frac{1}{\ln z} &= \frac{1}{\ln\big(1+(z-1)\big)} \\
&= \frac{1}{(z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 + \dots} \\
&= \frac{1}{z-1}\,\frac{1}{1 - \frac{1}{2}(z-1) + \frac{1}{3}(z-1)^2 + \dots} \\
&\frac{1}{z-1}\Big(1 + \frac{1}{2}(z-1) - \frac{1}{12}(z-1)^2 + \dots\Big) \\
&= \frac{1}{z-1} + \frac{1}{2} - \frac{1}{12}(z-1) + \dots.
\end{aligned}$$

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Linear Ordinary Differential Equations

The most general inhomogeneous n^{th} order linear ordinary differential equation may be written

$$\Big(a_n(x)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \ldots + a_0(x)\Big)f(x) = g(x). \tag{5.1}$$

Here g is known. It turns out that to solve this equation it is necessary to have completely solved the equation with $g = 0$. We will thus begin with

$$\Big(a_n(x)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \ldots + a_0(x)\Big)f(x) = 0. \tag{5.2}$$

(5.2) is a homogeneous equation.

We've already seen two things: there are in general many solutions to a linear ODE and linear combinations of solutions are also solutions. The immediate problem is to write down a general form for the solutions of the equation.

Regular Points:

A regular point for (5.2) is a point at which the $a_j(x)$ are all continuous and $a_n \neq 0$. The basic fact about linear ODEs is that at regular points a there is a unique solution f with prescribed values of $f(a)$, $f'(a)$, …, $f^{(n-1)}(a)$. Thus, given any n numbers $\alpha_0,\ \alpha_1,\ \ldots,\ \alpha_{n-1}$ and given any regular point a there is a solution whose first $n-1$ derivatives at a are the α_j. Since any solution obviously has $n-1$ derivatives at a there is a one to one correspondence between the set of n independent numbers, in other words $\mathbf{R}^n$ or $\mathbf{C}^n$, and the set of solutions to the linear ODE.

Imagine that we have somehow produced n linearly independent solutions of (5.2), $f_1,\ f_2,\ \ldots,\ f_n$. Then the linear combinations

$$f = c_1 f_1 + c_2 f_2 + \ldots + c_n f_n \tag{5.3}$$

are all solutions. The set of all such combinations is an n-dimensional vector space. In other words, the general such linear combination has n free parameters. One can hope that by choosing these parameters correctly one can give f whatever $n-1$ derivatives one wants at a prescribed point a. (To see that this is possible choose the f_k so that $f_k^{(j)}(a) = \delta_{k,j+1}$ so that $c_k = \alpha_{k+1}$. Any other choice for the f_k amounts to a change of basis.) Thus, in order to completely solve an n^{th} order linear ODE it is sufficient to produce n linearly independent solutions.

Example 5.1: Fortunately (or perhaps intentionally) one of the most commonly used equations in applications is the simplest: the 1-dimensional Helmholtz equation

$$\Big(\frac{d^2}{dx^2} + k^2\Big) f(x) = 0.$$

This equation is a linear ODE which is regular on all of $\mathbf{R}$. The general solution may be written

$$f(x) = c_1 \cos kx + c_2 \sin kx.$$

Note that $c_1 = f(0)$ and $c_2 = \frac{f'(0)}{k}$.

Linear superposition is not the only way to parameterize solutions to linear equations. A common parameterization of solutions to the 1-dimensional Helmholtz equation is

$$f(x) = A\cos(kx - \phi).$$

Here A and the phase ϕ are the parameters. One can easily show that

$$A^2 = c_1^2 + c_2^2$$

and

$$\tan\phi = \frac{c_2}{c_1}.$$

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Example 5.2: Consider time harmonic solutions of the 1-dimensional bending equation,

$$u(x,t) = f(x)\sin(\omega t).$$

With the notation $k = \sqrt{K}\,\omega$ one has

$$\Big(\frac{d^4}{dx^4} - k^2\Big) f = 0.$$

The four solutions $\cos(\sqrt{k}\ x)$, $\sin(\sqrt{k}\ x)$, $\cosh(\sqrt{k}\ x)$ and $\sinh(\sqrt{k}\ x)$ are clearly linearly independent since no combination of trigonometric and hyperbolic functions of a real variable can be zero everywhere. Thus

$$f(x) = c_1 \cos(\sqrt{k}\ x) + c_2 \sin(\sqrt{k}\ x) + c_3 \cosh(\sqrt{k}\ x) + c_4 \sinh(\sqrt{k}\ x).$$

Imagine that the rod is subjected to a periodic stress at $x = 0$ in such a way that the deflection at 0, α_0, is negative the amount of bending $\alpha_1 = -\alpha_0$. Assume further that the second and third derivatives, α_2 and α_3, are 0 at $x = 0$. Then

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & k^{\frac{1}{2}} & 0 & k^{\frac{1}{2}} \\ -k & 0 & k & 0 \\ 0 & -k^{\frac{3}{2}} & 0 & k^{\frac{3}{2}} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \alpha_0 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

This system is most easily solved directly (without trying to invert the matrix). One finds $c_3 = c_1$, $c_4 = c_2$, $c_1 = \frac{\alpha_0}{2}$ and $c_2 = -\frac{\alpha_0}{2\sqrt{k}}$. Thus

$$f(x) = \frac{\alpha_0}{2}\Big(\cos(\sqrt{k}\ x) + \cosh(\sqrt{k}\ x) - \frac{\sin(\sqrt{k}\ x) + \sinh(\sqrt{k}\ x)}{\sqrt{k}}\Big).$$

....

The fact that a solution is completely determined by it's 0^{th} through $(n-1)^{\text{st}}$ derivatives at one point allows one to study models in which different differential equations are solved in different regions. If the model is $L_1 f = 0$ for $x < a$ and $L_2 f = 0$ for $x > a$ the way to proceed is to find the general solution to $L_1 f_1 = 0$ and the general solution to $L_2 f_2 = 0$ and then impose the constraint that

$$f_1^{(j)}(a) = f_2^{(j)}(a)$$

for $j = 0, 1, \ldots, n-1$.

Example 5.3: Consider a model in which f satisfies 1-dimensional Helmholtz equations with different values of k for $x > 0$ and $x < 0$:

$$\frac{d^2 f}{dx^2} = \begin{cases} -k_-^2 f & \text{if } x < 0 \\ -k_+^2 f & \text{if } x > 0. \end{cases}$$

For $x < 0$ one has

$$f(x) = a_1 \cos(k_- x) + a_2 \sin(k_- x)$$

and for $x > 0$

$$f(x) = b_1 \cos(k_+ x) + b_2 \sin(k_+ x).$$

At $x = 0$ we must have

$$a_1 = b_1$$

and

$$k_- a_2 = k_+ b_2.$$

Thus the general solution may be written

$$f(x) = \begin{cases} a_1 \cos(k_- x) + a_2 \sin(k_- x) & \text{if } x < 0 \\ a_1 \cos(k_+ x) + \frac{k_-}{k_+} a_2 \sin(k_+ x) & \text{if } x > 0. \end{cases}$$

....

The set of equations $f^{(j)}(a) = \alpha_j$ is a set of n linear equations for the coefficients c_k. Explicitly

$$\begin{pmatrix} f_1(a) & f_2(a) & \cdots & f_n(a) \\ f_1'(a) & f_2'(a) & \cdots & f_n'(a) \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)}(a) & f_2^{(n-1)}(a) & \cdots & f_n^{(n-1)}(a) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix}.$$

This equation has a solution for any choice of the α_j only if the matrix on the left is invertible. One can conclude that the following statements are equivalent:

i) Any solution of (5.2) may be written in the form (5.3).

ii) The f_j are n linearly independent solutions of (5.2).

iii) The f_j are solutions of (5.2) for which the determinant

$$\mathcal{W}(f_1, \ldots, f_n) = \det \begin{pmatrix} f_1(a) & f_2(a) & \cdots & f_n(a) \\ f_1'(a) & f_2'(a) & \cdots & f_n'(a) \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)}(a) & f_2^{(n-1)}(a) & \cdots & f_n^{(n-1)}(a) \end{pmatrix} \tag{5.4}$$

$$\neq 0$$

for any a. The determinant in (5.4) is known as the *Wronskian* of $f_1, \ldots, f_n$.

In general, producing n linearly independent solutions to an n^{th} order ODE can be extremely difficult if $n > 1$. Often one needs to resort to numerical procedures. As a rule the difficulty increases with n. However, most of the problems we will encounter are second order.

The most general second order linear ODE is

$$\Big(a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x)\Big)f(x) = 0.$$

As indicated we restrict ourselves to x for which $a_2 \neq 0$. Thus we can divide by a_2 writing the equation

$$\Big(\frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)\Big)f(x) = 0 \tag{5.5}$$

with obvious definitions of p and q. The Wronskian of two functions has a simple form

$$\mathcal{W}(f_1, f_2) = f_1 f_2' - f_1' f_2.$$

If f_1 and f_2 are solutions of (5.2) then one finds by direct calculation that

$$\Big(\frac{d}{dx} + p\Big)\mathcal{W}(f_1, f_2) = 0.$$

This first order equation can be solved immediately giving *Abel's identity*

$$\mathcal{W}(f_1, f_2) = Ke^{-\int p\,dx}$$

where K is a constant. In particular, to determine $\mathcal{W}(f_1, f_2)$ one need only find K, and this can be done at any point x. Further, since e^x is never 0, f_1 and f_2 are linearly independent solutions if their Wronskian is non-zero anywhere.

Given one solution to a second order linear ODE a second solution can be constructed using the method called *reduction of order*. Let f_1 be a solution of (5.5). We will look for another solution of the form

$$f_2 = uf_1$$

and will obtain an equation for u by substituting f_2 into (5.5). One obtains

$$\Big(\frac{d^2}{dx^2} + (2\frac{f_1'}{f_1} + p)\frac{d}{dx}\Big)u = 0.$$

This is a first order equation for u' which can be solved immediately,

$$u' = \text{const}\ \frac{1}{(f_1)^2} e^{-\int p\,dx}.$$

The constant is uninteresting and can be set to 1.

A final comment about regular points. If the coefficients $a_j(x)$ are analytic at x then it turns out that the solutions f are analytic at x as well. (Note that this holds for the solutions of the Euler equations of Example 5.5 except at $x = 0$.) Since the solutions are analytic at any regular point they have convergent Taylor expansions about regular points.

Example 5.4: Consider a linear second order ODE with analytic coefficients. Let a be a regular point. Then the Taylor expansions of the solutions can be obtained directly from the differential equation as follows. Write the equation in the form (5.5) and expand everything in sight in a Taylor expansion about a:

$$p(x) = \sum_{j=0}^{\infty} p_j (x-a)^j,$$

$$q(x) = \sum_{j=0}^{\infty} q_j (x-a)^j$$

and

$$f(x) = \sum_{j=0}^{\infty} c_j (x-a)^j.$$

Then

$$\Big(\frac{d^2}{dx^2} + \sum_{k=0}^{\infty} p_k (x-a)^k \frac{d}{dx} + \sum_{m=0}^{\infty} q_m (x-a)^m\Big) \sum_{j=0}^{\infty} c_j (x-a)^j = 0$$

from which it follows that

$$\sum_{j=0}^{\infty} \Big[(j+2)(j+1)c_{j+2} + \sum_{k+m=j} \Big((m+1)c_{m+1}p_k + c_m q_k\Big)\Big](x-a)^j = 0.$$

In order for a power series to be 0 each coefficient must be 0. Thus

$$(j+2)(j+1)c_{j+2} + \sum_{k+m=j} \Big((m+1)c_{m+1}p_k + c_m q_k\Big) = 0.$$

At $j = 0$ one has

$$2c_2 + c_1 p_0 + c_0 q_0 = 0.$$

Note that c_2 can be expressed as a function of c_0 and c_1

$$c_2 = -\frac{p_0 c_1 + q_0 c_0}{2}.$$

At $j = 1$,

$$6c_3 + 2c_2 p_0 + c_1(q_0 + p_1) + c_0 q_1 = 0$$

giving c_3 as a function of c_0 and c_1,

$$c_3 = -\frac{(q_0 + p_1 - p_0^2)c_1 + (q_1 - p_0 q_0)c_0}{6}.$$

Ultimately, all the Taylor coefficients c_j can be expressed as some combination of the p_j and q_j times c_0 plus some other combination of the p_j and q_j times c_1, determining the solutions up to two free parameters, c_0 and c_1.

Note that the solution with $f(0) = 1$ and $f'(0) = 0$, call it $f_1(x)$, is obtained by setting $c_0 = 1$ and $c_1 = 0$ while the solution with $f(0) = 0$ and $f'(0) = 1$, call it $f_2(x)$ is obtained by setting $c_0 = 0$ and $c_1 = 1$. Then

$$f(x) = c_0\ f_1(x) + c_1\ f_2(x).$$

Of course the expressions for c_j generally get more complicated as j increases. As a general technique, this approach is useful only for obtaining the first few terms in the Taylor expansions of f. There are, however, special cases. One classic example is Airy's equation,

$$\Big(\frac{d^2}{dx^2} - x\Big) f(x) = 0. \tag{5.6}$$

One finds that

$$(j+2)(j+1)c_{j+2} - c_{j-1} = 0$$

for $j > 0$ and $2c_2 = 0$ for $j = 0$. This is a fairly simple recursion relation. Note that it relates coefficients c_{3j} to c_{3j-3}, c_{3j+1} to c_{3j-2} and c_{3j+2} to c_{3j-1}. One finds, working from $j = 1$ out, that

$$c_{3j} = \frac{1}{(3j)(3j-1)(3j-3)(3j-4)\cdots(3)(2)} c_0,$$

$$c_{3j+1} = \frac{1}{(3j+1)(3j)(3j-2)(3j-3)\ldots(4)(3)} c_1$$

and

$$c_{3j+2} = 0.$$

Thus we have produced two linearly independent solutions to Airy's equation: one by choosing $c_0 = 1$ and $c_1 = 0$,

$$f_1(x) = 1 + \frac{1}{3 \cdot 2}x^3 + \frac{1}{6 \cdot 5 \cdot 3 \cdot 2}x^6 + \frac{1}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2}x^9 + \cdots,$$

and another by choosing $c_0 = 0$ and $c_1 = 1$,

$$f_2(x) = x + \frac{1}{4 \cdot 3}x^4 + \frac{1}{7 \cdot 6 \cdot 4 \cdot 3}x^7 + \frac{1}{10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3}x^{10} + \cdots.$$

....

Regular Singular Points:

If $a_n(a) = 0$ in (5.2) then a is a singular point. We will restrict our attention to second order equations. In the form (5.5) a singular point x is a point at which p and q diverge. Loosely put, regular singular points are points at which the divergence of p is first order and the divergence of q is second order. We begin with an example which is characteristic of and neccesary for the whole theory.

Example 5.5: A class of exactly solvable linear ODEs are the Euler equations,

$$\left(ax^2\frac{d^2}{dx^2} + bx\frac{d}{dx} + c\right)f(x) = 0.$$

Note that at $x = 0$ the coefficient of $\frac{d^2}{dx^2}$ is 0 and the basic facts quoted above don't apply. $x = 0$ is said to be a *singular point* for these equations. For $x \neq 0$ we can expect that all solutions can be built from linear combinations of any two linearly independent solutions. Try the form $f(x) = x^r$. Then

$$ar(r-1) + br + c = 0$$

so that we find

$$r = \frac{a - b \pm \sqrt{(a-b)^2 - 4ac}}{2a}.$$

Thus, if $(a-b)^2 - 4ac \neq 0$ two solutions have been produced,

$$f_1(x) = x^{\frac{a-b+\sqrt{(a-b)^2-4ac}}{2a}}$$

and

$$f_2 = x^{\frac{a-b-\sqrt{(a-b)^2-4ac}}{2a}}.$$

If $(a-b)^2 - 4ac = 0$ only one solution

$$f_1(x) = x^{\frac{a-b}{2a}}$$

has been produced. The other is of the form

$$f_2(x) = u(x)x^{\frac{a-b}{2a}}$$

where

$$\begin{aligned} u'(x) &= x^{\frac{b-a}{a}} e^{-\frac{b}{a}\int \frac{1}{x}\,dx} \\ &= \frac{1}{x}. \end{aligned}$$

Thus we may choose

$$f_2(x) = x^{\frac{a-b}{2a}} \ln x$$

and the general solution can be written

$$f(x) = x^{\frac{a-b}{2a}} \big(c_1 + c_2 \ln x\big).$$

Note that even though the coefficient of $\frac{d^2}{dx^2}$ is 0 at $x = 0$ one can obtain solutions for all $x \neq 0$. Note that there are always solutions which are singular at $x = 0$.

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The general approach will be the same as taken for the Euler equations of Example 5.5: solve the equations near, but not at a, and see how they behave as x approaches a. We saw for the Euler equations that there is a variety of possible behavior: solutions can have poles or branch points at singular points or can be regular.

Example 5.6: Unfortunately singular ODEs arise often in applications. Consider the 3-dimensional Helmholtz equation

$$\left(\Delta + k^2\right)\Psi = 0.$$

In a homework problem you are asked to transform the Laplacian into cylindrical coordinates obtaining

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}. \tag{5.7}$$

In this coordinate system the Helmholtz equation is separable in the sense that solutions in the form of a product

$$\Psi(r, \phi, z) = R(r)\Phi(\phi)Z(z)$$

can be found, reducing a PDE to three ODEs. In fact, setting $\Phi(\phi) = e^{im\phi}$ and $Z(z) = e^{ik_z z}$ one finds that

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} - k_z^2 + k^2\right)R(r) = 0. \tag{5.8}$$

This equation is singular at $r = 0$.

In Example 3.4 the Laplacian was expressed in spherical coordinates, (3.1). Again, the Helmholtz equation is separable in this coordinate system. Setting

$$\Psi(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$$

one obtains solutions if $\Phi(\phi) = e^{im\phi}$,

$$\left(\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} - \frac{m^2}{\sin^2\theta}\right)\Theta(\theta) = -l(l+1)\Theta(\theta) \tag{5.9}$$

and then

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + k^2\right)R(r) = 0. \tag{5.10}$$

Equation (5.9) is singular at $\theta = 0$ and $\theta = \pi$. Equation (5.10) is singular at $r = 0$. The choice of $l(l+1)$ as separation parameter in (5.9) turns out to be convenient.

Equations (5.8), (5.9) and (5.10) will have to be dealt with in detail in this course.

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A singular point a of a second order linear ODE (5.5) is said to be a *regular singular point* if p and q are both meromorphic with a pole at a of order no larger than 1 for p and 2 for q. It follows that, for x close enough to a,

$$p(x) = \frac{p_{-1}}{x-a} + p_0 + p_1(x-a) + p_2(x-a)^2 + \dots$$

and

$$q(x) = \frac{q_{-2}}{(x-a)^2} + \frac{q_{-1}}{x-a} + q_0 + q_1(x-a) + q_2(x-a)^2 + \dots.$$

To understand the behavior of solutions to (5.5) as x approaches a it seems reasonable to approximate p and q by their most divergent terms,

$$p(x) \approx \frac{p_{-1}}{x-a}$$

and

$$q(x) \approx \frac{q_{-2}}{(x-a)^2}.$$

Substituting these approximations into (5.5) and shifting variables so that $a = 0$ one obtains an Euler equation

$$\Big(\frac{d^2}{dx^2} + \frac{p_{-1}}{x}\frac{d}{dx} + \frac{q_{-2}}{x^2}\Big)f(x) \approx 0.$$

Thus, it seems reasonable to expect that the behavior, as x approaches a regular singular point, of the correct solution f is similar to the behavior of the related Euler equation. Recalling Example 5.5 this behavior is $(x-a)^r$ where r satisfies the *indicial equation*

$$r(r-1) + p_{-1}r + q_{-2} = 0.$$

The facts are as follows. Let r_1 and r_2 be the roots of the indicial equation. Assume that $r_1 \geq r_2$ and set $a = 0$. Then

i) If $r_1 - r_2$ is not an integer then there are analytic functions g_1 and g_2 so that $g_1(0) = g_2(0) = 1$ and

$$f_1(x) = x^{r_1}\, g_1(x)$$

$$f_2(x) = x^{r_2}\, g_2(x)$$

are linearly independent solutions.

ii) If $r_1 = r_2$ then there are analytic functions g_1 and g_2 so that $g_1(0) = g_2(0) = 1$ and

$$f_1(x) = x^{r_1}\, g_1(x)$$

$$f_2(x) = f_1(x) \ln x + x^{r_1}\, g_2(x)$$

are linearly independent solutions.

iii) If $r_1 - r_2$ is a positive integer then there are analytic functions g_1 and g_1 and a constant c so that $g_1(0) = g_2(0) = 1$ and

$$f_1(x) = x^{r_1}\, g_1(x)$$

$$f_2(x) = c\, f_1(x) \ln x + x^{r_2}\, g_2(x)$$

are linearly independent solutions.

The Taylor coefficients of the g_j and the constant c can be determined by substituting into the differential equation as in Example 5.4. The procedure is tedious but straightforward. However, the Taylor expansion of the g_j is not very interesting. What is important about these results is that the nature of the solutions singularities is made explicit:

$$f_1(x) = x^{r_1}\Big(1 + a_1 x + \dots\Big)$$

and

$$f_2(x) = c f_1(x) \ln x + x^{r_2}\Big(1 + b_1 x + \dots\Big)$$

for some coefficients a_j, b_j and c. In case (i) $c = 0$ and in case (ii) $c = 1$.

Example 5.7: Bessel's differential equation is

$$\Big(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{m^2}{x^2} + 1\Big) f(x) = 0. \tag{5.11}$$

Choose $m > 0$. The solutions of this equation are known as *Bessel functions.* Note that if f is a solution to Bessel's equation then

$$R(r) = f\big(\sqrt{k^2 - k_z^2}\ r\big)$$

is a solution to the radial part of the Helmholtz equation in cylindrical coordinates, (5.8). Note that there is no *apriori* restriction on the sign of $k^2 - k_z^2$.

Bessel's equation is singular at 0 with

$$p(x) = \frac{1}{x}$$

and

$$q(x) = \frac{-m^2}{x^2} + 1.$$

The indicial equation is

$$\begin{aligned} 0 &= r(r-1) + r - m^2 \\ &= r^2 - m^2. \end{aligned}$$

Thus $r_1 = m$ and $r_2 = -m$.

The nature of the solutions depends on m. The case of interest for the Helmholtz equation is $m = 0, 1, 2, \ldots$. There is always a solution which is regular at 0, the f_1 given above. The standard notation for this solution is $J_m(x)$ and is called the Bessel function of order m of the first type. It is generally chosen so that

$$J_m(x) = \frac{1}{m!}\left(\frac{x}{2}\right)^m \Big(1 + c_1 x + c_2 x^2 + \ldots\Big).$$

The second solution which arises from the analysis above, the f_2, is called the m^{th} order Bessel function of the second type and is denoted by $Y_m(x)$ (although notation varies: Morse and Ingard call it N_m). It is generally chosen so that,

$$Y_m(x) = \frac{2}{\pi} J_m(x) \ln x - \frac{(m-1)!}{\pi}\left(\frac{x}{2}\right)^{-m}\Big(1 + d_1 x + d_2 x^2 + \ldots\Big).$$

The coefficients c_j are determined by substituting into the differential equation and setting the coefficients of the different powers of x equal to 0, beginning with the lowest power and working out. The d_j are more subtle since adding any multiple of J_m to Y_m doesn't effect the stated properties of Y_m: $Y_m + cJ_m$ is a solution to Bessel's equation which diverges as $x \to 0$ in the same way that Y_m does. Note that this has no effect on the coefficients d_j until $j = 2m$ (see problem 24).

Note that, up to a multiplicative constant, J_m is the only solution which is finite at $x = 0$. Further, for $m > 0$ the solution Y_m diverges like $\frac{1}{x^m}$ and for $m = 0$ like $\ln x$. J_m is analytic while Y_m, because of the natural logarithm, is multivalued with a branch point at $x = 0$.

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Example 5.8: Now consider (5.10). Solutions to this equation can be given by

$$R(r) = f\big(kr\big)$$

where f satisfies

$$\Big(\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} - \frac{l(l+1)}{x^2} + 1\Big)f(x) = 0. \tag{5.12}$$

Solutions to (5.12) are sometimes called *spherical Bessel functions.* This equation has a regular singular point at $x = 0$. The indicial equation is

$$\begin{aligned} 0 &= r(r-1) + 2r - l(l+1) \\ &= (r-l)(r+l+1). \end{aligned}$$

Thus, $r_1 = l$ and $r_2 = -l-1$. It follows that there is a one solution (up to an overall multiplicative constant) which behaves like x^l as $x \to 0$. It is generally given as

$$j_l(x) = \frac{x^l}{(2l+1)(2l-1)\cdots(3)(1)}\Big(1 + a_1x + a_2x^2 + \dots\Big).$$

Any other solution which is linearly independent of j_l must diverge as $x \to 0$, behaving like x^{-l-1}. Further, this second solution might also have a logarithmic divergence and thus be multi-valued about 0 since $r_1 - r_2 = 2l+1$ is an integer. In this case, however, we turn out to be lucky: there is no logarithmic term (the coefficient c in case (iii) turns out to be 0). A standard choice is

$$y_l(x) = -\frac{(2l+1)(2l-1)\cdots(3)(1)}{x^{l+1}}\Big(1 + b_1x + b_2x^2 + \dots\Big).$$

Note that replacing x by $-x$ in (5.12) doesn't change the differential equation. Thus $j_l(-x)$ and $y_l(-x)$ are solutions as well. But $j_l(-x) \sim x^l$ as $x \to 0$ implying that $j_l(x)$ and $j_l(-x)$ are linearly dependent, which means that they are proportional to each other. Thus

$$\begin{aligned} &\frac{x^l}{(2l+1)(2l-1)\cdots(3)(1)}\Big(1 + a_1x + a_2x^2 + \dots\Big) \\ &\qquad = \text{const } (-1)^l \frac{x^l}{(2l+1)(2l-1)\cdots(3)(1)}\Big(1 - a_1x + a_2x^2 + \dots\Big). \end{aligned}$$

This is only possible if const $= (-1)^l$ and $0 = a_1 = a_3 = a_5 = \ldots$.

For y_l the situation is a bit more subtle and depends on the choice of y_l. One can add factors of j_l to y_l without effecting the behavior of y_l at 0. Further, since $y_l(-x)$ is also a solution which diverges like $\frac{1}{x^{l+1}}$ at 0, the best one can say is that

$$y_j(-x) = c_1 y_l(x) + c_2 j_l(x).$$

However, one can insist that $y_l(-x) = (-1)^{l+1} y_l(x)$ since if this were not the case for y_l one could choose $y_l(x) + (-1)^{l+1} y_l(-x)$ as the second solution. Thus, with this choice for y_l, $0 = b_1 = b_3 = b_5 = \ldots$. The coefficients a_2 and b_2 can now be determined by substituting into the differential equation.

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Large x Asymptotics:

We've seen how to estimate the local behavior of solutions to second order linear ODEs at regular points and in the neighborhood of a regular singular point. We saw that near a regular point the solutions don't behave in any spectacular way in the sense that nothing particular stands out. The solutions are analytic and, if we want, we can generate their Taylor expansions. Near a singular point, on the other hand, the solutions can be singular. We can find the precise form of the singular parts and, since they dominate near a singularity, from them we know what the solution has to look like as one closes in on the singularity. Thus, in a sense, singular points make things easier: there is some distinctive asymptotic behavior which can be extracted without fully solving the equation.

A similar thing often happens as $x \to \infty$. There is often some explicit behavior which can be extracted without fully solving the equation. As a simple example consider

$$\Big(\frac{d^2}{dx^2} + \kappa^2 + \frac{\lambda}{x^2}\Big) f(x) = 0. \tag{5.13}$$

As $x \to \infty$ one expects that the term $\frac{\lambda}{x^2}$ might become negligible compared to κ^2. If so then one should have

$$f(x) \approx c_1 e^{i\kappa x} + c_2 e^{-i\kappa x}$$

for sufficiently large x. To see if this works let's try to generate a series

$$f(x) = e^{\pm i\kappa x}\Big(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots\Big).$$

Substituting into the differential equation one finds

$$\begin{aligned} 0 &= \Big(\frac{d^2}{dx^2} + \kappa^2 + \frac{\lambda}{x^2}\Big)e^{\pm i\kappa x}\Big(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots\Big) \\ &= e^{\pm i\kappa x}\Big(\frac{d^2}{dx^2} \pm 2i\kappa\frac{d}{dx} + \frac{\lambda}{x^2}\Big)\Big(1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots\Big) \\ &= e^{\pm i\kappa x}\Big((\mp 2i\kappa a_1 + \lambda)\frac{1}{x^2} + (2a_1 \mp 4i\kappa a_2 + \lambda a_1)\frac{1}{x^3} + \dots\ \Big). \end{aligned}$$

It follows that

$$a_1 = \pm\frac{\lambda}{2i\kappa},$$

$$a_2 = -\frac{(\lambda+2)\lambda}{8\kappa^2}$$

and so on. Thus there are two linearly independent solutions, $f_\pm$, whose asymptotic forms, as $x \to \infty$ are

$$f_\pm(x) = e^{\pm i\kappa x}\Big(1 \pm \frac{\lambda}{2i\kappa}\frac{1}{x} - \frac{(\lambda+2)\lambda}{8\kappa^2}\frac{1}{x^2} + \dots\Big). \tag{5.14}$$

The general term in this expansion is not hard to find. One has a one step recursion obtained from the coefficient of $\frac{1}{x^{j+2}}$,

$$\big(\lambda + j(j+1)\big)a_j \mp 2i\kappa(j+1)a_{j+1} = 0.$$

By starting at $j = 1$ and working up one finds

$$a_{j+1} = (\pm 1)^{(j+1)}\frac{\big(\lambda + j(j+1)\big)\big(\lambda + (j-1)j\big)\cdots(\lambda+2)\lambda}{(j+1)!(2i\kappa)^{j+1}}.$$

Note that the even and odd coefficents have different behavior. For j even, $j = 2k$,

$$a_{2k} = (-1)^k\frac{\big(\lambda + 2k(2k+2)\big)\big(\lambda + (2k-2)2k\big)\cdots(\lambda+2)\lambda}{(2k)!(2\kappa)^{2k}}$$

is real and independent of $\pm$. For j odd, $j = 2k+1$,

$$a_{2k+1} = \pm(-1)^{k+1}i\frac{\big(\lambda + (2k+1)(2k+3)\big)\big(\lambda + 2k(2k+1)\big)\cdots(\lambda+2)\lambda}{(2k+1)!(2\kappa)^{2k+1}}$$

is pure imaginary and proportional to $\pm$. In particular,

$$\overline{f_+(x)} = f_-(x)$$

so that

$$f_1 = \frac{f_+ + f_-}{2}$$

and

$$f_2 = \frac{f_+ - f_-}{2i}$$

are two real linearly independent solutions with leading order large x asymptotic behavior

$$f_1(x) \approx \cos x$$

and

$$f_2(x) \approx \sin x.$$

Note as well that f_1 is even in x while f_2 is odd.

Example 5.9: Consider Bessel's differential equation (5.11). First eliminate the $\frac{d}{dx}$ term using the transformation $f(x) = u(x)g(x)$ and choosing u so that the resulting equation for g has no first derivative term:

$$\begin{aligned} 0 &= \Big(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{m^2}{x^2} + 1\Big)ug \\ &= u\Big(\frac{d^2}{dx^2} + 2\frac{u'}{u}\frac{d}{dx} + \frac{u''}{u} + \frac{1}{x}\frac{d}{dx} + \frac{1}{x}\frac{u'}{u} - \frac{m^2}{x^2} + 1\Big)g \end{aligned}$$

so that choosing

$$2\frac{u'}{u} + \frac{1}{x} = 0 \tag{5.15}$$

one has

$$\Big(\frac{d^2}{dx^2} + \frac{u''}{u} + \frac{1}{x}\frac{u'}{u} - \frac{m^2}{x^2} + 1\Big)g = 0.$$

Solving (5.15) one finds

$$u = \text{const}\ \frac{1}{\sqrt{x}}.$$

An overall multiplicative constant is uninteresting so set const $= 1$.

Thus, solutions f to Bessel's equation can be written

$$f(x) = \frac{1}{\sqrt{x}} g(x)$$

where

$$\Big(\frac{d^2}{dx^2} - \frac{m^2 - \frac{1}{4}}{x^2} + 1\Big) g = 0.$$

This equation is precisely of the form (5.13) with $\kappa = 1$ and $\lambda = -m^2 + \frac{1}{4}$. Using (5.14) one finds that there are two linearly independent solutions to Bessel's equation with asymptotic forms $\frac{e^{\pm ix}}{\sqrt{x}}$. These are called *Hankel functions*, are denoted by $H_m^{(\pm)}$ and are chosen to be

$$H_m^{(\pm)}(x) = \sqrt{\frac{2}{\pi x}}\, e^{\pm i(x - \frac{m\pi}{2} - \frac{\pi}{4})} \Big(1 \mp \frac{m^2 - \frac{1}{4}}{2i}\frac{1}{x} - \frac{(m^2 - \frac{9}{4})(m^2 - \frac{1}{4})}{8}\frac{1}{x^2} + \ldots\Big).$$

There are constants $a_\pm$ and $b_\pm$ for which

$$H_m^{(\pm)} = a_\pm J_m + b_\pm Y_m.$$

To see find what these constants are requires knowledge about J_m, Y_m and $H_m^{(\pm)}$ for the same range of x. Short distance asymptotics for J_m and Y_m and long distance asymptotics for $H_m^{(\pm)}$ will not suffice. Note that both J_m and Y_m are real so that one is proportional to $\operatorname{Re} H_m^{(\pm)}$, the other to $\operatorname{Im} H_m^{(\pm)}$.

It turns out (in a homework problem) that

$$H_m^{(\pm)} = J_m \pm i Y_m.$$

In particular, for large x

$$J_m(x) \approx \sqrt{\frac{2}{\pi x}} \cos(x - \frac{m\pi}{2} - \frac{\pi}{4})$$

and

$$Y_m(x) \approx \sqrt{\frac{2}{\pi x}} \sin(x - \frac{m\pi}{2} - \frac{\pi}{4}).$$

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Example 5.10: The same style of large x analysis can be applied to the spherical Bessel equation (5.12). Again, setting $f(x) = u(x)g(x)$ and choosing u so that the resulting equation for g has no first derivative term one finds that

$$u\Big(\frac{d^2}{dx^2} + 2\frac{u'}{u}\frac{d}{dx} + \frac{u''}{u} + \frac{2}{x}\frac{d}{dx} + \frac{2}{x}\frac{u'}{u} - \frac{l(l+1)}{x^2} + 1\Big) g = 0$$

so that

$$2\frac{u'}{u} + \frac{2}{x} = 0.$$

Solving for u one finds that one can choose

$$u = \frac{1}{x}$$

and then, with this choice, that

$$\Big(\frac{d^2}{dx^2} - \frac{l(l+1)}{x^2} + 1\Big)g = 0.$$

It follows that two linearly independent solutions to the spherical Bessel equation can be found with asymptotic form $\frac{1}{x}e^{\pm x}$. These are called spherical Hankel functions and are given in standard form by

$$h_l^{(\pm)}(x) = \frac{i^{\mp(l+1)}}{x}e^{\pm ix}\Big(1 \mp \frac{l(l+1)}{2i}\frac{1}{x} + \dots\Big).$$

The spherical Bessel functions can actually be given in elementary form:

$$j_l(x) = (-1)^l x^l \big(\frac{1}{x}\frac{d}{dx}\big)^l \frac{\sin x}{x} \tag{5.16}$$

and

$$y_l(x) = (-1)^{(l+1)} x^l \big(\frac{1}{x}\frac{d}{dx}\big)^l \frac{\cos x}{x}. \tag{5.17}$$

It follows that

$$\begin{aligned} j_0(x) &= \frac{\sin x}{x} \\ y_0(x) &= -\frac{\cos x}{x} \\ j_1(x) &= \frac{\sin x}{x^2} - \frac{\cos x}{x} \\ y_1(x) &= -\frac{\cos x}{x^2} - \frac{\sin x}{x} \end{aligned}$$

and so on.

Note that the leading order large x asymptotics for the spherical Bessel functions can be obtained from (5.16) and (5.17) by letting the derivatives act on as few $\frac{1}{x}$ terms as possible. It follows that for x large

$$j_l(x) \approx \frac{(-1)^l}{x}\frac{d^l \sin x}{dx^l}$$

and

$$y_l(x) \approx \frac{(-1)^{l+1}}{x} \frac{d^l \cos x}{dx^l}$$

from which it follows that

$$h_l^{(\pm)} = j_l \pm i y_l.$$

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Example 5.11: As a final example lets look at large positive x asymptotics for solutions to Airy's equation (5.6). Here, for large x we can't expect behavior like $e^{\text{const } x}$. Instead try

$$f(x) \approx x^p e^{\kappa x^\alpha} \Big(1 + \frac{a_1}{x^\delta} + \frac{a_2}{x^{2\delta}} + \dots\Big). \tag{5.18}$$

Substituting into the differential equation

$$\Big(\frac{d^2}{dx^2} - x\Big) x^p e^{\kappa x^\alpha} = x^{p-2} e^{\kappa x^\alpha} \Big(p(p-1) + \alpha\kappa(2p + \alpha - 1)x^\alpha + (\alpha\kappa)^2 x^{2\alpha} - x^3\Big).$$

The largest terms are the x^3 and $x^{2\alpha}$ terms. They must cancel each other. This implies that

$$\alpha = \frac{3}{2}$$

and $(\alpha\kappa)^2 = 1$ so that $\alpha\kappa = \pm 1$ and then

$$\kappa = \pm\frac{2}{3}.$$

The next largest term is the x^α term which must vanish. Thus $2p + \alpha - 1 = 0$ so that

$$p = -\frac{1}{4}.$$

The remaining term will be canceled by part of the a_1 term in the asymptotic expansion, and so on.

Thus, there are two linearly independent solutions to Airy's equation which, as $x \to \infty$, have asymptotic form $x^{-\frac{1}{4}} e^{\pm\frac{2}{3}x^{\frac{3}{2}}}$. The standard solutions are

$$\mathrm{Ai}(x) \approx \frac{1}{2}\pi^{-\frac{1}{2}} x^{-\frac{1}{4}} e^{-\frac{2}{3}x^{\frac{3}{2}}}$$

and

$$\mathrm{Bi}(x) \approx \pi^{-\frac{1}{2}} x^{-\frac{1}{4}} e^{\frac{2}{3}x^{\frac{3}{2}}}.$$

The large $-x$ behavior is somewhat different: as $x \to -\infty$ it can be shown that

$$\mathrm{Ai}(x) \approx \pi^{-\frac{1}{2}} x^{-\frac{1}{4}} \sin(\frac{2}{3} x^{\frac{3}{2}})$$

and

$$\mathrm{Bi}(x) \approx \pi^{-\frac{1}{2}} x^{-\frac{1}{4}} \cos(\frac{2}{3} x^{\frac{3}{2}}).$$

....

Example 5.12: Imagine a sphere of radius $r = R + \epsilon \sin(\omega t)$ with $\epsilon \ll R$. Consider the pressure amplitude induced in free space.

Recall linear lossless acoustics (Example 1.8),

$$\frac{1}{c^2}\frac{\partial P'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}' = 0$$

and

$$\rho_0 \frac{\partial \mathbf{v}'}{\partial t} + \nabla P' = 0.$$

Differentiating the first equation with respect to t and substituting in from the second one has

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}) P' = 0.$$

Assuming harmonic time dependence

$$P'(\mathbf{x}, t) = \mathrm{Re}\, P_A(\mathbf{x}) e^{-i\omega t}$$

and

$$\mathbf{v}'(\mathbf{x}, t) = \mathrm{Re}\, \mathbf{v}_A(\mathbf{x}) e^{-i\omega t}$$

one has

$$(\nabla^2 + \frac{\omega^2}{c^2}) P_A = 0$$

$$-i\omega \rho_0 \mathbf{v}_A = \nabla P_A.$$

Since $\epsilon \ll R$ one can model the oscillating sphere as a velicity condition: $\mathbf{v}_A(r,\theta,\phi) = \epsilon\omega$. This translates to a condition on P_A,

$$-\frac{\partial P_A}{\partial r}\Big|_{r=R} = -i\omega^2\rho_0\epsilon. \tag{5.19}$$

P_A satisfies the Helmholtz equation and, since the boundary conditions are independent of θ and ϕ, can be chosen independent of θ and ϕ. It follows that

$$P_A(r) = c_1 h_0^{(+)}(kr) + c_2 h_0^{(-)}(kr).$$

Further, one expects an outward travelling wave at large r so that $c_2 = 0$. Thus

$$\begin{aligned} P_A(r) &= c_1 h_0^{(+)}(kr) \\ &= c_1 \frac{e^{ikr}}{r}. \end{aligned}$$

To determine c_1 use (5.19) from which one finds

$$c_1 k h_0^{(+)\prime}(kR) = i\omega^2\rho_0\epsilon.$$

Thus

$$c_1 = i\omega^2\rho_0\epsilon \frac{R^2 e^{-ikR}}{k(ikR-1)}$$

so that

$$P_A(r) = i\omega^2\rho_0\epsilon \frac{R^2}{k(ikR-1)e^{ikR}} \frac{e^{ikr}}{r}.$$

....

Example 5.13: Now consider an infinitely long cylinder whose radius is $r = R + \epsilon\sin(\omega t)$ with $\epsilon \ll R$ and $kR \ll 1$. Then

$$P_A(r,\theta,z) = c_1 H_0^{(+)}(kr)$$

with

$$-\frac{\partial P_A}{\partial r}\Big|_{r=R} = -i\omega^2\rho_0\epsilon.$$

It follows that

$$-c_1 k H_0^{(+)\prime}(kR) = -i\omega^2 \rho_0 \epsilon.$$

To evaluate $H_0^{(+)\prime}(kR)$ we use the assumption that $kR \ll 1$ and the small x asymptotics for Bessels equation. One has

$$\begin{aligned} H_0^{(+)\prime}(x) &= \frac{d}{dx}(J_0(x) + iY_0(x)) \\ &\approx \frac{d}{dx} i \frac{2}{\pi} \ln(x) \\ &= \frac{2i}{\pi x}. \end{aligned}$$

It follows that

$$c_1 \approx \frac{\pi}{2} \omega^2 \rho_0 \epsilon R.$$

....

Boundary Value Problems for O.D.E.s

Consider the second order homogeneous ordinary differential equation (5.5),

$$\Big(\frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)\Big) f(x) = 0,$$

for $x \in (a, b)$. A boundary value problem for this differential equation on (a, b) is two conditions

$$F\big(f(a), f'(a)\big) = 0$$

and

$$G\big(f(b), f'(b)\big) = 0.$$

The boundary conditions are said to be linear and/or homogeneous if F and G are.

We will only be concerned with linear boundary conditions,

$$\alpha_a f(a) + \beta_a f'(a) = A$$

$$\alpha_b f(b) + \beta_b f'(b) = B.$$

The boundary conditions are homogeneous only if $A = B = 0$. To see how to solve such a boundary value problem let f_1 and f_2 be linearly independent solutions of (5.5). Then

$$f = c_1 f_1 + c_2 f_2$$

for some constants c_1 and c_2. The boundary conditions become the following equation for c_1 and c_2

$$\begin{pmatrix} \alpha_a f_1(a) + \beta_a f_1'(a) & \alpha_a f_2(a) + \beta_a f_2'(a) \\ \alpha_b f_1(b) + \beta_b f_1'(b) & \alpha_b f_2(b) + \beta_b f_2'(b) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}.$$

If the boundary conditions are inhomogeneous, so that the vector $\begin{pmatrix} A \\ B \end{pmatrix}$ is non-zero then this equation has a solution, that is, one can produce a unique pair of coefficients c_1 and c_2, only if the matrix is invertible. If so, then one can explicitly produce a solution, f, satisfying the boundary conditions. Note that this solution is unique: multiples of f by a constant, cf, will not satisfy the boundary conditions unless the constant $c = 1$.

Example 6.1: For $x \in (0, L)$ consider the one dimensional Helmholtz equation

$$(\frac{d^2}{dx^2} + k^2)f(x) = 0.$$

Let f satisfy the boundary conditions

$$f'(0) = A$$

and

$$f(L) = 0.$$

Writing

$$f(x) = c_1 \sin(kx) + c_2 \cos(kx)$$

one has

$$\begin{pmatrix} k & 0 \\ \sin(kL) & \cos(kL) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} A \\ 0 \end{pmatrix}.$$

It follows that if $k\cos(kL) \neq 0$ then

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{k\cos(kL)} \begin{pmatrix} \cos(kL) & 0 \\ -\sin(kL) & k \end{pmatrix} \begin{pmatrix} A \\ 0 \end{pmatrix}$$

so that

$$f(x) = \frac{A}{k}\sin(kx) - \frac{A\tan(kL)}{k}\cos(kx).$$

If $k = 0$ then $\frac{d^2 f}{dx^2} = 0$ so that $f(x) = Ax - AL$. If $\cos(kL)$ then $\sin(kL) = \pm 1$ so that the boundary conditions lead to $kc_1 = A$ and $\pm c_1 = 0$. Thus, unless $A = 0$ there is no solution. If $A = 0$ then $c_1 = 0$ and c_2 is arbitrary.

....

Example 6.2: Consider lossless acoustics in one spatial dimension. The steady state reponse at frequency ω reduces to the study of

$$(\frac{d^2}{dx^2} + \frac{\omega^2}{c^2})P_A = 0$$

and

$$-i\omega\rho_0 v_A = -\frac{dP_A}{dx}.$$

Note that a relation between P_A and v_A is equivalent to a relation between P_A and P'_A.

Often one models the action of a loudspeaker as a velocity specification at the position of the loudspeaker. Consider a cylindrical tube of length L with a loudspeaker mounted at $x = 0$. Let the end at $x = L$ be closed and let the velocity of the face of the loudspeaker be

$$u_0 \cos(\omega t).$$

Then $v_A(L) = 0$ since the end is closed and $v_A(0) = u_0$. This translates to a boundary value problem for P_A: $P'_A(0) = -i\omega\rho_0 u_0$ and $P'_A(L) = 0$. Writing

$$P_A(x) = c_1 \sin(\frac{\omega}{c}x) + c_2 \cos(\frac{\omega}{c}x)$$

one has

$$\begin{pmatrix} \frac{\omega}{c} & 0 \\ \cos(\frac{\omega}{c}L) & -\sin(\frac{\omega}{c}L) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} i\omega\rho_0 u_0 \\ 0 \end{pmatrix}.$$

Thus, if $\sin(\frac{\omega}{c}L) \neq 0$ then

$$P_A(x) = -i\rho_0 c u_0 \Big(\sin(\frac{\omega}{c}x) + \cot(\frac{\omega}{c}L)\cos(\frac{\omega}{c}x)\Big).$$

For frequencies with $\sin(\frac{\omega}{c}L) = 0$ there is no choice of c_1 and c_2 which satisfy the boundary conditions unless $u_0 = 0$ when $c_1 = 0$ and c_2 is arbitrary. At such a frequency the system is at resonance. Note that the form given for P_A above becomes infinite as ω approaches a resonant frequency. Physically this means that in this lossless model the response to a periodic excitation at a resonant frequency is infinite in the sense that there is no steady state.

....

Homogeneous boundary conditions are more subtle and can not always be satisfied. For these one has

$$\alpha_a f(a) + \beta_a f'(a) = 0$$

$$\alpha_b f(b) + \beta_b f'(b) = 0.$$

Again let f_1 and f_2 be linearly independent solutions of (5.5) so that

$$f = c_1 f_1 + c_2 f_2$$

for some constants c_1 and c_2. The boundary conditions become the following equation for c_1 and c_2

$$\begin{pmatrix} \alpha_a f_1(a) + \beta_a f_1'(a) & \alpha_a f_2(a) + \beta_a f_2'(a) \\ \alpha_b f_1(b) + \beta_b f_1'(b) & \alpha_b f_2(b) + \beta_b f_2'(b) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The only way for there to be non-zero solutions (that is, at least one of the c_j is non-zero) is for

$$\det \begin{pmatrix} \alpha_a f_1(a) + \beta_a f_1'(a) & \alpha_a f_2(a) + \beta_a f_2'(a) \\ \alpha_b f_1(b) + \beta_b f_1'(b) & \alpha_b f_2(b) + \beta_b f_2'(b) \end{pmatrix} = 0. \tag{6.1}$$

As we saw in the last two examples this condition is rarely satisfied. If the differential equation (and thus the solutions) has a free parameter (such as k in the one-dimensional Helmholtz equation) then a given homogeneous boundary value problem can typically be satisfied only for a discrete set of values of the free parameter. Further, if f satisfies a homogeneous boundary value problem then cf does as well for any constant c. Finally, if (6.1) is satisfied for any pair of linearly independent solutions f_1 and f_2 then it is satisfied for all pairs of linearly independent solutions (changing the pairs of solution is just a change of basis).

Consider the simple example given by the Helmholtz equation

$$(\frac{d^2}{dx^2} + k^2) f = 0$$

with the boundary conditions $f(0) = f(L) = 0$. Writing

$$f(x) = c_1 \sin(kx) + c_2 \cos(kx)$$

the boundary condition can be satisfied only if

$$\det \begin{pmatrix} 0 & 1 \\ \sin(kL) & \cos(kL) \end{pmatrix} = -\sin(kL) = 0.$$

This is only possible for $k = \frac{n\pi}{L}$ where n is an integer. If so then $c_2 = 0$ and

$$f(x) = c_1 \sin(\frac{n\pi}{L} x)$$

for any c_1.

Homogeneous boundary value problems lead to the notion of eigen-functions for differential operators. Let

$$L = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)$$

be a second order differential operator and let

$$\alpha_a f(a) + \beta_a f'(a) = 0$$

$$\alpha_b f(b) + \beta_b f'(b) = 0$$

be homogeneous boundary conditions. The differential equation

$$(L - \lambda)f(x) = 0 \tag{6.2}$$

restricted to functions satisfying the boundary conditions is called an eigenvalue problem for L corresponding to the stated boundary conditions. Note that there will be a solution f only for certain values of λ for which (6.1) is satisfied. These values of λ are called eigenvalues of L and the corresponding solutions f are called eigenfunctions. The equation (6.1) appropriate to (6.2) is called an *eigenvalue condition*.

An example is provided by the functions $\sin(\frac{n\pi}{L}x)$ for the one dimensional Helmholtz equation in $(0, L)$ with $f(0) = f(L) = 0$. In that example $\lambda = k^2$ and the eigenvalue condition is $-\sin(kL) = 0$.

Another type of homogeneous boundary value problem is given by insisting that the solution in $(0, L)$ be a periodic function on all of $\mathbf{R}$ with period L. This is insured if $f(0) = f(L)$ and $f'(0) = f'(L)$. These are called periodic boundary conditions. If f_1 and f_2 are linearly independent solutions then $f = c_1 f_1 + c_2 f_2$ and one has

$$\begin{pmatrix} f_1(0) - f_1(L) & f_2(0) - f_2(L) \\ f_1'(0) - f_1'(L) & f_2'(0) - f_2'(L) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This system has a non-trivial solution only if

$$\det \begin{pmatrix} f_1(0) - f_1(L) & f_2(0) - f_2(L) \\ f_1'(0) - f_1'(L) & f_2'(0) - f_2'(L) \end{pmatrix} = 0. \tag{6.3}$$

Again, if the differential equation is put in the form (6.2) one has an eigenvalue problem for $\frac{d^2}{dx^2}$ with eigenvalue condition (6.3).

For the Helmholtz equation one has

$$f(x) = c_1 e^{ikx} + c_2 e^{-ikx}$$

so that periodic boundary conditions lead to the eigenvalue condition

$$\begin{aligned}\det\begin{pmatrix} 1-e^{ikL} & 1-e^{-ikL} \\ ik(1-e^{ikL}) & -ik(1-e^{-ikL}) \end{pmatrix} &= -2ik(1-e^{ikL})(1-e^{-ikL}) \\ &= -4ik(1-\cos kL) \\ &= 0.\end{aligned}$$

Thus the eigenvalue condition becomes $\cos kL = 1$ so that one has

$$k = \frac{2n\pi}{L}$$

for $n \in \{\ldots,\ -2,\ -1,\ 0,\ 1,\ 2,\ \ldots\}$. The eigenfunctions are

$$e^{2in\pi\frac{x}{L}}.$$

Recall that the eigen values are $\lambda = k^2$. Thus, for this example there are two eigenfunctions, $e^{\pm 2i|n|\pi\frac{x}{L}}$, for each eigenvalue $\frac{4n^2\pi^2}{L^2}$. Note that the general solution, at a given eigenvalue k is a superposition of the two eigenfunctions and thus still contains two free parameters.

Example 6.3: Consider the free vibration of a circular membrane of radius R. In the linear approximation the dispacement of the membrane, $u(\mathbf{x},t)$, satisfies the wave equation

$$\Big(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)u(\mathbf{x},t) = 0.$$

If the membrane is clamped along it's edge then there is the additional condition $u(\mathbf{x},t) = 0$ if $\|\mathbf{x}\| = R$. Transforming to polar coordinates and assuming harmonic time dependence,

$$u(\mathbf{x},t) = \operatorname{Re} u_A(r,\theta)e^{-i\omega t},$$

one has the Helmholz equation

$$\Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\omega^2}{c^2}\Big)u_A(r,\theta) = 0$$

with the boundary condition

$$u_A(R,\theta) = 0.$$

We will look for separable solutions

$$u_A(r,\theta) = X(r)Y(\theta)$$

with $X(R) = 0$, $X(0)$ finite and $Y(\theta)$ periodic with period 2π. Substituting into the Helmholtz equation one has

$$Y\Big(X'' + \frac{1}{r}X'\Big) + \frac{1}{r^2}XY'' + \frac{\omega^2}{c^2}XY = 0.$$

We get a solution if

$$Y'' + \lambda_1 Y = 0,$$

$$X'' + \frac{1}{r}X' - \frac{\lambda_1}{r^2}X + \lambda_2 X = 0,$$

and

$$\frac{\omega^2}{c^2} = \lambda_2.$$

The equation for Y is a one dimensional Helmholtz equation with periodic boundary conditions. Thus $\lambda_1 = \pm m$ for $m = 0,\ 1,\ 2,\ \ldots$ corresponding to solutions

$$Y(\theta) = a\ e^{im\theta} + b\ e^{-im\theta}.$$

Given this choice for Y the equation for X leads to Bessel's equation and has solutions

$$X(r) = a\ J_m(\sqrt{\lambda_2}\, r) + b\ Y_m(\sqrt{\lambda_2}\, r).$$

The condition that X be finite at $r = 0$ implies that $b = 0$. The condition that $X(R) = 0$ implies that

$$J_m(\sqrt{\lambda_2}\, R) = 0.$$

It follows that $\sqrt{\lambda_2}\, R$ must be a root of J_m. There are an infinite number of these roots. Let $x_{m,j}$ be these roots, $J_m(x_{m,j}) = 0$, listed in increasing order. For $x_{m,j}$ large enough they can be estimated using the large x asymptotic form for J_m. This leads to the approximation

$$\cos(x_{m,j} - \frac{m\pi}{2} - \frac{\pi}{4}) \approx 0$$

which gives $x_{m,j} \approx (n_m + \frac{1+m}{2} + \frac{1}{4})\pi$ for large j. n_m is an integer. For small values of j the zeros must be approximated numerically. One finds

$$x_{0,1} = 2.405, \quad x_{0,2} = 5.520, \quad x_{0,3} = 8.654, \quad \ldots$$

$$x_{1,1} = 3.832, \quad x_{1,2} = 7.016, \quad x_{1,3} = 10.173, \quad \ldots$$

$$x_{2,1} = 5.136, \quad x_{2,2} = 8.417, \quad x_{2,3} = 11.620, \quad \ldots$$

and so on. In general, setting $k_{m,j} = \sqrt{\lambda_2}$ one has

$$k_{m,j} = \frac{x_{m,j}}{R}$$

and solutions $X(r) = J_m(k_{m,j}r)$.

Thus we have solutions

$$u_{A\ m,j}(r,\theta) = J_m(k_{m,j}r)\big(ae^{im\theta} + be^{-im\theta}\big)$$

as long as

$$\frac{\omega^2}{c^2} = k_{m,j}^2.$$

The frequencies

$$\omega_{m,j} = ck_{m,j}$$

are called the resonant frequencies of the system.

....

Example 6.4: Now consider a related problem: the vibration of a circular membrane of radius R whose center is being vibrated harmonically by a small piston of radius r_0 attatched to the center of the membrane. The piston moves through a distance $\epsilon\cos(\omega t)$. This leads to the condition

$$u(\mathbf{x}, t) = \epsilon\cos(\omega t)$$

for $\|\mathbf{x}\| = r_0$ as well as $u(\mathbf{x}, t) = 0$ if $\|\mathbf{x}\| = R$. Proceeding as in the previous example one has

$$u(\mathbf{x}, t) = \mathrm{Re}\, u_A(r,\theta)e^{-i\omega t}$$

with $u_A(r_0,\theta)=\epsilon$ and $u_A(R,\theta)=0$. Since $\epsilon\neq 0$ the solution must be independent of θ. It follows that

$$u_A(r,\theta)=a\ J_0(\frac{\omega}{c}r)+b\ Y_0(\frac{\omega}{c}r).$$

The boundary conditions become

$$\begin{pmatrix} J_0(\frac{\omega}{c}r_0) & Y_0(\frac{\omega}{c}r_0) \\ J_0(\frac{\omega}{c}R) & Y_0(\frac{\omega}{c}R) \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} \epsilon \\ 0 \end{pmatrix}.$$

This system has a non-trivial solution only if

$$J_0(\frac{\omega}{c}r_0)Y_0(\frac{\omega}{c}R)-J_0(\frac{\omega}{c}R)Y_0(\frac{\omega}{c}r_0)\neq 0.$$

Frequencies for which this determinant is 0 are precisely the resonant frequencies of the system. If the system is not at resonance one has

$$\begin{pmatrix} a \\ b \end{pmatrix}=\frac{1}{J_0(\frac{\omega}{c}r_0)Y_0(\frac{\omega}{c}R)-J_0(\frac{\omega}{c}R)Y_0(\frac{\omega}{c}r_0)}\begin{pmatrix} Y_0(\frac{\omega}{c}R) & -Y_0(\frac{\omega}{c}r_0) \\ -J_0(\frac{\omega}{c}R) & J_0(\frac{\omega}{c}r_0) \end{pmatrix}\begin{pmatrix} \epsilon \\ 0 \end{pmatrix}$$

leading to

$$u(\mathbf{x},t)=\epsilon\,\mathrm{Re}\ \frac{Y_0(\frac{\omega}{c}R)J_0(\frac{\omega}{c}r)-J_0(\frac{\omega}{c}R)Y_0(\frac{\omega}{c}r)}{J_0(\frac{\omega}{c}r_0)Y_0(\frac{\omega}{c}R)-J_0(\frac{\omega}{c}R)Y_0(\frac{\omega}{c}r_0)}e^{-i\omega t}.$$

If ω is a resonant frequency then one of the assumtions are wrong: there are no solutions whose time dependence is given by $e^{-i\omega t}$. In fact, a lossless resonant system has no steady state: the amplitude of vibration in such a system grows with time.

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Eigenfunction Expansions I: finite domains

Recall that given an $n \times n$ matrix $\mathbf{A}$ the equation

$$\big(\mathbf{A} - \lambda\big)\mathbf{v} = 0$$

has solutions only for those values of λ for which $\det(\mathbf{A} - \lambda) = 0$. Recall further that if $\mathbf{A}$ is self-adjoint then it has n orthogonal eigenvectors which can be normalized and used as an orthonormal basis. Note that in general

$$\mathbf{w}^*\mathbf{A}\mathbf{v} = (\mathbf{A}^*\mathbf{w})^*\mathbf{v}.$$

Thus, if $\langle \mathbf{w}, \mathbf{v}\rangle = \mathbf{w}^*\mathbf{v}$ is the standard inner product on $\mathbf{C}^n$ then $\mathbf{A}$ is self-adjoint if and only if $\langle \mathbf{w}, \mathbf{A}\mathbf{v}\rangle = \langle \mathbf{A}\mathbf{w}, \mathbf{v}\rangle$.

Now consider a second order differential operator

$$L(\lambda) = \frac{d}{dx}P(x)\frac{d}{dx} + Q(x) - \lambda R(x)$$

on an interval (a, b) with P, Q and R real and P and R positive on (a, b). Consider functions

$$f, g : (a, b) \longrightarrow \mathbf{C}$$

and introduce the inner product

$$\langle f, g\rangle = \int_a^b \overline{f(x)}g(x)\,dx.$$

Consider

$$\langle f, \Big(\frac{d}{dx}P(x)\frac{d}{dx} + Q(x) - \lambda R(x)\Big)g\rangle$$

for functions f and g which are sufficiently well behaved. Integrating by parts twice one has

$$\begin{aligned}\int_a^b \overline{f(x)}\frac{d}{dx}P(x)\frac{d}{dx}g(x)\,dx &= \bar{f}Pg'\big|_a^b - \int_a^b \frac{d\bar{f}}{dx}P\frac{dg}{dx}\,dx \\ &= \bar{f}Pg'\big|_a^b - \bar{f}'Pg\big|_a^b + \int_a^b \Big(\frac{d}{dx}P\frac{d}{dx}\bar{f}\Big)g\,dx\end{aligned}$$

so that

$$\langle f, \Big(\frac{d}{dx}P(x)\frac{d}{dx} + Q(x) - \lambda R(x)\Big)g\rangle = P(\bar{f}g' - \bar{f}'g)\big|_a^b + \langle\Big(\frac{d}{dx}P(x)\frac{d}{dx} + Q(x) - \lambda R(x)\Big)f, g\rangle.$$

Thus if

$$P(\bar{f}g' - \bar{f}'g)\Big|_a^b = 0$$

then L is self-adjoint in the sense that

$$\langle f, Lg \rangle = \langle Lf, g \rangle.$$

This is where the boundary conditions come in. If one restricts attention to a set of functions which satisfy some conditions at a and b which make $P(\bar{f}g' - \bar{f}'g)\Big|_a^b = 0$ then L is self-adjoint on this restricted set. The most common examples of such a condition are
i) The Sturm-Liouville problem: any linear homogeneous two point boundary condition

$$\alpha_a f(a) + \beta_a f'(a) = 0$$

$$\alpha_b f(b) + \beta_b f'(b) = 0$$

with α_a, β_a, α_b and β_b all real.
ii) Periodic boundary conditions with $P = 1$:

$$f(a) = f(b)$$

and

$$f'(a) = f'(b).$$

The essential facts are
i) The equation

$$L(\lambda)f = 0$$

has non-zero solutions only for λ in some discrete, infinite set $\{\lambda_1, \lambda_2, \ldots\}$. The λ_j are called the eigenvalues of L. The set of all the eigenvalues is called the *spectrum* of L. The solutions f_j to

$$L(\lambda_j)f_j = 0$$

which satisfy the boundary conditions are called the eigenvectors of L. They are of finite norm. For the Sturm-Liouville problem there is only one eigenfunction for each eigenvalue λ_j. For periodic boundary conditions there can be two eigenfunctions per eigenvalue.

ii) The f_j form a basis for the set of all functions on (a, b) in the sense that given any g there is a sequence of coefficients c_j with

$$g(x) = \sum_j c_j f_j(x).$$

This is called an eigenfunction expansion of g.

iii) The f_j are an orthogonal basis in the sense that

$$\int_a^b \overline{f_j(x)} f_k(x)\, R(x)\, dx = \delta_{j,k} \int_a^b |f_j(x)|^2\, R(x)\, dx.$$

This gives a simple formula for the c_j:

$$c_j = \frac{1}{\int_a^b |f_j(x)|^2\, R(x)\, dx} \int_a^b \overline{f_j(x)} g(x)\, R(x)\, dx.$$

Example 7.1: Periodic boundary conditions on the one dimensional Helmholtz equation leads to the Fourier series. Consider

$$\Big(\frac{d^2}{dx^2} - \lambda\Big) f(x) = 0$$

on (a, b) with the boundary conditions

$$f(a) = f(b)$$

and

$$f'(a) = f'(b).$$

Then writing $f(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$ one has

$$\begin{pmatrix} e^{\sqrt{\lambda}a} - e^{\sqrt{\lambda}b} & e^{-\sqrt{\lambda}a} - e^{-\sqrt{\lambda}b} \\ \sqrt{\lambda}\big(e^{\sqrt{\lambda}a} - e^{\sqrt{\lambda}b}\big) & -\sqrt{\lambda}\big(e^{-\sqrt{\lambda}a} - e^{-\sqrt{\lambda}b}\big) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This leads to the eigenvalue condition

$$\sqrt{\lambda}\Big(1 - \cosh\sqrt{\lambda}(b-a)\Big) = 0.$$

Thus the eigenvalues are

$$\lambda_j = -\Big(\frac{2j\pi}{b-a}\Big)^2.$$

For $j = 0$ there is one eigenfunction, chosen normalized,

$$f_0(x) = \frac{1}{\sqrt{b-a}}.$$

For each $j > 0$ there are two eigenfunctions

$$f_{\pm j}(x) = \frac{1}{\sqrt{b-a}} e^{\pm \frac{2\pi i j}{b-a} x}.$$

The eigenfunction expansion of an arbitrary function g on (a, b) with respect to these eigenfunctions leads the the Fourier series for g,

$$g(x) = \sum_{j=-\infty}^{\infty} \frac{\langle f_j, g \rangle}{\sqrt{b-a}} e^{\frac{2\pi i j}{b-a} x}.$$

....

Example 7.2: Consider a clamped vibrating string described by the wave equation

$$\Big(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big) u(t, x) = 0$$

where $u(t, 0) = 0 = u(t, L)$. Impose the initial conditions $u(0, x) = g(x)$ and $\frac{\partial u}{\partial t}\big|_{t=0} = h(x)$ for some functions g and h with $g(0) = 0 = g(L)$ and $h(0) = 0 = h(L)$.

First let's develop the eigenfunction expansion for $\frac{d^2}{dx^2}$ with the boundary conditions $f(0) = 0 = f(L)$ (there are referred to as *Dirichlet boundary conditions*). Again, we need to consider

$$\Big(\frac{d^2}{dx^2} - \lambda\Big) f(x) = 0$$

on $(0, L)$. We have already seen that the eigenvalues for this problem are

$$\lambda_j = -\Big(\frac{j\pi}{L}\Big)^2$$

corresponding to the (normalized) eigenfunction

$$f_j(x) = \sqrt{\frac{2}{L}} \sin \frac{j\pi}{L} x.$$

Here $j \in \{1, 2, 3, \ldots\}$. Then one can write

$$u(t, x) = \sqrt{\frac{2}{L}} \sum_{j=1}^{\infty} c_j(t) \sin \frac{j\pi}{L} x$$

where

$$c_j(t) = \sqrt{\frac{2}{L}} \int_0^L \sin \frac{j\pi}{L} x \; u(t, x) \, dx.$$

Substituting this form for u into the wave equation one finds

$$\sum_{j=1}^{\infty} \Big((\frac{j\pi}{L})^2 + \frac{1}{c^2} \frac{d^2}{dt^2} \Big) c_j(t) \sin \frac{j\pi}{L} x = 0.$$

Taking the inner product of this equation $f_k(x) = \sqrt{\frac{2}{L}} \sin \frac{k\pi}{L} x$ yields the following equation for c_k,

$$\Big((\frac{k\pi}{L})^2 + \frac{1}{c^2} \frac{d^2}{dt^2} \Big) c_k(t) = 0.$$

It follows that

$$c_k(t) = a_k \sin(\frac{kc\pi}{L} t) + b_k \cos(\frac{kc\pi}{L} t).$$

To determine a_k and b_k one uses the initial conditions. One has

$$\sqrt{\frac{2}{L}} \sum_{j=1}^{\infty} c_j(0) \sin \frac{j\pi}{L} x = g(x)$$

and

$$\sqrt{\frac{2}{L}} \sum_{j=1}^{\infty} c_j'(0) \sin \frac{j\pi}{L} x = h(x).$$

Again, taking the inner product of these equations with f_k one obtains initial conditions for the c_k,

$$c_k(0) = \langle f_k, g \rangle$$

and

$$c_k'(0) = \langle f_k, h \rangle.$$

It follows that

$$c_k(t) = \frac{L}{kc\pi} \langle f_k, h \rangle \sin(\frac{kc\pi}{L} t) + \langle f_k, g \rangle \cos(\frac{kc\pi}{L} t)$$

so that

$$u(t, x) = \sqrt{\frac{2}{L}} \sum_{j=1}^{\infty} \Big(\frac{L}{jc\pi} \langle f_j, h \rangle \sin(\frac{jc\pi}{L} t) + \langle f_j, g \rangle \cos(\frac{jc\pi}{L} t) \Big) \sin \frac{j\pi}{L} x.$$

Now consider the case in which the string is plucked at the center, $x = \frac{L}{2}$, by being pulled a distance ϵ from equilibrium and then released. One then has

$$g(x) = \frac{2\epsilon}{L} \begin{cases} x & \text{if } 0 < x < \frac{L}{2} \\ L - x & \text{if } \frac{L}{2} < x < L \end{cases}$$

and

$$h(x) = 0.$$

One finds that

$$\langle f_j, g \rangle = \sqrt{\frac{2}{L}} \frac{2L\epsilon}{j^2\pi^2} \sin \frac{j\pi}{2}$$

so that

$$u(t,x) = \frac{4\epsilon}{\pi^2} \sum_{j=1}^{\infty} \frac{1}{(2j+1)^2} \cos\big(\frac{(2j+1)c\pi}{L} t\big) \sin\big(\frac{(2j+1)\pi}{L} x\big).$$

....

Example 7.3: Consider the time dependent Schrödinger equation for a particle of mass m confined to a one dimensional interval $[a,b]$ with rigid walls

$$-i\hbar \frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t)$$

with $\psi(a,t) = 0 = \psi(b,t)$. One can solve this problem by expanding $\psi(x,t)$ with respect to the eigenfunctions of $\frac{\partial^2}{\partial x^2}$ with Dirichlet boundary conditions at a and b. One has

$$\psi(x,t) = \sqrt{\frac{2}{b-a}} \sum_{j=1}^{\infty} c_j(t) \sin \frac{j\pi x}{b-a}$$

where the coefficients $c_j(t)$ satisfy the first order equation

$$-i\hbar \frac{dc_j}{dt} = \frac{\hbar^2}{2m} \frac{j^2\pi^2}{(b-a)^2} c_j$$

with the initial condition

$$c_j(0) = \sqrt{\frac{2}{b-a}} \int_a^b \psi(0,x) \ \sin \frac{j\pi x}{b-a} \, dx.$$

It follows that

$$c_j(t) = c_j(0) e^{i \frac{j\pi\hbar}{2m(b-a)^2} t}$$

so that

$$\psi(x,t) = \sqrt{\frac{2}{L}} \sum_{j=1}^{\infty} c_j(0) e^{i\frac{j\pi\hbar}{2m(b-a)^2}t} \sin\frac{j\pi x}{b-a}.$$

If one denotes the eigenfunctions by

$$\psi_j(x) = \sqrt{\frac{2}{b-a}} \sin\frac{j\pi x}{b-a}$$

then one has

$$\begin{aligned}\psi(x,t) &= \sum_{j=1}^{\infty} e^{i\frac{j\pi\hbar}{2m(b-a)^2}t} \psi_j(x)\langle\psi_j, \psi(x,0)\rangle \\ &= e^{i\hbar\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)t}\psi(x,0)\end{aligned}$$

with the operator $e^{i\hbar\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)t}$ defined by the eigenfunction expansion

$$e^{i\hbar\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right)t}\phi = \sum_{j=1}^{\infty} e^{i\frac{j\pi\hbar}{2m(b-a)^2}t}\psi_j\langle\psi_j, \phi\rangle.$$

....

Now consider the differential equation

$$\Big(\frac{d^2}{dx^2} + p\frac{d}{dx} + q - \lambda\Big)f(x) = 0 \tag{7.1}$$

corresponding to a second order O.D.E. in standard form. To relate this equation to an eigenvalue equation for a self-adjoint operator functions P, Q and R must be found for which

$$\frac{d^2}{dx^2} + p\frac{d}{dx} + q - \lambda = \frac{1}{P}\Big(\frac{d}{dx}P\frac{d}{dx} + Q - \lambda R\Big).$$

This equation gives

$$p = \frac{P'}{P},$$
$$q = \frac{Q}{P}$$

and

$$1 = \frac{R}{P}.$$

It follows that

$$P = e^{\int p\,dx}$$

so that (7.1) becomes

$$\Big(\frac{d}{dx}e^{\int p\,dx}\frac{d}{dx}+e^{\int p\,dx}q-\lambda e^{\int p\,dx}\Big)f=0. \tag{7.2}$$

Given either Sturm-Liouville or P-periodic boundary conditions this now becomes a self-adjoint eigenvalue problem whose eigenfunctions satisfy the orthogonality condition

$$\int_a^b \overline{f_j(x)}f_k(x)\,e^{\int p\,dx}\,dx=\delta_{j,k}\int_a^b |f_j(x)|^2\,e^{\int p\,dx}\,dx. \tag{7.3}$$

Sturm-Liouville problems can also be posed for differential equations which have a regular singular point at an endpoint. Consider the differential operator

$$L=\frac{d^2}{dx^2}+p\frac{d}{dx}+q$$

on (a,b). Let L have a regular singular point at a. One's chief concern is that L should be related to a (possibly) self-adjoint operator of the form given in (7.3),

$$\tilde{L}=\frac{d}{dx}e^{\int p\,dx}\frac{d}{dx}+e^{\int p\,dx}q.$$

One needs $\langle f,\tilde{L}g\rangle=\langle\tilde{L}f,g\rangle$. There is a difficulty with this expression independent of boundary conditions: if L has a regular singular point at a then $\langle f,\tilde{L}g\rangle$ might well not exist unless f and g are chosen from some restricted set. More explicitly, p is allowed to have a simple pole at $x=a$, $p(x)\sim\frac{p_{-1}}{x-a}$ as $x\downarrow a$, and q is allowed to have a second order pole at a, $q(x)\sim\frac{q_{-2}}{(x-a)^2}$ as $x\downarrow a$. Thus, for x close to a

$$e^{\int p\,dx}\sim(x-a)^{p_{-1}}$$

and then

$$e^{\int p\,dx}q\sim q_{-2}(x-a)^{p_{-1}-2}.$$

Thus, depending on p_{-1}, it is possible to have $\langle f,e^{\int p\,dx}qg\rangle=\infty$ unless f and g go to 0 rapidly enough as $x\downarrow a$.

There are several ways out of this dilemma. For our purposes the most straightforward is to restrict the set of functions we deal with to those whose behavior near $x=a$ is that given by the indicial equation. Let r_1 and r_2 be the roots of the indicial equation for L. Conditions of the form

$$\lim_{x\downarrow a}\Big(f(x)-c_1(x-a)^{r_1}-c_2(x-a)^{r_2}\Big)=0$$

if $r_1\neq r_2$ and

$$\lim_{x\downarrow a}\Big(f(x)-c_1(x-a)^{r}-c_2(x-a)^{r}\ln|x-a|\Big)=0$$

if $r_1=r_2=r$ can be imposed on both f and g. If one insists that $\alpha_a c_1+\beta_a c_2=0$ for real values α_a and β_a then both $\tilde{L}f$ and $\tilde{L}g$ go to 0 as $x\downarrow a$ and, if there is some real, homogeneous boundary condition at b,

$$\alpha_b f(b)+\beta_b f'(b)=0,$$

then one can easily check that $(\bar{f}g'-\bar{f}'g)\big|_a^b=0$. Thus $\tilde{L}$ is self-adjoint and its eigenfunctions can be made into an ortho-normal basis. (Note that if a is a regular point then $r_1=0$ and $r_2=1$ and the classical Sturm-Liouville theory is recovered.)

Example 7.4: Recall Example 6.3, the free vibrations of a clamped circular membrane. One must solve the two dimensional wave equation in polar coordinates,

$$\Big(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)u(r,\theta,t)=0$$

with the boundary condition $u(R,\theta,t)=0$ and initial conditions $u(r,\theta,0)=g(r,\theta)$ and $\frac{\partial u}{\partial t}\big|_{t=0}=h(r,\theta)$. Rather than proceeding as in Example 6.3, where some solutions were found (although it was not made clear to what extent all solutions were found), let's try to find an eigenfunction expansion for $\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$.

The normalized eigenfunctions of $\frac{d^2}{d\theta^2}$ with periodic boundary conditions are $\frac{1}{\sqrt{2\pi}}e^{im\theta}$ for $m\in\{0,\pm1,\pm2,\ldots\}$ corresponding to the eigenvalue m^2. It follows that any function of θ, in particular $u(r,\theta,t)$, can be expanded in an eigenfunction expansion in the functions $\frac{1}{\sqrt{2\pi}}e^{im\theta}$:

$$u(r,\theta,t)=\frac{1}{\sqrt{2\pi}}\sum_{m=-\infty}^{\infty}u_m(r,t)e^{im\theta}$$

with

$$u_m(r,t)=\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}u(r,\theta,t)e^{im\theta}\,d\theta.$$

Substituting into the wave equation, multiplying by $\frac{1}{\sqrt{2\pi}}e^{im\theta}$ and integrating over θ one finds that $u_m(r,t)$ satisfies the equation

$$\Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)u_m(r,t) = 0$$

with the conditions $u_m(R,t) = 0$ and $u_m(0,t)$ finite. The operator $L_m = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2}$ has a regular singular point at $r = 0$, however the condition that u_m be finite at $r = 0$ means we can restrict to functions that behave like r^m as $r \downarrow 0$ and are 0 at $r = R$. On this restricted set of functions L_m is self-adjoint. The eigenfunctions of L_m were found in Example 6.3. Using the notation developed there the eigenvalues are $-k_{m,j}^2 = -\frac{\omega_{m,j}^2}{c^2}$ corresponding to normalized eigenfunctions

$$f_{m,j}(r) = N_{m,j} J_{|m|}(k_{|m|,j} r)$$

where the normalization factor, $N_{m,j}$, is given by (7.3),

$$\begin{aligned} N_{m,j} &= \frac{1}{\sqrt{\int_0^R \big(J_{|m|}(k_{|m|,j} r)\big)^2 \, r\, dr}} \\ &= \frac{x_{|m|,j}}{R\sqrt{\int_0^{x_{|m|,j}} \big(J_{|m|}(z)\big)^2 \, z\, dz}} \end{aligned}$$

and the orthogonality relation by

$$\int_0^R f_{m,j}(r) f_{m,k}(r)\, r\, dr = \delta_{jk}.$$

The integral in the normalization factor $N_{m,j}$ can be computed either numerically, or using the identity

$$\int_0^a \big(J_m(z)\big)^2 \, z\, dz = \frac{a^2}{2}\begin{cases} J_m(a)^2 - J_{m-1}(a)J_{m+1}(a) & \text{if } m > 0 \\ J_0(a)^2 + J_1(a)^2 & \text{if } m = 0 \end{cases} \tag{7.4}$$

Note that the functions

$$u_{A\,m,j}(r,\theta) = \frac{1}{\sqrt{2\pi}} N_{m,j} J_{|m|}(k_{|m|,j} r) e^{im\theta}$$

are the sought after eigenfunctions of $\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$. They satisfy the orthogonality condition

$$\int_0^R \int_0^{2\pi} \overline{u_{A\,m,j}(r,\theta)}\, u_{A\,n,k}(r,\theta)\, r\, d\theta\, dr = \delta_{nm}\delta_{jk}.$$

Setting

$$c_{m,j}(t) = \int_0^R \int_0^{2\pi} \overline{u_{A\,m,j}(r,\theta)}\, u(r,\theta,t)\, r\, d\theta\, dr$$

one has

$$u(r,\theta,t) = \sum_{m=-\infty}^{\infty} \sum_{j=1}^{\infty} c_{m,j}(t)\, u_{A\,m,j}(r,\theta).$$

The coefficients $c_{m,j}(t)$ satisfy the equation

$$\Big(\frac{d^2}{dt^2} + \omega_{m,j}^2\Big) c_{m,j}(t) = 0$$

with the initial conditions

$$c_{m,j}(0) = \int_0^R \int_0^{2\pi} \overline{u_{A\,m,j}(r,\theta)}\, g(r,\theta)\, r\, d\theta\, dr$$

and

$$c'_{m,j}(0) = \int_0^R \int_0^{2\pi} \overline{u_{A\,m,j}(r,\theta)}\, h(r,\theta)\, r\, d\theta\, dr.$$

it follows that

$$c_{m,j}(t) = c_{m,j}(0)\cos(\omega_{m,j}t) + \frac{c'_{m,j}(0)}{\omega_{m,j}} \sin(\omega_{m,j}t)$$

so that

$$u(r,\theta,t) = \sum_{m=-\infty}^{\infty} \sum_{j=1}^{\infty} \Big(c_{m,j}(0)\cos(\omega_{m,j}t) + \frac{c'_{m,j}(0)}{\omega_{m,j}} \sin(\omega_{m,j}t) \Big) u_{A\,m,j}(r,\theta).$$

....

In the last example the eigenvalues and eigenfunctions of the Laplacian in a disk were found. The geometry, that of a disk, is special because a coordinate system can be found, polar coordinates, in which the Laplacian is separable so that the problem reduces to solving O.D.E.'s. In more complicated geometries, in which the Laplacian is not separable, the eigenvalue problem can still be posed. The discussion here will be restricted to two dimensions although it is readily extended to arbitrary dimension.

Let A be a region in $\mathbf{R}^2$. There are some restrictions that need to be put on A, but they are difficult to state mathematically. Loosely put, attention will be restricted to regions A in which Gauss' law can be applied. Note that

$$\begin{aligned}\int_A \Big(\bar{\psi}\nabla^2\phi - \phi\nabla^2\bar{\psi}\Big)\, dA &= \int_A \nabla\cdot\Big(\bar{\psi}\nabla\phi - \phi\nabla\bar{\psi}\Big)\, dA \\ &= \int_{\partial A} \Big(\bar{\psi}\nabla\phi - \phi\nabla\bar{\psi}\Big)\cdot d\mathbf{l}.\end{aligned}$$

Thus, if there are boundary conditions on imposed on ψ and ϕ on ∂A so that

$$\int_{\partial A} \left(\bar{\psi}\nabla\phi - \phi\nabla\bar{\psi}\right) \cdot \mathbf{n}\, dl = 0$$

then ∇^2 is self-adjoint in A with respect to the inner product

$$\langle \psi, \phi \rangle = \int_A \bar{\psi}\, \phi\, dA$$

in the sense that $\langle \psi, \nabla^2\phi \rangle = \langle \nabla^2\psi, \phi \rangle$. A large class of self-adjoint boundary conditions for ∇^2 is given by

$$(\alpha\phi + \beta\mathbf{n} \cdot \nabla\phi)\big|_{\partial A} = 0$$

for real constants α and β. The two most important special cases are *Dirichlet boundary conditions*, $\phi\big|_{\partial A} = 0$ and *Neumann boundary conditions*, $\mathbf{n} \cdot \nabla\phi\big|_{\partial A} = 0$.

Given self adjoint boundary conditions the eigenfunctions of ∇^2 are complete and can be chosen orthonormal. In the case of either Dirichlet or Neumann boundary conditions more can be said:

$$\begin{aligned} \langle \phi, \nabla^2\phi \rangle &= -\int_A \nabla\bar{\phi} \cdot \nabla\phi\, dA + \int_{\partial A} \bar{\phi}\nabla\phi \cdot \mathbf{n}\, dl \\ &= -\|\nabla\phi\|^2 \\ &\leq 0. \end{aligned}$$

It follows that if λ is an eigenvalue of ∇^2 with eigenvector ϕ then

$$\lambda = \frac{\langle \phi, \nabla^2\phi \rangle}{\langle \phi, \phi \rangle} \leq 0.$$

Let λ_j be a listing of the eigenvalues of $-\nabla^2$. In more than one dimension it is possible for a single eigenvalue to have more than one eigenfunction so that some of the λ_j might repeat. Such an eigenvalue is called degenerate and the span of its eigenfunctions is called its eigenspace. The dimension of the eigenspace is called the multiplicity of the eigenvalue. In general the λ_j form an unbounded increasing sequence, $\lambda_j \leq \lambda_{j+1}$ and $\lambda_j \to \infty$ as $j \to \infty$. Finally, note that for Neumann boundary conditions 0 is always an eigenvalue corresponding to the constant eigenfunction $\frac{1}{\sqrt{|A|}}$, where $|A|$ is the area of A. For Dirichlet boundary conditions 0 is never an eigenvalue of the Laplacian.

Example 7.5: Consider a free particle of mass m confined to a two dimensional region A. The quantum-mechanical Hamiltonian for this situation is

$$-\frac{\hbar^2}{2m}\nabla^2$$

with Dirichlet boundary conditions on ∂A. The the ground state eigenvalue is greater than zero. Thus the particle is never at rest.

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Example 7.6: Consider sound propagation in a long straight pipe with cross section A. Let the axis of the pipe be the z axis and let

$$\nabla_\perp = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}.$$

Then the linear pressure satisfies the wave equation

$$\Big(\frac{\partial^2}{\partial z^2} + \nabla_\perp^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)P(x, y, z, t) = 0$$

and the velocity is related to the pressure through Euler's equation

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla P.$$

The condition that the normal component of the velocity to the wall of the pipe must be zero,

$$\mathbf{n} \cdot \mathbf{v}\big|_{\partial A} = 0,$$

translates to a boundary condition on P,

$$\mathbf{n} \cdot \nabla P\big|_{\partial A} = 0.$$

This boundary condition can be satisfied, and the x, y part of the equation can be separated from the z, t part by expanding in an eigenfunction expansion in the Neumann eigenfunctions of $\nabla_\perp^2$. More explicitly, let $-\lambda_j$ be a listing of the Neumann eigenvalues and let ψ_j be the corresponding eigenfunctions of $\nabla_\perp^2$. Then

$$P(x,y,z,t) = \sum_j f_j(z,t)\,\psi_j(x,y)$$

with

$$\Big(\frac{\partial^2}{\partial z^2} - \lambda_j - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big) f_j(z,t) = 0.$$

Consider the steady state solutions of frequency ω. For these

$$f_j(z,t) = \Big(a_j\ e^{i\sqrt{\frac{\omega^2}{c^2}-\lambda_j}\,z} + b_j\ e^{-i\sqrt{\frac{\omega^2}{c^2}-\lambda_j}\,z}\Big)e^{-i\omega t}$$

so that

$$P(x,y,z,t) = \sum_j \psi_j(x,y)\,\Big(a_j\ e^{i\sqrt{\frac{\omega^2}{c^2}-\lambda_j}\,z} + b_j\ e^{-i\sqrt{\frac{\omega^2}{c^2}-\lambda_j}\,z}\Big)e^{-i\omega t}.$$

Note that only those terms with $\frac{\omega^2}{c^2} > \lambda_j$ propagate.

The coefficients a_j and b_j are determined by the boundary conditions. If the pipe is infinitely long then, for large z, the sound wave must be a traveling wave propagating to the right. For such a wave the coefficients b_j are 0. The a_j can be determined if, for example, the frequency ω component of the pressure amplitude $P_A(x,y,z)$, satisfying

$$P(x,y,z,t) = P_A(x,y,z)\,e^{-i\omega t},$$

is known at $z = 0$. Then

$$a_j = \int_A \overline{\psi_j(x,y)}\,P_A(x,y,0)\,dx\,dy.$$

As a concrete example consider a cylindrical pipe with radius R. Let there be a piston mounted at $z = 0$ with radius $\frac{R}{2}$ and velocity (along the y-axis) $u_0\cos(\omega t)$. The pressure field of the resulting sound wave is calculated.

A is a disk of radius R. Let $x_{m,j}^N$ be the j solution of $J_m'(x) = 0$, let $k_{m,j}^N = \frac{x_{m,j}^N}{R}$ and $\omega_{m,j}^N = ck_{m,j}^N$. Then the Neumann eigenvalues of $-\nabla_\perp^2$ in A are

$$\lambda_{m,j} = (k_{m,j}^N)^2$$

corresponding to the eigenfunctions

$$\psi_{m,j}(r,\theta) = \frac{1}{\sqrt{2\pi}} N_{m,j}^N J_{|m|}(k_{|m|,j}^N r)e^{im\theta}.$$

For $m = 0 = j$ one has $k^N_{0,0} = 0$ and then $N^N_{0,0} = \frac{\sqrt{2}}{R}$. Otherwise

$$N^N_{m,j} = \frac{1}{\sqrt{\int_0^R \left(J_{|m|}(k^N_{|m|,j} r)\right)^2 r\,dr}}$$
$$= \frac{x^N_{|m|,j}}{R\sqrt{\int_0^{x^N_{|m|,j}} \left(J_{|m|}(z)\right)^2 z\,dz}}.$$

The pressure wave is given by

$$P(r,\theta,z,t) = \mathrm{Re} \sum_{m=-\infty}^{\infty} \sum_{j=0}^{\infty} \psi_{m,j}(r,\theta)\, a_{m,j}\; e^{\frac{i}{c}\sqrt{\omega^2-(\omega^N_{m,j})^2}\, z - i\omega t}.$$

The coefficients $a_{m,j}$ are determined by the velocity boundary condition at $z = 0$:

$$\frac{\partial P}{\partial z}\Big|_{z=0} = \omega\rho_0 u_0 \sin(\omega t) \cdot \begin{cases} 1 & \text{if } 0 \le r < \frac{R}{2} \\ 0 & \text{if } \frac{R}{2} < r < R \end{cases}$$
$$= \mathrm{Re} \sum_{m=-\infty}^{\infty} \sum_{j=0}^{\infty} \psi_{m,j}(r,\theta)\, \frac{i}{c}\sqrt{\omega^2 - (\omega^N_{m,j})^2}\; a_{m,j}\; e^{-i\omega t}$$

which gives

$$a_{m,j} = \frac{\rho_0 c u_0 \omega}{\sqrt{\omega^2 - (\omega^N_{m,j})^2}} \int_0^{\frac{R}{2}} \int_0^{2\pi} \overline{\psi_{m,j}(r,\theta)}\, r\,d\theta\,dr$$
$$= \delta_{m,0}\sqrt{2\pi}\, N^N_{0,j}\rho_0 c u_0 \frac{\omega}{\sqrt{\omega^2 - (\omega^N_{0,j})^2}} \int_0^{\frac{R}{2}} J_0(k^N_{0,j} r)\, r\,dr.$$

Thus, P is independent of θ and is given by (noting that $\mathrm{Re}(ie^{-i\omega t}) = \sin\omega t$)

$$P(r,z,t) = \rho_0 c u_0 R\ \mathrm{Re}\Big[\sqrt{\frac{\pi}{2}}\; e^{i\frac{\omega}{c}z - i\omega t}$$
$$+ \sum_{j=1}^{\infty} \frac{\omega}{\sqrt{\omega^2 - (\omega^N_{0,j})^2}} \frac{\int_0^{\frac{x^N_{0,j}}{2}} J_0(x)\,x\,dx}{x^N_{0,j}\sqrt{\int_0^{x^N_{0,j}} \left(J_0(x)\right)^2 x\,dx}} J_0(k^N_{0,j} r) e^{\frac{i}{c}\sqrt{\omega^2-(\omega^N_{0,j})^2}\, z - i\omega t}\Big].$$

Note that for $j > 0$ for which $\omega^N_{0,j} \neq 0$ the speed of propagation is not c, but is frequency dependent, given by

$$c_{0,j} = \frac{c\omega}{\sqrt{\omega^2 - (\omega^N_{0,j})^2}}.$$

This phenomenon, the speed of propagation being frequency dependent, is known as dispersion.

Finally, the integrals over the Bessel functions can be done numerically, or by using the identities (7.4) and

$$\int_0^a J_0(z)\, z\, dz = a\, J_1(a).$$

To find the zeros x^N_{mj} of J'_m one can use the identities

$$J'_0(x) = -J_1(x)$$

and, for $m \geq 0$,

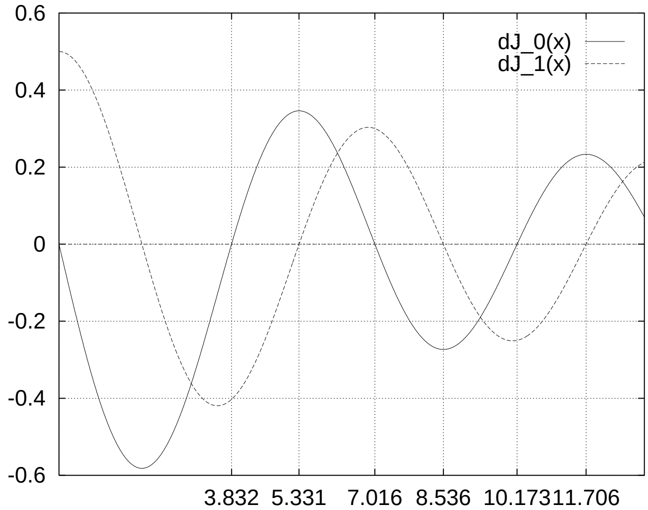
$$J'_m(x) = J_{m-1}(x) - \frac{m}{x} J_m(x).$$

Below, J'_m is plotted for $m = 0$ and $m = 1$. The zeros are labelled on the horizontal axis. Note that

$$x^N_{0,1} = 3.832$$
$$x^N_{0,2} = 7.016$$
$$x^N_{0,3} = 10.173$$

and

$$x^N_{1,1} = 5.331$$
$$x^N_{1,2} = 8.536$$
$$x^N_{1,3} = 11.706.$$



Finding the zeros of J'_0 and J'_1.

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The notion of self-adjointness is not restricted to second order operators. Consider

$$\int_a^b \bar{f}\frac{d^4g}{dx^4}\,dx = \Big(\bar{f}\frac{d^3g}{dx^3} - \frac{d\bar{f}}{dx}\frac{d^2g}{dx^2} + \frac{d^2\bar{f}}{dx^2}\frac{dg}{dx} - \frac{d^3\bar{f}}{dx^3}g\Big)\Big|_a^b + \int_a^b \frac{d^4\bar{f}}{dx^4}g\,dx$$

so that

$$\langle f, \frac{d^4}{dx^4}g\rangle = \Big(\bar{f}\frac{d^3g}{dx^3} - \frac{d\bar{f}}{dx}\frac{d^2g}{dx^2} + \frac{d^2\bar{f}}{dx^2}\frac{dg}{dx} - \frac{d^3\bar{f}}{dx^3}g\Big)\Big|_a^b + \langle\frac{d^4}{dx^4}f, g\rangle.$$

Thus, $\frac{d^4}{dx^4}$ is self adjoint on (a, b) if

$$\Big(\bar{f}\frac{d^3g}{dx^3} - \frac{d\bar{f}}{dx}\frac{d^2g}{dx^2} + \frac{d^2\bar{f}}{dx^2}\frac{dg}{dx} - \frac{d^3\bar{f}}{dx^3}g\Big)\Big|_a^b = 0.$$

For us the most important use of $\frac{d^4}{dx^4}$ is to model the vibration of a bar using the one dimensional plate equation,

$$\Big(\frac{d^4}{dx^4} + K\frac{\partial^2}{\partial t^2}\Big)u(x,t) = 0.$$

The most common self adjoint boundary conditions are

i) clamped at a (or b):

$$u(a,t) = 0 = \frac{\partial u}{\partial x}\Big|_{x=a},$$

ii) free at a (or b):

$$\frac{\partial^2 u}{\partial x^2}\Big|_{x=a} = 0 = \frac{\partial^3 u}{\partial x^3}\Big|_{x=a},$$

iii) periodic:

$$\begin{aligned} u(a,t) &= u(b,t) \\ \frac{\partial u}{\partial x}\Big|_{x=a} &= \frac{\partial u}{\partial x}\Big|_{x=b} \\ \frac{\partial^2 u}{\partial x^2}\Big|_{x=a} &= \frac{\partial^2 u}{\partial x^2}\Big|_{x=b} \\ \frac{\partial^3 u}{\partial x^3}\Big|_{x=a} &= \frac{\partial^3 u}{\partial x^3}\Big|_{x=b}. \end{aligned}$$

In general, any self adjoint boundary conditions for a fourth order operator must consist of four equations, and the eigenvalue condition will be a determinant of a four by four matrix. Thus, the eigenvalue problem for a fourth order operator is considerably more complicated than that for a second order operator. There is no way to avoid some amount of numerical work, and in special cases tricks are very important.

Example 7.7: Consider the vibrations of a bar of length L clamped at both ends. In solving the one dimensional plate equation we begin by generating an eigenfunction expansion for $\frac{d^4}{dx^4}$. Consider

$$\left(\frac{d^4}{dx^4} - \lambda\right)\psi(x) = 0$$

with the boundary conditions $\psi(0) = 0 = \psi(L)$, $\psi'(0) = 0 = \psi'(L)$. The general solution may be written

$$\psi(x) = a\cos kx + b\sin kx + c\cosh kx + d\sinh kx$$

with $k^4 = \lambda$. At 0 one has

$$0 = a + c$$

and

$$0 = kb + kd$$

so that

$$\psi(x) = a(\cos kx - \cosh kx) + b(\sin kx - \sinh kx).$$

At L one has

$$0 = a(\cos kL - \cosh kL) + b(\sin kL - \sinh kL)$$

and

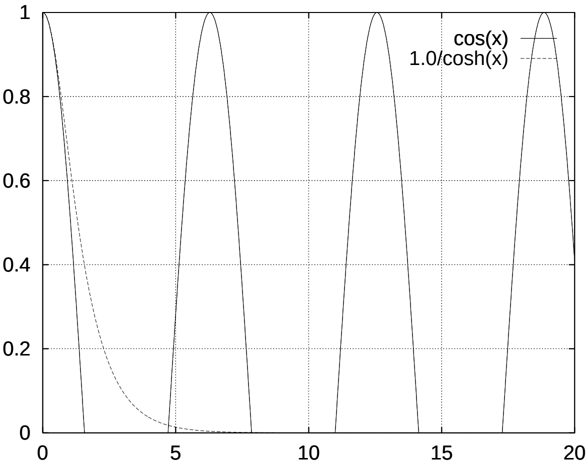
$$0 = ka(-\sin kL - \sinh kL) + kb(\cos kL - \cosh kL).$$

Thus the eigenvalue condition may be written

$$\begin{aligned} 0 &= (\cos kL - \cosh kL)^2 + (\sin^2 kL - \sinh^2 kL) \\ &= 2 - 2\cos kL\cosh kL, \end{aligned}$$

or

$$\cos kL\cosh kL = 1.$$

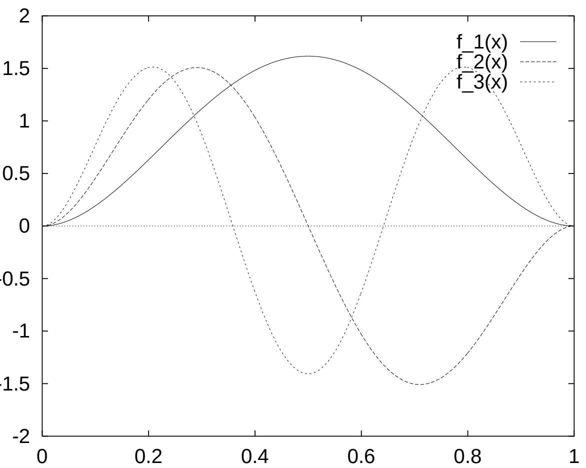


$$x = kL$$

The plot above shows that there are eigenvalues when $kL = 4.73,\ 7.851,\ 10.995, \ldots$. Given one of these values for kL the corresponding (unnormalized) eigenfunction is

$$f(x) = \Big(-\frac{\sin kL - \sinh kL}{\cos kL - \cosh kL}(\cos kx - \cosh kx) + \sin kx - \sinh kx \Big).$$

Below the first three (unnormalized) eigenfunctions are plotted with $L = 1$. Note that they are not sinusoidal.



Let the values of kL which satisfy the eigenvalue condition be labeled ξ_j. The eigenvalues are $\lambda_j = \frac{\xi_j^4}{L^4}$. Let the normalized eigenfunction corresponding to λ_j be

$$\psi_j(x) = N_j \Big(-\frac{\sin \xi_j - \sinh \xi_j}{\cos \xi_j - \cosh \xi_j}(\cos \frac{\xi_j x}{L} - \cosh \frac{\xi_j x}{L}) + \sin \frac{\xi_j x}{L} - \sinh \frac{\xi_j x}{L} \Big)$$

where $N_j = \langle f_j, f_j \rangle^{-\frac{1}{2}}$. Then the displacement is

$$u(x,t) = \sum_{j=1}^{\infty} \psi_j(x) \Big(a_j \cos \frac{\xi_j^2 t}{L^2 \sqrt{K}} + b_j \sin \frac{\xi_j^2 t}{L^2 \sqrt{K}} \Big).$$

The coefficients a_j and b_j can be determined by initial conditions:

$$a_j = \langle \psi_j, u(x,0) \rangle$$

and

$$b_j = \frac{L^2\sqrt{K}}{\xi_j^2} \langle \psi_j, \frac{\partial u}{\partial t}\big|_{t=0} \rangle.$$

Note that for kL large enough the eigenvalue condition becomes $\cos kL + \ldots = 0$, correct to order e^{-kL}. Further

$$\frac{\sin kL - \sinh kL}{\cos kL - \cosh kL} = 1 - 2 \sin kL \; e^{-kL} + \ldots$$

correct to order e^{-2kL}. Substituting this approximation into the eigenfunctions one has

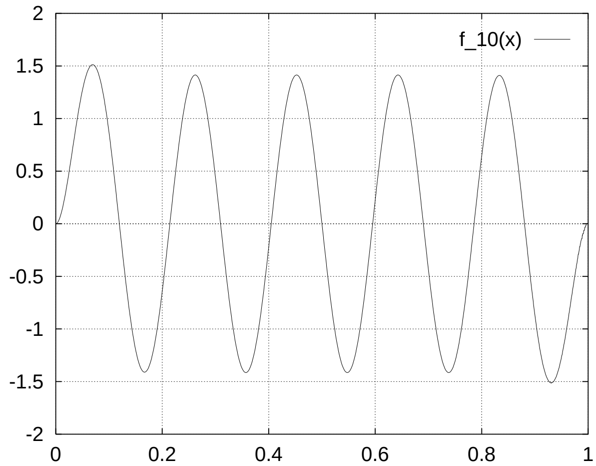
$$f(x) = \sin kx - \cos kx + e^{-kx} - \sin kL \; e^{-k(L-x)} + \ldots$$

correct to order e^{-kL} uniformly on $[0, L]$. Note that the exponential terms are negligable except when x is not much further than $\frac{1}{k}$ from either 0 or L.

Solving the approximate eigenvalue condition, $\cos kL = 0$, gives $kL = (n + \frac{1}{2})\pi$. This approximation is valid as long as $e^{-(n+\frac{1}{2})\pi} \ll 1$ (this is not a very stringent condition since for $n \geq 2$ $e^{-(n+\frac{1}{2})\pi} \leq e^{-\frac{3}{2}\pi} < 0.009$). The eigenfunctions are given approximately by

$$f_n(x) = \sin\left((n + \frac{1}{2})\pi\frac{x}{L}\right) - \cos\left((n + \frac{1}{2})\pi\frac{x}{L}\right) + e^{-(n+\frac{1}{2})\pi\frac{x}{L}} - (-1)^{n-1} \; e^{-(n+\frac{1}{2})\pi\frac{L-x}{L}} + \ldots \; .$$

The exponential terms are important only if x is no more than $\frac{2L}{(n+\frac{1}{2})}$ from either 0 or L (since $e^{-4\pi}$ is tiny). Thus f_n is essentially sinusoidal except for the first and last cycles. Below is a plot of f_{10} with $L = 1$.



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The last topic of this chapter is something we've been avoiding: how to deal with the angular part of the Laplacian in spherical coordinates. Recall that, in spherical coordinates ((3.1)),

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Big(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\Big).$$

We need to study the operator

$$L = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{7.5}$$

The range of θ and ϕ is chosen so that, for fixed r, the surface of a sphere is covered. The two basic conditions on functions $f(\theta,\phi)$ that we will act on with L is that they be single valued and finite. In terms of θ and ϕ this dictates that $\phi \in (0, 2\pi)$ with periodic boundary conditions imposed, and that $\theta \in (0,\pi)$ with no condition other than that functions are finite.

Subject to these conditions we look for eigenfunctions of L,

$$Lf(\theta,\phi) = \lambda f(\theta,\phi).$$

The ϕ dependence can be dealt with by using the periodic eigenfunctions of $\frac{\partial^2}{\partial\phi^2}$,

$$\frac{1}{\sqrt{2\pi}}e^{im\phi}.$$

Then look for eigenfunctions of the form

$$f(\theta,\phi) = f_m(\theta)\frac{1}{2\pi}e^{im\phi}.$$

Then

$$\Big(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{m^2}{\sin^2\theta} - \lambda\Big)f_m(\theta) = 0.$$

This equation has regular singular points at 0 and π and is usually analysed by making the change of variables

$$x = \cos\theta.$$

Then we need to solve the equation

$$\Big(\frac{d}{dx}(1-x^2)\frac{d}{dx} - \frac{m^2}{1-x^2} - \lambda\Big)g(x) = 0 \tag{7.6}$$

for $x \in (-1,1)$ and set $f_m(\theta) = g(\cos\theta)$.

The solutions of this equation are called Generalized Legendre functions. The complete study of these functions is quite involved. We won't get into it. Here we'll just notice that the indicial equation at both ± 1 is

$$r(r-1) + r - \frac{m^2}{4} = 0$$

which gives $r = \pm m$. In particular, for $m = 0$ $f(\pm 1)$ is non-zero while for $m \neq 0$ $f(\pm 1) = 0$.

The facts are that for each m the eigenvalues of (7.6) are given by $\lambda = -l(l+1)$ for $l \in \{|m|, |m|+1, |m|+2, \ldots\}$. The corresponding eigenfunctions are the generalized Legendre polynomials

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{\frac{m}{2}} \frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l.$$

None of the other solutions of (7.6) are finite at both $x = \pm 1$. They satisfy the orthogonality realtion

$$\int_{-1}^{1} P_{l'}^m(x) P_l^m(x)\, dx = \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{l'l}.$$

The standard choice for normalized mutually orthogonal eigenfunctions of L, (7.5), are called spherical harmonics and are given by

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)e^{im\phi}.$$

Here

$$LY_{lm} = -l(l+1)Y_{lm}$$

for $l \in \{0,1,2,3,\ldots\}$ and $m \in \{-l, -l+1, \ldots, l-1, l\}$. They satisfy

$$\int_0^{2\pi}\int_0^{\pi} \overline{Y_{l'm'}(\theta,\phi)}\, Y_{lm}(\theta,\phi)\ \sin\theta\, d\theta\, d\phi = \delta_{l'l}\,\delta_{m'm}.$$

The eigenfunction expansion in spherical harmonics is given by

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_{lm} Y_{lm}(\theta,\phi)$$

with

$$c_{lm} = \int_0^{2\pi} \int_0^{\pi} \overline{Y_{lm}(\theta,\phi)}\, f(\theta,\phi) \sin\theta \, d\theta \, d\phi.$$

A few spherical harmonics are given below.

$$Y_{00} = \sqrt{\frac{1}{4\pi}}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\, \sin\theta \, e^{i\phi}$$

$$Y_{10} = -\sqrt{\frac{3}{4\pi}}\, \cos\theta$$

$$Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\, \sin^2\theta \, e^{2i\phi}$$

$$Y_{21} = -\sqrt{\frac{15}{8\pi}}\, \sin\theta\cos\theta \, e^{i\phi}$$

$$Y_{20} = \sqrt{\frac{5}{4\pi}}\, (\frac{3}{2}\cos^2\theta - \frac{1}{2}).$$

For the $-m$ spherical harmonics use

$$Y_{l\,-m} = (-1)^m \overline{Y_{lm}}.$$

Example 7.8: Consider the radiation from a compact source. Away from the source the acoustic pressure satisfies the wave equation, written here in spherical coordinates

$$\Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}L - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big) P(r,\theta,\phi,t) = 0.$$

Expanding in spherical harmonics one has

$$P(r,\theta,\phi,t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{lm}(r,t) Y_{lm}(\theta,\phi)$$

where

$$\Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big) f_{lm}(r,t) = 0.$$

The $l = 0$ equation has exact solutions. In general let

$$f_{lm}(r,t) = \frac{u_{lm}(r,t)}{r}.$$

Then

$$\Big(\frac{\partial^2}{\partial r^2} - \frac{l(l+1)}{r^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)u_{lm}(r,t) = 0.$$

For $l = 0$ one has

$$u_{00}(r,t) = g(r-ct) + h(r-ct)$$

for arbitrary differentiable functions g and h. In particular

$$f_{00}(r,t) = \frac{g(r-ct) + h(r+ct)}{r}.$$

Note that for arbitrary l, m the radial dependence has the asymptotic form

$$f_{lm}(r,t) \approx \frac{g_{lm}(r-ct) + h_{lm}(r+ct)}{r}$$

for large r (large enough so that the $\frac{l(l+1)}{r^2}$ term can be ignored).

Now consider the frequency ω response. Setting

$$P(r,\theta,\phi,t) = P_A(r,\theta,\phi)e^{-i\omega t}$$

the pressure amplitude P_A satisfies the Helmholtz equation, here writtten in spherical coordinates with $k = \frac{\omega}{c}$

$$\Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r}L + k^2\Big)P_A(r,\theta,\phi) = 0. \tag{7.8}$$

Expanding P_A in spherical harmonics one has

$$P_A(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{A\,lm}(r)Y_{lm}(\theta,\phi)$$

and

$$\Big(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + k^2\Big)f_{A\,lm}(r) = 0. \tag{7.9}$$

This is the spherical Bessel equation. Thus

$$f_{A\,lm}(r) = a_{lm}h_l^{(+)}(kr) + b_{lm}h_l^{(-)}(kr)$$

so that

$$P_A(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\Big(a_{lm}h_l^{(+)}(kr) + b_{lm}h_l^{(-)}(kr)\Big)Y_{lm}(\theta,\phi). \tag{7.10}$$

Consider the case in which the source is radiating into free space. Recall the large x asymptotic form for the spherical Hankel functions,

$$h_l^{(\pm)}(x) \approx \frac{i^{\mp(l+1)}}{x} e^{\pm ix}.$$

Note that the h_l^+ represent outgoing spherical waves while the h_l^- represent incoming spherical waves. Thus, since there is nothing to reflect waves back at the source, in (7.10) the coefficients $b_{lm} = 0$ so that

$$P_A(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} h_l^{(+)}(kr) Y_{lm}(\theta,\phi).$$

It follows that for kr sufficiently large

$$\begin{aligned} P_A(r,\theta,\phi) &\approx \frac{e^{ikr}}{kr} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^{-(l+1)} a_{lm} Y_{lm}(\theta,\phi) \\ &= \mathcal{F}(\theta,\phi) \frac{e^{ikr}}{kr} \end{aligned}$$

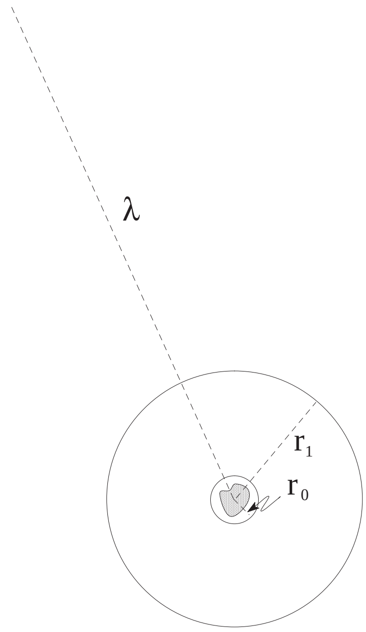
with

$$\mathcal{F}(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^{-(l+1)} a_{lm} Y_{lm}(\theta,\phi).$$

To estimate $\mathcal{F}$ it is sufficient to estimate the a_{lm}. To make things concrete let the source be some object, S, which is vibrating. Let the object be small enough to fit into a sphere of radius r_0. In the case in which

$$kr_0 \ll 1$$

(physically this is the case in which the wavelength is much greater than r_0) the a_{lm} can be estimated. Introduce a second sphere, of radius r_1, with $r_0 \ll r_1 \ll \frac{1}{k} = \frac{\lambda}{2\pi}$ and center both spheres at the origin of coordinates.



If $r_0 < r < r_1$ the k^2 term in (7.8) can be ignored so that $\nabla^2 P_A \approx 0$ and then (7.9) becomes

$$\Big(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2}\Big) f_{A\,lm}(r) = 0.$$

This is an Euler equation. The solutions which decrease as r increases are

$$f_{A\,lm}(r) = \frac{\alpha_{lm}}{r^{l+1}}$$

(this also follows from the small x asymptitics for $h_l^{(+)}(x)$) so that, for $r_0 \le r \le r_1$,

$$P_A(r,\theta,\phi) \approx \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \sum_{m=-l}^{l} \alpha_{lm} Y_{lm}(\theta,\phi).$$

For r_0 small enough the acoustic velocity amplitude at $r = r_0$ should be close to the velocity amplitude of the surface of the resonator. Further, through Euler's equation, the acoustic velocity amplitude at $r = r_0$ is proportional to $\nabla P_A\big|_{r=r_0}$. Thus

$$\begin{aligned}\alpha_{lm} &= -\frac{r_0^{l+2}}{l+1}\int_0^{\pi}\int_0^{2\pi} \overline{Y_{lm}(\theta,\phi)}\ \frac{\partial P_A}{\partial r}\Big|_{r=r_0} \sin\theta\, d\phi\, d\theta \\ &= -i\omega\rho_0\,\frac{r_0^{l+2}}{l+1}\int_0^{\pi}\int_0^{2\pi} \overline{Y_{lm}(\theta,\phi)}\ \hat{\mathbf{r}}\cdot\mathbf{v}(r_0,\theta,\phi)\,\sin\theta\, d\phi\, d\theta\end{aligned}$$

is close to $\omega\rho_0\, r_0^{l+2}$ times the velocity amplitude of the surface of the resonator. Since $r_1 \ll r_0$ it follows that

$$P_A(r_1,\theta,\phi) \approx \frac{\alpha_{00}}{\sqrt{4\pi}\, r_1} + \frac{1}{r_1^2}\Big(\alpha_{1,1}Y_{1,1} + \alpha_{1,0}Y_{1,0} + \alpha_{1,-1}Y_{1,-1}\Big) + \dots$$

where the omitted terms are smaller by a factor of $\frac{r_0}{r_1}$. Now using

$$a_{lm} = \frac{1}{h_l^{(+)}(kr_1)} \int_0^{2\pi} \int_0^{\pi} \overline{Y_{lm}(\theta,\phi)}\, P_A(r_1,\theta,\phi) \sin\theta \, d\theta \, d\phi$$

one has

$$a_{0,0} = k\alpha_{00}.$$

Finally, to determine α_{00} use

$$\nabla^2 P_A \approx 0$$

and Gauss's law giving

$$\int_{\partial S} \mathbf{n} \cdot \nabla P_A d\sigma = r_0^2 \int_0^{2\pi} \int_0^{\pi} \frac{\partial P_A}{\partial r}\Big|_{r=r_0} \sin\theta \, d\theta \, d\phi$$

from which it follows, using Euler's equation, that

$$a_{0,0} = \frac{-i\rho_0\omega^2}{\sqrt{4\pi}\, c} \int_{\partial S} \mathbf{n} \cdot \mathbf{v}_A d\sigma.$$

As long as $\alpha_{00} \neq 0$ the leading term in P_A is the first term so that, for $r \gg \lambda$,

$$P_A(r,\theta,\phi) \approx \frac{-i\rho_0\omega^2}{4\pi c} \Big(\int_{\partial S} \mathbf{n} \cdot \mathbf{v}_A d\sigma \Big) \frac{e^{ikr}}{kr}.$$

In this case note that since

$$(\nabla^2 + k^2) \frac{e^{ikr}}{kr} = -4\pi\delta(\mathbf{x})$$

one sees that P_A is approximately the solution of

$$(\nabla^2 + k^2) P_A = \frac{-i\rho_0\omega^2}{c} \Big(\int_{\partial S} \mathbf{n} \cdot \mathbf{v}_A d\sigma \Big) \delta(\mathbf{x})$$

which goes to zero as $r \to \infty$.

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Example 7.9: Consider the non-relativistic quantum mechanics of a free particle of mass m confined in a rigid spherical cavity of radius R. Given an initial state $\phi(r,\theta,\phi)$ the time evolved state $\psi(r,\theta,\phi,t)$ is the solution of the time dependent Schrödinger equation

$$-i\hbar \frac{\partial \psi}{\partial t} = H\psi$$

where $\psi(r,\theta,\phi,0) = \phi(r,\theta,\phi)$ and

$$H = -\frac{\hbar^2}{2m}\Big[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Big(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\Big)\Big]$$

in a sphere $0 \le r \le R$ subject to Dirichlet boundary conditions $\psi(R,\theta,\phi,t) = 0$ on the boundary.

Given the eigenvalues and eigenfunctions λ, ψ_λ of H one has

$$\psi(r,\theta,\phi,t) = \sum_\lambda \langle\psi_\lambda,\phi\rangle\, e^{\frac{i}{\hbar}\lambda t}\psi_\lambda(r,\theta,\phi).$$

Thus, the problem reduces to the eigenvalue problem for H,

$$(H-\lambda)\psi = 0.$$

To solve by separation of variables write

$$\psi(r,\theta,\phi) = f_{lm}(r)Y_{lm}(\theta,\phi).$$

One finds that

$$\Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2} + \frac{2m\lambda}{\hbar^2}\Big)f_{lm}(r) = 0.$$

One has

$$f_{lm}(r) = N_{lm}j_l(\frac{\sqrt{2m\lambda}}{\hbar}\,r)$$

with the eigenvalues λ determined by

$$j_l(\frac{\sqrt{2m\lambda}}{\hbar}\,R) = 0.$$

If $x_{l,j}$ is the j^{th} zero of $j_l(x)$ then the eigenvalues are

$$\lambda_{l,j} = \frac{(\hbar R x_{l,j})^2}{2m}$$

....

Example 7.10: We find the solution of

$$(\nabla^2 + k^2)f(\mathbf{x}) = 0$$

which satisfies

$$f\Big|_{\|\mathbf{x}\|=r_0} = 1 + \sin\theta\cos\phi.$$

and has the large $\|\mathbf{x}\| = r$ asymptotic form

$$f(\mathbf{x}) \approx \frac{e^{ikr}}{r}.$$

This example is rigged to be easy:

$$1 + \sin\theta\cos\phi = \sqrt{4\pi}\,Y_{0,0} - \sqrt{\frac{2\pi}{3}}\Big(Y_{1,1}(\theta,\phi) - Y_{1,-1}(\theta,\phi)\Big).$$

Thus

$$f(\mathbf{x}) = f_{0,0}(r)Y_{0,0} + f_{1,1}(r)Y_{1,1}(\theta,\phi) + f_{1,-1}(r)Y_{1,-1}(\theta,\phi)$$

with

$$\Big(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + k^2\Big)f_{0,0}(r) = 0,$$

$f_{0,0}(r_0) = \sqrt{4\pi}$ and $f(r) \to 0$ as $r \to \infty$ and

$$\Big(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{2}{r^2} + k^2\Big)f_{1,\pm1}(r) = 0,$$

$f_{1,\pm1}(r_0) = \mp\sqrt{\frac{2\pi}{3}}$ and $f_{1,\pm1}(r) \to 0$ as $r \to \infty$. It follows that

$$f_{0,0}(r) = \frac{\sqrt{4\pi}}{h_0^{(+)}(kr_0)}h_0^{(+)}(kr)$$

and

$$f_{1,\pm1}(r) = \mp\frac{\sqrt{\frac{2\pi}{3}}}{h_1^{(+)}(kr_0)}h_1^{(+)}(kr).$$

so that

$$\begin{aligned}
f(\mathbf{x}) &= \frac{h_0^{(+)}(kr)}{h_0^{(+)}(kr_0)} + \frac{h_1^{(+)}(kr)}{h_1^{(+)}(kr_0)}\sin\theta\cos\phi \\
&= \frac{r_0}{r}e^{ik(r-r_0)} + \frac{\frac{i}{k^2r^2} + \frac{1}{kr}}{\frac{i}{k^2r_0^2} + \frac{1}{kr_0}}e^{ik(r-r_0)}\sin\theta\cos\phi.
\end{aligned}$$

....

Eigenfunction Expansions II: infinite domains

Consider a second order ordinary differential operator

$$L = \frac{d}{dx} P \frac{d}{dx} + Q$$

on an infinite domain, say (a, ∞). Let

$$\|f\| = \sqrt{\int_a^\infty |f(x)^2 \, dx}$$

be a norm. Note that if f, g have finite norm then $f, g \to 0$ as $x \to \infty$. If f, g have finite norm and P is bounded for large enough x then

$$\begin{aligned} \langle f, Lg \rangle &= \int_a^\infty \overline{f(x)} Lg(x) \, dx \\ &= \langle Lf, g \rangle - P(a) \Big(\overline{f(a)} g'(a) - \overline{f'(a)} g(a) \Big) \end{aligned}$$

so that L is self-adjoint on (a, ∞) if

$$P(a) \Big(\overline{f(a)} g'(a) - \overline{f'(a)} g(a) \Big) = 0.$$

Note that a Sturm-Liouville boundary condition at a,

$$\alpha f(a) + \beta f'(a) = 0$$

with α and β both real, is sufficient to make L self-adjoint.

Self-adjoint operators on infinite domains have *generalized eigenfunction expansions.* Consider the differential equation

$$L\psi = \epsilon R \psi$$

where $R(x) > 0$ and $\epsilon \in \mathbf{R}$. If, given ϵ, there is a solution ψ with finite norm, $\|\psi\| < \infty$, then ϵ is said to be an eigenvalue and ψ an eigenfunction. Typically, the set of eigenvalues is discrete. If, given ϵ, there are no solutions with finite norm, but there is at least one solution which is polynomially bounded,

$$|\psi(x)| \leq \text{const } (1 + |x|^n)$$

for some n, then ϵ is said to be a generalized (or continuum) eigenvalue and ψ a generalized (or continuum) eigenfunction. Typically, the set of generalized eigenvalues is a continuum.

Let the eigenvalues and corresponding eigenfunctions be given by ϵ_j and ψ_j respectively. Let the continuum eigenfunctions be $\psi_n(\epsilon, \cdot)$ and let I_n be the domain over which ϵ varies. The facts are that one has the orthogonality relations

$$\langle \psi_j, \psi_k \rangle_R = \delta_{jk}, \qquad \langle \psi_n(\epsilon, \cdot), \psi_k \rangle_R = 0, \qquad \langle \psi_n(\epsilon, \cdot), \psi_m(\eta, \cdot) \rangle_R = \delta_{nm} \delta(\epsilon - \eta),$$

with

$$\langle \psi, \phi \rangle_R = \int_a^\infty \overline{\psi(x)} \phi(x) \, R(x) \, dx,$$

and the completeness relation

$$\sum_j \psi_j(x) \, \overline{\psi_j(y)} R(y) + \sum_n \int_{I_n} \psi_n(\epsilon, x) \, \overline{\psi_n(\epsilon, y)} R(y) \, d\epsilon = \delta_{(a,\infty)}(x - y).$$

An immediate consequence is the generalized eigenfunction expansion of a function f relative to L and R:

$$f(x) = \sum_j c_j \psi_j(x) + \sum_n \int_{I_n} \tilde{f}_n(k) \psi_n(\epsilon, x) \, d\epsilon$$

with

$$c_j = \langle \psi_j, f \rangle_R$$

and

$$\tilde{f}_n(\epsilon) = \langle \psi_n(\epsilon, \cdot), f \rangle_R.$$

The operator L is diagonolized by the eigenfunction expansion in the sense that

$$Lf(x) = \sum_j \epsilon_j c_j R(x) \psi_j(x) + \sum_n \int_{I_n} \epsilon \tilde{f}_n(\epsilon) R(x) \psi_n(\epsilon, x) \, d\epsilon.$$

Note that solving the differential equation $(L - R\epsilon)\psi = 0$ can only determine ψ up to a multiplicative constant (constant in x, but possibly dependent on ϵ). In order that the orthogonality and completeness relations return one times the delta function the multiplicative constant must be chosen appropriately. This is completely analogous to normalizing an eigenvector, and the constant will be called a normalization factor.

Note that the normalizations are determined only up to an arbitrary complex phase since $\langle e^{i\theta}\psi, e^{i\theta}\psi\rangle_R = \langle\psi,\psi\rangle_R$. Further, it is common in the integral over the continuous spectrum to integrate over some variable other than the eigenvalue parameter ϵ (a frequent choice is wavenumber k given by $\epsilon = k^2$). If λ is some parameter related differentiably to ϵ then the completeness relation can be written

$$\sum_j \psi_j(x)\,\overline{\psi_j(y)}R(y) + \sum_n \int_{\epsilon^{-1}(I_n)} \psi_n(\epsilon(\lambda),x)\,\overline{\psi_n(\epsilon(\lambda),y)}R(y)\,\frac{d\epsilon}{d\lambda}d\lambda = \delta_{(a,\infty)}(x-y)$$

so that the continuum eigenfunctions normalized with respect to λ, $\phi_n(\lambda,x)$, are given by

$$\phi_n(\lambda,x) = \sqrt{\frac{d\epsilon}{d\lambda}}\,\psi_n(\epsilon(\lambda),x).$$

Example 8.1: For $L = \frac{d^2}{dx^2}$ on $(-\infty,\infty)$ all of the solutions to $\psi'' = \epsilon\psi$ are of the form

$$\psi(x) = a\,e^{\sqrt{\epsilon}\,x} + b\,e^{-\sqrt{\epsilon}\,x}.$$

The only way for these functions to be polynomially bounded for all x is if $\epsilon \in (-\infty,0)$. In particular, there are no eigenvalues but there are continuum eigenvalues $(-\infty,0)$ and, given $\epsilon \in (-\infty,0)$, continuum eigenfunctions

$$N_\pm(\epsilon)e^{\pm\sqrt{\epsilon}\,x}.$$

The normalization factor can be determined either by the orthogonality relation,

$$\begin{aligned}\overline{N_\pm(\epsilon)}N_\pm(\eta)\int_{-\infty}^{\infty} e^{\pm i(-\sqrt{\epsilon}+\sqrt{\eta})\,x}\,dx &= 2\pi\overline{N_\pm(\epsilon)}N_\pm(\eta)\delta(\sqrt{\epsilon}-\sqrt{\eta})\\ &= 4\pi\sqrt{\epsilon}\;\overline{N_\pm(\epsilon)}N_\pm(\eta)\delta(\epsilon-\eta)\\ &= \delta(\epsilon-\eta)\end{aligned}$$

so that one may choose

$$N_\pm(\epsilon) = \sqrt{\frac{1}{4\pi\sqrt{\epsilon}}},$$

or by the completeness relation,

$$\begin{aligned}\sum_\pm \int_0^\infty |N_\pm(\epsilon)|^2 e^{\pm i\sqrt{\epsilon}\,(x-y)}\,d\epsilon &= \sum_\pm \int_0^\infty |N_\pm(k^2)|^2 e^{\pm ik\,(x-y)}\,2kdk\\ &= \int_{-\infty}^{\infty} |N_\pm(k^2)|^2 e^{ik\,(x-y)}\,2|k|dk\\ &= \delta(x-y)\end{aligned}$$

so that, again, one may choose

$$N_{\pm}(\epsilon) = \sqrt{\frac{1}{4\pi\sqrt{\epsilon}}}.$$

Setting $\epsilon = -k^2$ the generalized eigenfunction expansion is just the Fourier transform. The completeness relation becomes

$$\begin{aligned}\delta(x-y) &= \sum_{\pm}\int_0^{\infty} \frac{1}{4\pi\sqrt{\epsilon}} e^{\pm i\sqrt{\epsilon}\,(x-y)}\,d\epsilon \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-y)}\,dk.\end{aligned}$$

....

Example 8.2: For $L = \frac{d^2}{dx^2}$ on $(0,\infty)$ with Dirichlet boundary conditions $\psi(0) = 0$ all of the solutions of the differential equation $\psi'' = \epsilon\psi$ which satisfy the boundary conditions are of the form

$$\psi(x) = a\,\sin(\sqrt{-\epsilon}\,x).$$

They are are polynomially bounded for all $x > 0$ only if $\epsilon \in (-\infty, 0)$. Setting $\epsilon = -k^2$ and noting that changing k to $-k$ gives nothing new, one obtains the generalized eigenfunction expansion, often called the Fourier sine transform,

$$f(x) = \int_0^{\infty} \hat{f}(k)\,a\,\sin(kx)\,dk$$

with

$$\hat{f}(k) = \int_0^{\infty} f(x)\,a\,\sin(kx)\,dx.$$

The *normalization constant*, a, can be determined by the completeness condition

$$\begin{aligned}\delta_{(0,\infty)}(x-y) &= \int_0^{\infty} |a|^2\,\sin(kx)\,\sin(ky)\,dk \\ &= -\frac{1}{4}\int_0^{\infty} |a|^2\,(e^{ik(x+y)} - e^{ik(x-y)} - e^{-ik(x-y)} - e^{-ik(x+y)})\,dk \\ &= \frac{\pi}{2}|a|^2\,\delta(x-y).\end{aligned}$$

Since x and y are both positive, $\delta(x+y) = 0$ so that it is sufficient to choose

$$a = \sqrt{\frac{2}{\pi}}.$$

Similarly there is the Fourier cosine transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}(k)\, \cos(kx)\, dk$$

with

$$\hat{f}(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\, \cos(kx)\, dx$$

corresponding to Neumann boundary conditions at $x = 0$.

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Example 8.3: As an application consider sound propagation near a wall. Let the wall be at $x = 0$, let the pressure be constant in the y direction and suppose the sound field at $z = 0$ is known for all time and that the sound is propagating in the z direction. Then

$$P(x, z, t) = \frac{1}{\pi} \int_{-\infty}^\infty \int_0^\infty a(q, z, \omega)\, \cos(qx) e^{-i\omega t}\, dq\, d\omega.$$

Applying the wave equation to P one finds that

$$\left(\frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2} - q^2\right) a(q, z, \omega) = 0$$

so that

$$a(q, z, \omega) = \alpha(q, \omega) e^{i\sqrt{\frac{\omega^2}{c^2} - q^2}\, z} + \beta(q, \omega) e^{-i\sqrt{\frac{\omega^2}{c^2} - q^2}\, z}$$

and thus

$$P(x, z, t) = \frac{1}{\pi} \int_{-\infty}^\infty \int_0^\infty \left(\alpha(q, \omega) e^{i\sqrt{\frac{\omega^2}{c^2} - q^2}\, z} + \beta(q, \omega) e^{-i\sqrt{\frac{\omega^2}{c^2} - q^2}\, z}\right) \cos(qx) e^{-i\omega t}\, dq\, d\omega.$$

For the sound to propagate and remain finite in the direction of increasing z one must have $\alpha(q, \omega) = 0$ for $\omega < 0$ and $\beta(q, \omega) = 0$ for $\omega > 0$ and the square roots must be defined so that, for $q > |\frac{\omega}{c}|$, the exponentials decrease with increasing z. Then, taking into account the reality of p, one may write

$$P(x, z, t) = \frac{2}{\pi} \operatorname{Re} \int_0^\infty \int_0^\infty \alpha(q, \omega) e^{i\sqrt{\frac{\omega^2}{c^2} - q^2}\, z - i\omega t} \cos(qx)\, dq\, d\omega$$

choosing the square root here to have non-negative imaginary part. Since $P(x,0,t) = f(x,t)$ is known one has

$$\alpha(q,\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} f(x,t)\cos(qx)e^{i\omega t}\,dx\,dt.$$

Substituting back into the integral for $P(x,z,t)$ one can compute $P(x,z,t)$ for all t and $z > 0$. In particular, one can study diffraction near a wall: choose

$$f(x,t) = \begin{cases} \cos(\omega_0 t) & \text{if } 0 < x < x_0 \\ 0 & \text{if } x > x_0. \end{cases}$$

Then

$$\alpha(q,\omega) = \frac{\sin(qx_0)}{q}\Big(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\Big)$$

and

$$P(x,z,t) = \frac{2}{\pi}\,\mathrm{Re}\int_0^\infty \frac{\sin(qx_0)}{q} e^{i\sqrt{\frac{\omega^2}{c^2}-q^2}\,z - i\omega_0 t}\cos(qx)\,dq.$$

....

Example 8.4: Consider

$$L = -\frac{d^2}{dx^2}$$

on $(0,\infty)$ with the boundary condition $\psi'(0) = -a\psi(0)$ for some positive constant a. The eigenvalue problem

$$(-\frac{d^2}{dx^2} - \epsilon)\psi = 0$$

has the general solution

$$\psi(x) = \mathcal{N}\big(\sqrt{\epsilon}\,\cos(\sqrt{\epsilon}\,x) - a\sin(\sqrt{\epsilon}\,x)\big).$$

These solutions are bounded for $\epsilon \geq 0$ so that the continuous spectrum is $(0,\infty)$. Normalizing with respect to $k = \sqrt{\epsilon}$ one has

$$\begin{aligned}\psi(x) &= \mathcal{N}\big(k\,\cos(kx) - a\sin(kx)\big)\\ &= \mathcal{N}\sqrt{k^2+a^2}\cos(kx - \arctan(\frac{a}{k}))\end{aligned}$$

so that (by comparison with the cosine transform)

$$\mathcal{N} = \sqrt{\frac{2}{\pi(k^2+a^2)}}.$$

In addition to the continuous spectrum there is one eigenvalue corresponding to the square integrable solution obtained when $\epsilon = -a^2$ or $\sqrt{\epsilon} = ia$. The corresponding normalized eigenfunction is

$$\phi(x) = \sqrt{2a}\, e^{-ax}.$$

The resulting eigenfunction expansion is

$$f(x) = c_0\sqrt{2a}\, e^{-ax} + \sqrt{\frac{2}{\pi}} \int_0^\infty c(k) \cos(kx - \arctan(\frac{a}{k}))$$

with

$$c_0 = \int_0^\infty \sqrt{2a}\, e^{-ax} f(x)\, dx$$

and

$$c(k) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(kx - \arctan(\frac{a}{k}))f(x)\, dx.$$

Note that when $a = 0$ one obtains the cosine transform and in the limit $a \to \infty$ one obtains the sine transform.

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Example 8.5: Now consider

$$L = -\frac{d^2}{dx^2} + u(x)$$

where

$$u(x) = \begin{cases} 0 & \text{if } |x| > 1 \\ -D & \text{if } |x| \le 1 \end{cases}.$$

Then the solutions of

$$\big(L - \epsilon\big)\psi = 0$$

can be constructed as follows. Let $k = \sqrt{\epsilon}$. One has

$$\psi(x) = \begin{cases} A_- e^{ikx} + B_- e^{-ikx} & \text{if } x < -1 \\ a e^{i\sqrt{k^2+D}\ x} + b e^{-i\sqrt{k^2+D}\ x} & \text{if } -1 \le x \le 1 \\ A_+ e^{ikx} + B_+ e^{-ikx} & \text{if } 1 < x \end{cases}$$

with the conditions that ψ and its derivative be continuous at $x = \pm 1$. It follows that

$$\begin{pmatrix} e^{-ik} & e^{ik} \\ ike^{-ik} & -ike^{ik} \end{pmatrix} \begin{pmatrix} A_- \\ B_- \end{pmatrix} = \begin{pmatrix} e^{-i\sqrt{k^2+D}} & e^{i\sqrt{k^2+D}} \\ i\sqrt{k^2+D}\, e^{-i\sqrt{k^2+D}} & -i\sqrt{k^2+D}\, e^{i\sqrt{k^2+D}} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

and

$$\begin{pmatrix} e^{ik} & e^{-ik} \\ ike^{ik} & -ike^{-ik} \end{pmatrix} \begin{pmatrix} A_+ \\ B_+ \end{pmatrix} = \begin{pmatrix} e^{i\sqrt{k^2+D}} & e^{-i\sqrt{k^2+D}} \\ i\sqrt{k^2+D}\, e^{i\sqrt{k^2+D}} & -i\sqrt{k^2+D}\, e^{-i\sqrt{k^2+D}} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

so that

$$\begin{pmatrix} A_- \\ B_- \end{pmatrix} = T \begin{pmatrix} A_+ \\ B_+ \end{pmatrix}$$

where

$$\begin{aligned} T = &\begin{pmatrix} e^{-ik} & e^{ik} \\ ike^{-ik} & -ike^{ik} \end{pmatrix}^{-1} \begin{pmatrix} e^{-i\sqrt{k^2+D}} & e^{i\sqrt{k^2+D}} \\ i\sqrt{k^2+D}\, e^{-i\sqrt{k^2+D}} & -i\sqrt{k^2+D}\, e^{i\sqrt{k^2+D}} \end{pmatrix} \\ &\cdot \begin{pmatrix} e^{i\sqrt{k^2+D}} & e^{-i\sqrt{k^2+D}} \\ i\sqrt{k^2+D}\, e^{i\sqrt{k^2+D}} & -i\sqrt{k^2+D}\, e^{-i\sqrt{k^2+D}} \end{pmatrix}^{-1} \begin{pmatrix} e^{ik} & e^{-ik} \\ ike^{ik} & -ike^{-ik} \end{pmatrix} \\ &= \begin{pmatrix} T_{11}(k) & T_{12}(k) \\ T_{21}(k) & T_{22}(k) \end{pmatrix} \end{aligned}$$

with

$$T_{11}(k) = \frac{1}{4k\sqrt{k^2+D}} \Big((k+\sqrt{k^2+D})^2 e^{2i(k-\sqrt{k^2+D})} - (k-\sqrt{k^2+D})^2 e^{2i(k+\sqrt{k^2+D})} \Big)$$

$$T_{12}(k) = -\frac{D}{2k\sqrt{k^2+D}} \cos(\sqrt{k^2+D})$$

$$T_{21}(k) = -\frac{D}{2k\sqrt{k^2+D}} \cos(\sqrt{k^2+D})$$

and

$$T_{22}(k) = \frac{1}{4k\sqrt{k^2+D}} \Big((k+\sqrt{k^2+D})^2 e^{2i(-k+\sqrt{k^2+D})} - (k-\sqrt{k^2+D})^2 e^{-2i(k+\sqrt{k^2+D})} \Big).$$

If $\epsilon > 0$ then k is real and ψ is polynomially bounded, but does not have finite norm. Thus $(0,\infty)$ are all continuum eigenvalues and, given $\epsilon \in (0,\infty)$, there are two corresponding continuum eigenfunctions. (They can be taken to be what one obtains with $A_- \neq 0$, $B_- = 0$ and $A_- = 0$, $B_- \neq 0$, or whatever linear combinations are convenient).

If $\epsilon < 0$ then k is pure imaginary and can be chosen to have positive imaginary part. Then ψ grows exponentially unless both $A_- = 0$ and $B_+ = 0$. But this is possible only if $T_{11}(k) = 0$. Given such a k, ψ decreases exponentially and is of finite norm so that $\epsilon = k^2$ is an eigenvalue. Setting $k = i\sqrt{D}\,\kappa$ the condition $T_{11}(k) = 0$ becomes

$$e^{-i\sqrt{D}\,\sqrt{1-\kappa^2}} = -\kappa - i\sqrt{1-\kappa^2}$$

with $\kappa \neq 1$.

For $\kappa > 1$ one would have

$$e^{\sqrt{D}\sqrt{\kappa^2-1}} = -\kappa + \sqrt{\kappa^2 - 1},$$

but the left side is positive while the right is negative, so there are no solutions with $\kappa > 1$. For $0 < \kappa < 1$ one can break into real and imaginary parts. One finds

$$\cos\left(\sqrt{D}\sqrt{1-\kappa^2}\,\right) = -\kappa$$

and

$$\sin\left(\sqrt{D}\sqrt{1-\kappa^2}\,\right) = \sqrt{1-\kappa^2}.$$

In principle, since this is two equations one can not expect a real solution, but these two equations are actually equivalent, since $\cos\theta = -\kappa$ implies that $\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\kappa^2}$. The most convenient one to solve is the second. Solving amounts to finding the values of $x \in (0,1)$ for which $\sin(\sqrt{D}\,x) = x$. For $\sqrt{D} \leq 1$ there are no solutions. For $\sqrt{D} > 1$ there is a least one solution, with the number of solutions increasing as $\sqrt{D}$ increases. For each solution x there is an eigenvalue $\epsilon = -(1-x^2)D$.

Now let $D > 1$ and let $\psi_1, \ldots, \psi_N$ be the normalized eigenfunctions corresponding to eigenvalues $\epsilon_1, \ldots, \epsilon_N$. For the continuum eigenfunctions let $\psi_-(k,\cdot)$ be the normalized solution with $A_- = 0$ and $B_- = B$ and let $\psi_+(k,\cdot)$ be the normalized solution with $B_- = 0$ and $A_- = A$. To determine the normalization constant for the eigenfunctions it is sufficient to compute $\|\psi_j\|$ and check that it is 1. For the continuum eigenfunctions we will use the completeness relation

$$\begin{aligned}
\delta(x-y) &= \sum_{j=1}^{N} \psi_j(x)\overline{\psi_j(y)} + \sum_{\pm}\int_0^\infty \psi_\pm(k,x)\overline{\psi_\pm(k,y)}\,dk \\
&= \sum_{j=1}^{N} \psi_j(x)\overline{\psi_j(y)} + \int_{-\infty}^\infty \psi_+(k,x)\psi_-(k,y)\,dk.
\end{aligned}$$

Since the ψ_j go to zero exponentially as $x \to -\infty$ if we let both x and y go to $-\infty$ we get

$$\delta(x-y) = \int_{-\infty}^{\infty} BAe^{-ik(x-y)}\,dk$$

so that it is sufficient to choose $A = B = \frac{1}{\sqrt{2\pi}}$.

Applications: This operator arises both in the quantum mechanics of a particle of mass m in a square well in which the Hamiltonian is $H = \frac{\hbar^2}{2m} L$ and in the one dimensional Helmholtz equation with slower wave speed in a finite region in which

$$c(x) = \begin{cases} c_0 & \text{if } |x| > 1 \\ c_0 - \delta & \text{if } |x| < 1 \end{cases}$$

and

$$u(x) = \frac{\omega^2}{c_0^2} - \frac{\omega^2}{c(x)^2}.$$

In the former case a representation of the time evolution operator may be obtained by eigenfunction expansion:

$$\langle x | e^{\frac{i}{\hbar}\frac{\hbar^2}{2m} L t} | y \rangle = \sum_{j=1}^{N} e^{\frac{i}{\hbar}\frac{\hbar^2}{2m}\epsilon_j t} \psi_j(x) \overline{\psi_j(y)} + \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}\frac{\hbar^2}{2m} k^2 t} \psi_+(k,x) \psi_-(k,y)\, dk.$$

In the later case the possible free vibration fields $y(x, \omega)$ at (angular) frequency ω of the string may be obtained by eigenfunction expansion:

$$y(x,\omega) = \sum_{j=1}^{N} c_j\, \psi_j(x) + \int_{-\infty}^{\infty} c(k)\, \psi_+(k,x)\, dk.$$

....

Now let's apply this machinery to Bessel's equation,

$$\Big(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{m^2}{x^2} - \epsilon\Big)\psi(x) = 0.$$

Thus, we are considering the operator

$$L = \frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{m^2}{x^2}.$$

Note that L can be related to the standard form

$$L + k^2 = \frac{1}{P}\Big(\frac{d}{dx} P \frac{d}{dx} + Q + k^2 R\Big)$$

by choosing $\frac{P'}{P} = \frac{1}{x}$ so that

$$P = x, \quad Q = -\frac{m^2}{x}, \quad R = x.$$

The most common example is to take $x \in (0, \infty)$ and, as a boundary condition, insist that ψ be finite. Then

$$\psi(\sqrt{-\epsilon}\,, x) = a\, J_m(\sqrt{-\epsilon}\, x).$$

This function is polynomially bounded only if $\sqrt{-\epsilon}$ is real. Thus, L has only continuum eigenvalues; $(0, \infty)$, for $m > 0$, and $[0, \infty)$ for $m = 0$. The completeness relation gives, setting $k = \sqrt{-\epsilon}$,

$$\int_0^\infty |a|^2 J_m(kx) J_m(ky)\, y\, 2k dk = \delta(x - y).$$

To determine a let $x, y \to \infty$. Then, recalling the large x asymptotic form for J_m from Example 5.9,

$$\delta(x - y) = \frac{4}{\pi}\sqrt{\frac{y}{x}} \int_0^\infty |a|^2 \cos(kx - \frac{m\pi}{2} - \frac{\pi}{4}) \cos(ky - \frac{m\pi}{2} - \frac{\pi}{4})\, dk$$

from which it follows, writing the cosines as sums of exponentials and using the Fourier transform completeness relation, that

$$|a|^2 = \frac{1}{2}.$$

Thus one can write the completeness relation as

$$\int_0^\infty J_m(kx) J_m(ky)\, y\, k\, dk = \delta(x - y).$$

The transform obtained in this way is known as the *Hankel transform.* Given a function f the Hankel transform is usually written

$$\tilde{f}(k) = \int_0^\infty x\, J_m(kx)\, f(x)\, dx$$

so that the inverse transform is given by

$$f(x) = \int_0^\infty k\, J_m(kx)\, \tilde{f}(k)\, dk.$$

Other transforms can be obtained by choosing different boundary conditions. For example, consider $x \in (a, \infty)$ for $a > 0$ with the boundary condition

$$\psi(a) = 0.$$

Then

$$\psi(\sqrt{-\epsilon}\,, x) = A\, J_m(\sqrt{-\epsilon}\, x) + B Y_m(\sqrt{-\epsilon}\, x)$$

with

$$A\, J_m(\sqrt{-\epsilon}\, a) + B Y_m(\sqrt{-\epsilon}\, a) = 0$$

so that

$$\psi(\sqrt{-\epsilon}\,, x) = A\Big(J_m(\sqrt{-\epsilon}\, x) - \frac{J_m(\sqrt{-\epsilon}\, a)}{Y_m(\sqrt{-\epsilon}\, a)} Y_m(\sqrt{-\epsilon}\, x)\Big).$$

This function is polynomially bounded only if $\sqrt{-\epsilon}$ is real. Thus, L has only continuum eigenvalues; $(0, \infty)$, for $m > 0$, and $[0, \infty)$ for $m = 0$. The completeness relation gives, setting $k = \sqrt{-\epsilon}$,

$$2\int_0^\infty |A|^2 \Big(J_m(kx) - \frac{J_m(ka)}{Y_m(ka)} Y_m(kx)\Big)\Big(J_m(ky) - \frac{J_m(ka)}{Y_m(ka)} Y_m(ky)\Big)\, ykdk = \delta(x-y).$$

Again, to determine $|A|^2$ let $x, y \to \infty$. Recalling the large x asymptotic form for J_m and Y_m,

$$\begin{aligned}\delta(x-y) = \frac{4}{\pi}\sqrt{\frac{y}{x}} \int_0^\infty |A|^2 \Big(\cos(kx - \frac{m\pi}{2} - \frac{\pi}{4}) - \frac{J_m(ka)}{Y_m(ka)} \sin(kx - \frac{m\pi}{2} - \frac{\pi}{4})\Big) \\ \cdot \Big(\cos(ky - \frac{m\pi}{2} - \frac{\pi}{4}) - \frac{J_m(ka)}{Y_m(ka)} \sin(ky - \frac{m\pi}{2} - \frac{\pi}{4})\Big)\, dk.\end{aligned}$$

Using the identity

$$\alpha \cos\theta + \beta \sin\theta = \sqrt{\alpha^2 + \beta^2}\, \cos(\theta - \tan^{-1} \frac{\beta}{\alpha})$$

one finds

$$\delta(x-y) = \frac{4}{\pi}\sqrt{\frac{y}{x}} \int_0^\infty |A|^2 \Big(1 + (\frac{J_m(ka)}{Y_m(ka)})^2\Big) \cos(kx - \phi(k)) \cos(ky - \phi(k))\, dk$$

with

$$\phi(k) = \tan^{-1} \frac{J_m(ka)}{Y_m(ka)} - \frac{m\pi}{2} + \frac{\pi}{4}.$$

Choosing

$$|A|^2 = \frac{1}{2\Big(1 + \big(\frac{J_m(ka)}{Y_m(ka)}\big)^2\Big)}$$

and noting that

$$\begin{aligned}
&\lim_{x+y\to\infty}\int_0^\infty \Big(e^{ik(x+y)-2i\phi(k)} + e^{-ik(x+y)+2i\phi(k)}\Big)\,dk \\
&\qquad = -\lim_{x+y\to\infty}\frac{1}{i(x+y)}\int_0^\infty \Big(e^{ik(x+y)}\frac{d}{dk}e^{-2i\phi(k)} - e^{-ik(x+y)}\frac{d}{dk}e^{2i\phi(k)}\Big)\,dk \\
&\qquad = 0
\end{aligned}$$

the completeness relation becomes

$$\int_0^\infty \frac{\Big(J_m(kx) - \frac{J_m(ka)}{Y_m(ka)}Y_m(kx)\Big)\Big(J_m(ky) - \frac{J_m(ka)}{Y_m(ka)}Y_m(ky)\Big)}{1 + \big(\frac{J_m(ka)}{Y_m(ka)}\big)^2}\,ykdk = \delta(x-y).$$

If the transform associated with this completeness relation is chosen to be

$$\tilde f(k) = \int_a^\infty f(x)\,\Big(J_m(kx) - \frac{J_m(ka)}{Y_m(ka)}Y_m(kx)\Big)\,xdx$$

then the inverse transform is

$$f(x) = \int_0^\infty \tilde f(k)\,\Big(J_m(kx) - \frac{J_m(ka)}{Y_m(ka)}Y_m(kx)\Big)\frac{k}{1 + \big(\frac{J_m(ka)}{Y_m(ka)}\big)^2}\,dk.$$

Example 8.6: Consider sound propagating from the ground. Let the normal component of the acoustic velocity field be known at z=0,

$$v_z(r,\theta,0,t) = f(r,\theta,t).$$

Let

$$\Big(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)P = 0.$$

Writing

$$P(r,\theta,z,t) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^\infty \tilde P_m(k,z,\omega)\,e^{im\theta - i\omega t}\,J_{|m|}(kr)\,kdk\,d\omega$$

one has

$$\Big(\frac{d^2}{dz^2} - k^2 + \frac{\omega^2}{c^2}\Big)\tilde P_m(k,z,\omega) = 0$$

so that

$$\tilde{P}_m(k,z,\omega) = A_m(k,\omega)\, e^{i\sqrt{\frac{\omega^2}{c^2}-k^2}\, z}.$$

To determine $A_m(k,\omega)$ use the boundary condition at $z=0$,

$$-\rho_0 \frac{\partial f(r,\theta,t)}{\partial t} = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\infty} A_m(k,\omega)\, i\sqrt{\frac{\omega^2}{c^2} - k^2}\; e^{im\theta - i\omega t}\, J_{|m|}(kr)\, kdk\, d\omega$$

so that

$$A_m(k,\omega) = \frac{\rho_0 \omega}{\sqrt{\frac{\omega^2}{c^2} - k^2}} \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{2\pi} f(r,\theta,t)\, e^{-im\theta + i\omega t}\, J_{|m|}(kr)\, rd\theta\, dr\, dt.$$

As a concrete example let

$$f(r,\theta,t) = \frac{1}{r}\delta(r)\delta(t).$$

Then

$$A_m(k,\omega) = 2\pi\rho_0 c\delta_{m,0} \frac{\omega}{\sqrt{\omega^2 - k^2 c^2}}$$

so that

$$P(r,\theta,z,t) = 2\pi\rho_0 c \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\omega}{\sqrt{\omega^2 - k^2c^2}}\; e^{i\sqrt{\frac{\omega^2}{c^2}-k^2}\, z - i\omega t}\, J_0(kr)\, kdk\, d\omega.$$

....

Integral Transforms

An integral transform of a function $f(x)$ relative to a family of functions $h(k,x)$ is the linear transformation

$$(\mathcal{T}f)(k) = \int h(k,x) f(x)\, dx.$$

In most cases there is also an inverse transform from which f can be recovered.

$$f(x) = \int \tilde{h}(k,x)(\mathcal{T}f)(k)\, dk.$$

The class of functions for which the transform exists depends on h.

The most common integral transform is the Fourier transform:

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}\, f(x)\, dx. \tag{9.1}$$

Given the Fourier transform, $\hat{f}$, the original function, f, can be reconstructed using the Fourier inversion formula,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\, \hat{f}(k)\, dk. \tag{9.2}$$

Note that substituting (9.1) into (9.2) gives the completeness relation

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')}\, dk = \delta(x-x')$$

and substituting (9.2) into (9.1) gives the orthogonality relation

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k-k')x}\, dx = \delta(k-k').$$

Note that the norm

$$\|f\| = \sqrt{\int_{-\infty}^{\infty} |f(x)|^2\, dx} \tag{9.3}$$

is preserved by the Fourier transform in the sense that

$$\begin{aligned}
\|f\|^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} e^{ikx} \overline{\hat{f}(k)}\, dk \Big) \Big(\int_{-\infty}^{\infty} e^{-iqx} \hat{f}(q)\, dq \Big)\, dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{\hat{f}(k)} \hat{f}(q) \delta(k-q)\, dq\, dk \\
&= \int_{-\infty}^{\infty} |\hat{f}(k)|^2\, dk
\end{aligned}$$

so that we end up with the relation

$$\|f\| = \|\hat{f}\|.$$

Another identity arising for Fourier integrals is the relation between convolution and multiplication. The convolution of f with g is given by

$$(f * g)(x) = \int_{-\infty}^{\infty} g(x-y) f(y)\, dy.$$

Substituting the Fourier decomposition of g and f into the convolution gives

$$\begin{aligned}
(f * g)(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} e^{-ik(x-y)} \hat{g}(k)\, dk \Big) \Big(\int_{-\infty}^{\infty} e^{-iqx} \hat{f}(q)\, dq \Big)\, dy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-ikx} \hat{g}(k) \hat{f}(q)\, \delta(k-q)\, dq\, dk \\
&= \int_{-\infty}^{\infty} e^{-ikx} \hat{g}(k) \hat{f}(k)\, dk
\end{aligned}$$

so that

$$(\widehat{f * g})(k) = \sqrt{2\pi}\; \hat{g}(k) \hat{f}(k).$$

Example 9.1: Some examples of Fourier transforms: First, a Gaussian,

$$f(x) = e^{-\lambda x^2}$$

gives

$$\begin{aligned}
\hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx - \lambda x^2}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\lambda (x - \frac{ik}{2\lambda})^2 - \frac{k^2}{4\lambda}}\, dx \\
&= \frac{1}{\sqrt{2\lambda}}\; e^{-\frac{k^2}{4\lambda}}.
\end{aligned}$$

Note that the Fourier transform of a Gaussian is a Gaussian. Further, the broader f is the more sharply peaked $\hat{f}$ is, and conversely.

Now consider a Lorentzian,

$$f(x) = \frac{1}{x^2 + \lambda^2}.$$

To compute the Fourier transform it's best to use the calculus of residues. One finds

$$\begin{aligned}\hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \, \frac{1}{x^2+\lambda^2} \; dx \\ &= \sqrt{\frac{\pi}{2}} \, \frac{1}{\lambda} \, e^{-\lambda|k|}.\end{aligned}$$

Again, the more sharply peaked the Lorentzian, the broader it's Fourier transform. Note that the Fourier inversion formula shows that the Fourier transform of $e^{-\lambda|x|}$ is

$$\big(\widehat{e^{-\lambda|x|}}\big)(k) = \sqrt{\frac{2}{\pi}} \, \frac{\lambda}{k^2+\lambda^2}.$$

Finally, consider a square pulse,

$$f(x) = \begin{cases} 1 & \text{if } a < x < b \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}\hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_a^b e^{ikx} \; dx \\ &= \frac{1}{ik\sqrt{2\pi}} \Big(e^{ikb} - e^{ika}\Big).\end{aligned}$$

....

Not all functions can be Fourier transformed. For example, $f(x) = e^x$ has no Fourier transform. In general any integrable function has a bounded Fourier transform, and any function for which the norm (9.3) is finite has a Fourier transform with equal norm. The general question of what functions can be Fourier transformed and what the Fourier transform means (recall that the Fourier transform of 1 is $\delta(k)$ so that it doesn't exist even though we know how to make sense of it) is involved. Further, given a function which can be Fourier transformed there is an extensive theory which indicates how regular a function the Fourier transform is. Let's see what can be said of the examples above.

Note that for the square pulse $\hat{f}$ goes to 0 as $k \to \infty$ much more slowly than in the previous two examples. This is typical of Fourier transforms of discontinuous functions. In fact there is a relation between differentiability of f and the rate at which $\hat{f}(k) \to 0$ for large k. Explicitly, if f is n times differentiable on $\mathbf{R}$ but not $n+1$ times differentiable, with $f^{(j)}$ bounded and

$$\lim_{x\to\infty} f^{(j)}(x) = 0$$

for $j \in \{0, 1, 2, \ldots, n\}$ then one can integrate by parts n times giving

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}\ (ik)^n} \int_{-\infty}^{\infty} e^{ikx}\, f^{(n)}(x)\, dx.$$

Assume further that the non-differentiability of $f^{(n)}$ arises from a jump discontinuity at $x = a$. Then

$$\begin{aligned}\hat{f}(k) =& \frac{1}{\sqrt{2\pi}\ (ik)^n} \Big(\int_{-\infty}^{a} e^{ikx}\, f^{(n)}(x)\, dx + \int_{a}^{\infty} e^{ikx}\, f^{(n)}(x)\, dx \Big) \\ =& \frac{1}{\sqrt{2\pi}\ (ik)^{n+1}} \Big(e^{ika} \big(f^{(n)}(a - 0^+) - f^{(n)}(a + 0^+) \big) \\ & \qquad + \int_{-\infty}^{a} e^{ikx}\, f^{(n+1)}(x)\, dx + \int_{a}^{\infty} e^{ikx}\, f^{(n+1)}(x)\, dx \Big).\end{aligned}$$

Note that additional integrations by parts won't increase the power of $\frac{1}{k}$ so that the rate at which $\hat{f}(k) \to 0$ as $k \to \infty$ is $\frac{1}{k^n}$. If there are additional isolated singular points the integration region can be broken up further, and the singularities dealt with individually.

The small k behavior of $\hat{f}(k)$ is determined by the large x behavior of $f(x)$ in the sense that

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, dx.$$

Further,

$$\hat{f}'(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x\, f(x)\, dx$$

so that the differentiability of $\hat{f}(k)$ at $k = 0$ is determined by the rate at which $f(x) \to 0$ as $x \to \infty$. It is a fact that $\hat{f}(k)$ is analytic if $f(x) \to 0$ exponentially fast as $x \to \infty$. The converse is true if f is analytic and integrable in a strip: the rate of exponential decrease of $\hat{f}(k)$ is the distance from the real line to the closest singularity of f. This is precisely what happened for the Lorentzian.

Example 9.2: Consider a problem in one dimensional acoustics: a sound source at $x = 0$ given by a velocity condition

$$v(0, t) = f(t).$$

Then the acoustic pressure satisfies the wave equation

$$\Big(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \Big) P(x, t) = 0$$

with the boundary condition

$$\frac{\partial P}{\partial x}\Big|_{x=0} = -\rho_0 f'(t).$$

Since this is one dimensional we can solve the problem:

$$P(x,t) = \rho_0 c f(t - \frac{x}{c}).$$

However, this approach cannot be generalized to higher dimensions, so we will see how to use Fourier transforms to solve this problem.

Write

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} \hat{f}(\omega)\, d\omega$$

where

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} f(t)\, dt$$

is the Fourier transform of f. Similarly write

$$P(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} P_A(x,\omega)\, d\omega.$$

Here $P_A(x,\omega)$ is the Fourier transform of $P(x,t)$ with respect to t. Then

$$\begin{aligned} 0 &= \Big(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} P_A(x,\omega)\, d\omega \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} \Big(\frac{\partial^2}{\partial x^2} + \frac{\omega^2}{c^2}\Big) P_A(x,\omega)\, d\omega \end{aligned}$$

with the boundary condition

$$\frac{\partial}{\partial x}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} P_A(x,\omega)\, d\omega\Big|_{x=0} = -\rho_0 \frac{\partial}{\partial t}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} \hat{f}(\omega)\, d\omega.$$

It follows that

$$\Big(\frac{\partial^2}{\partial x^2} + \frac{\omega^2}{c^2}\Big) P_A(x,\omega) = 0$$

with the boundary condition

$$\frac{\partial P_A(x,\omega)}{\partial x}\Big|_{x=0} = i\rho_0 \omega \hat{f}(\omega). \tag{9.4}$$

Solving one finds

$$P_A(x,\omega) = \rho_0 c \hat{f}(\omega)\ e^{i\frac{\omega}{c}x}$$

so that

$$P(x,t) = \frac{\rho_0 c}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\; e^{-i\omega(t-\frac{x}{c})}\, d\omega$$
$$= \rho_0 c f(t - \frac{x}{c}).$$

Now, rather than radiating into free space, imagine the source is mounted at one end of a closed resonator of length L. Then in addition to (9.4) one has

$$\frac{\partial P_A(x,\omega)}{\partial x}\Big|_{x=L} = 0$$

so that

$$P_A(x,\omega) = \frac{i\rho_0 c\hat{f}(\omega)}{\sin(\frac{\omega}{c}L)}\;\cos\big(\frac{\omega}{c}(L-x)\big).$$

....

Example 9.3: Now consider a radiator in 3 dimensions. Let S be the surface of the radiator. For a point $\mathbf{s} \in S$ let the normal component of the velocity of the radiator be $u(\mathbf{s},t)$. Then the acoustic pressure satisfies

$$\Big(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)P(\mathbf{x},t) = 0$$

with the boundary condition

$$-\mathbf{n}\cdot\nabla P\big|_{\mathbf{x}\in S} = \rho_0 \frac{\partial u}{\partial t}.$$

Fourier transforming with respect to time,

$$u(\mathbf{s},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{u}(\mathbf{s},\omega)e^{-i\omega t}\, d\omega$$

and

$$P(\mathbf{x},t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} P_A(\mathbf{x},\omega)e^{-i\omega t}\, d\omega,$$

one has

$$\Big(\nabla^2 + \frac{\omega^2}{c^2}\Big)P_A(\mathbf{x},\omega) = 0$$

with the boundary condition

$$\mathbf{n}\cdot\nabla P_A\big|_{\mathbf{x}\in S} = i\rho_0\omega\hat{u}.$$

If the radiator is in free space then there is an additional boundary condition at spatial ∞: the pressure wave must be outgoing so that it satisfies the *Somerfeld radiation condition*

$$P_A(\mathbf{x},\omega) \sim \text{const}\ \frac{e^{i\frac{\omega}{c}\|\mathbf{x}\|}}{\|\mathbf{x}\|}$$

for large $\|\mathbf{x}\|$. This problem was studied in Example 7.8. Change to spherical coordinates and recall that for $\frac{\omega}{c}r \gg 1$ one has (if the integral is not 0)

$$P_A(\mathbf{x},\omega) \approx \frac{-i\rho_0\omega}{4\pi}\Big(\int_S \hat{u}(\mathbf{s},\omega)\ d\sigma(\mathbf{s})\Big)\ \frac{e^{i\frac{\omega}{c}r}}{r}.$$

It follows that

$$P(\mathbf{x},t) \approx \frac{-i\rho_0}{2^{\frac{5}{2}}\pi^{\frac{3}{2}}}\int_{-\infty}^{\infty}\omega\int_S \hat{u}(\mathbf{s},\omega)\ d\sigma(\mathbf{s})\frac{e^{i\frac{\omega}{c}(r-ct)}}{r}\ d\omega$$

so that

$$\begin{aligned} P(\mathbf{x},t) &\approx \frac{\rho_0}{2^{\frac{5}{2}}\pi^{\frac{3}{2}}}\frac{1}{r}\frac{\partial}{\partial t}\int_S\int_{-\infty}^{\infty}\hat{u}(\mathbf{s},\omega)e^{i\frac{\omega}{c}(r-ct)}\ d\omega\ d\sigma(\mathbf{s}) \\ &= \frac{\rho_0}{4\pi}\frac{\frac{\partial}{\partial t}\int_S u(\mathbf{s},t-\frac{r}{c})\ d\sigma(\mathbf{s})}{r}. \end{aligned}$$

As a concrete example consider the case in which the source begins vibrating suddenly at $t=0$. Assume the vibration is at frequency ω_0, but with exponentially decreasing amplitude so that

$$\int_S u(\mathbf{s},t)\ d\sigma(\mathbf{s}) = \begin{cases} 0 & \text{if } t<0 \\ A\ e^{-\gamma t}\cos(\omega_0 t) & \text{if } t\ge 0 \end{cases} \tag{9.5}$$

for some $\omega_0 \in \mathbf{R}$ and $A,\gamma \in (0,\infty)$. It follows that

$$P(\mathbf{x},t) \approx A\frac{\rho_0}{4\pi}\left[\delta(t) - \text{Re}(i\omega_0+\gamma)\begin{cases} 0 & \text{if } t<\frac{r}{c} \\ \frac{1}{r}e^{-(i\omega_0+\gamma)(t-\frac{r}{c})} & \text{if } t\ge\frac{r}{c} \end{cases}\right]$$

if r is large enough. The $\delta(t)$ is the spike in P which arises from turning the velocity on suddenly at $t=0$.

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Example 9.4: The previous example solved a boundary value problem by Fourier transforming in time. Now consider an initial value problem. Imagine that the acoustic pressure and it's rate of change are known at $t=0$,

$$P(\mathbf{x},0) = f(\mathbf{x})$$

and

$$\frac{\partial P}{\partial t}\Big|_{t=0} = g(\mathbf{x}).$$

Then

$$\Big(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)P(\mathbf{x},t) = 0$$

with the initial conditions given above. Fourier transforming with respect to $\mathbf{x}$ one has

$$P(\mathbf{x},t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbf{R}^3} p(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}}\,d^3k$$

with

$$\Big(\mathbf{k}\cdot\mathbf{k} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)p(\mathbf{k},t) = 0.$$

Setting

$$k = \sqrt{\mathbf{k}\cdot\mathbf{k}}$$

there are coefficients $a(\mathbf{k})$ and $b(\mathbf{k})$ with

$$P(\mathbf{x},t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbf{R}^3}\Big(a(\mathbf{k})\cos(ckt) + b(\mathbf{k})\sin(ckt)\Big)e^{i\mathbf{k}\cdot\mathbf{x}}\,d^3k.$$

Applying the initial conditions gives

$$a(\mathbf{k}) = \hat{f}(\mathbf{k})$$

and

$$b(\mathbf{k}) = \frac{1}{ck}\hat{g}(\mathbf{k}).$$

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Example 9.5: A linear integral operator acting on a function f is a transformation of the form

$$(\mathcal{F}f)(t) = \int_{-\infty}^{\infty} G(t,s)f(s)\,ds.$$

Sometimes G is referred to as the integral kernel of this linear operator. Now consider a linear, time translation invariant operator. Translation invariance means that

$$\int_{-\infty}^{\infty} G(t,s)f(s)\,ds = \int_{-\infty}^{\infty} G(t+\tau,s+\tau)f(s)\,ds$$

for any τ. Choosing $\tau = -s$ one sees that the only way for this transformation to be time translation invariant is if

$$G(t, s) = g(t - s)$$

for $g(\tau) = G(\tau, 0)$. But then

$$(\mathcal{F}f)(t) = \int_{-\infty}^{\infty} g(t - s) f(s)\, ds$$

is just a convolution so that Fourier transforming yields

$$\widehat{(\mathcal{F}f)}(\omega) = \sqrt{2\pi}\ \hat{g}(\omega)\hat{f}(\omega).$$

Linear time translation invariant filters abound in practice. The impedance in a one-dimensional acoustics problem is an example. If a linear relation exists between $P(x, t)$ and $v(x, t)$ at some point x_0 then, assuming that the mean state of the fluid in question is constant in time, the relation between P and v must be time translation invariant so that on Fourier transforming one finds that there is a function $Z(\omega)$ with

$$\hat{P}(x_0, \omega) = Z(\omega)\hat{v}(x_0, \omega).$$

Similarly, if some electro-acoustic transducer is sufficiently linear then there is a time translation invariant (assuming the characteristics of the transducer don't change with time, which is in practice never true, but reasonable over short enough spans of time) linear relation between the pressure on the face of the transducer, $P_f(t)$ and the voltage, $V(t)$ (or current, $I(t)$,) produced by the transducer. Fourier transforming yields the relation

$$\hat{P}_f(\omega) = M(\omega)\hat{V}(\omega).$$

Perhaps the most well known examples are provided by A.C. circuit theory. Given a voltage source $V(t)$, the current $I(t)$ at some point in the circuit is related to V through a frequency dependent impedance $Z(\omega)$ through Ohm's law

$$\hat{V}(\omega) = Z(\omega)\hat{I}(\omega).$$

Circuit diagrams provide an algorithm for producing Z from the impedances of individual circuit elements.

As a concrete example consider a resonator of length L with

$$\Big(\frac{d^2}{dx^2}+(\frac{\omega}{c}+i\alpha)^2\Big)P_A(x)=0,$$

$$P_A'(L)=0$$

and

$$P_A'(0)=i\omega\rho_0\hat{u}_0(\omega).$$

Here $\hat{u}_0(\omega)$ is the Fourier transform of the piston motion and α is small. Then

$$P_A(x)=\frac{i\omega\rho_0\hat{u}_0(\omega)\cos\big((\frac{\omega}{c}+i\alpha)(L-x)\big)}{(\frac{\omega}{c}+i\alpha)\sin\big((\frac{\omega}{c}+i\alpha)L\big)}.$$

It follows that the impudence at $x=0$ is

$$\frac{P_A(0)}{\hat{u}_0(\omega)}=\frac{i\omega\rho_0}{(\frac{\omega}{c}+i\alpha)}\cot\big((\frac{\omega}{c}+i\alpha)L\big).$$

In particular the acoustic pressure at $x=0$ is

$$P(0,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{i\omega\rho_0}{(\frac{\omega}{c}+i\alpha)}\cot\big((\frac{\omega}{c}+i\alpha)L\big)\hat{u}_0(\omega)\,e^{-i\omega t}\,d\omega.$$

Some words about doing the integral above. If $\hat{u}_0(\omega)$ is analytic in the lower half-plain then the integrand has a second order pole at $\omega=-ic\alpha$ and simple poles at $\omega=j\frac{c\pi}{L}-ic\alpha$ for $j\in\{\pm1,\pm2,\ldots\}$. Further, if $\hat{u}_0(\omega)$ is polynomially bounded then (since $t>0$) one can close the ω integration contour in the lower half-plane. One obtains

$$\begin{aligned}P(0,t)&=\sqrt{2\pi}\frac{\rho_0c^2}{L}\sum_{j\neq0}\frac{j-i\frac{\alpha L}{\pi}}{j}\hat{u}_0(j\frac{c\pi}{L}-ic\alpha)e^{-(ij\frac{c\pi}{L}+c\alpha)t}\\&\quad+\sqrt{2\pi}\frac{\rho_0c^2}{L}\Big((1-\alpha ct)\hat{u}_0(-ic\alpha)-i\alpha c\hat{u}_0'(-ic\alpha)\Big)e^{-c\alpha t}.\end{aligned}$$

Note that

$$\hat{u}(0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(t)\,dt.$$

In practice $\hat{u}(\omega)$ is 0 near $\omega=0$ because so that the piston, on the average, stays at $x=0$. Thus the $j=0$ term is usually negligible. Further, $\hat{u}(\omega)$ is often not analytic everywhere, but just in a neighborhood of some of the poles $j\frac{c\pi}{L}-ic\alpha$. There are two extreme cases to consider.

In the first, the piston vibrates at essentially one frequency, say ω_0, so that $\hat{u}(\omega)$ is essentially a delta function,

$$\hat{u}(\omega) = A\delta(\omega - \omega_0)$$

and the integral above is straightforward.

In the other case $\hat{u}(\omega)$ is non-zero and slowly varying relative to the cotangent. In this case, the integral is dominated by the regions close to the poles where $\hat{u}(\omega)$ is non-zero. Then the cotangent can be approximated by it's leading term. Assuming that $\hat{u}(\omega)$ is non-zero only in a neighborhood of $\omega_j = j\frac{c\pi}{L} - ic\alpha$ one can use the asymptotic formula

$$\begin{aligned} P(0,t) &\approx \frac{1}{\sqrt{2\pi}} \frac{i\omega_j \rho_0}{(\frac{\omega_j}{c} + i\alpha)} \frac{L}{c} \hat{u}_0(\omega_j)\, e^{-i\omega_j t} \int_{-\infty}^{\infty} \frac{1}{\omega - \omega_j + i\alpha c}\, d\omega \\ &= \frac{\rho_0 c^2}{L} \sqrt{2\pi} \frac{j - i\frac{\alpha L}{\pi}}{j} \hat{u}_0(j\frac{c\pi}{L} - ic\alpha) e^{-(ij\frac{c\pi}{L} + c\alpha)t} \end{aligned}$$

....

The last transform we'll look at is the *Laplace transform.* The Laplace transform is not directly related to a self adjoint operator. Instead it is a variation of the Fourier transform. The Laplace transform of a function f defined on $(0, \infty)$ is

$$\tilde{f}(s) = \int_0^{\infty} f(x)\, e^{-sx}\, dx.$$

The transform can actually be inverted. The inversion formula is

$$f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \tilde{f}(s) e^{sx}\, ds.$$

Any γ which is to the right of all the poles of $\tilde{f}(s)$ will work. To see that this inversion formula works note that

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \tilde{f}(s) e^{sx}\, ds &= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \int_0^{\infty} f(y)\, e^{s(x-y)}\, dy\, ds \\ &= \frac{1}{2\pi} e^{\gamma x} \int_{-\infty}^{\infty} \int_0^{\infty} e^{-\gamma y} f(y)\, e^{i\omega(x-y)}\, dy\, d\omega \\ &= \begin{cases} 0 & \text{if } x < 0 \\ f(x) & \text{if } x > 0. \end{cases} \end{aligned}$$

Note that another form of the restriction on γ is that the function

$$e^{-\gamma x} f(x) \cdot \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

have a Fourier transform.

Green's Functions

Let L be a partial differential operator in variables $x_1,\ x_2,\ \ldots,\ x_n$. In general, a Green's function for L is a solution of the equation

$$LG(x_1, y_1, \ldots, x_n, y_n) = \delta(x_1 - y_1)\delta(x_2 - y_2)\cdots\delta(x_n - y_n).$$

Note that a Green's function is not unique. Given a Green's function $G(x_1, y_1, \ldots, x_n, y_n)$ the function $G(x_1, y_1, \ldots, x_n, y_n) + f(x_1, x_2, \ldots, x_n)$ is also a Green's function whenever

$$Lf = 0.$$

Example 10.1: Let

$$L = \frac{d^2}{dx^2}.$$

If

$$\frac{d^2}{dx^2}G(x, y) = \delta(x, y)$$

then

$$G(x, y) = \frac{1}{2}|x - y| + ax + b$$

for any constants a and b.

If

$$L = \nabla^2$$

then in $\mathbf{R}^2$

$$G(\mathbf{x}, \mathbf{y}) = \frac{-1}{2\pi}\ln(\|\mathbf{x} - \mathbf{y}\|)$$

is a Green's functions and in $\mathbf{R}^3$

$$G(\mathbf{x}, \mathbf{y}) = \frac{-1}{4\pi\|\mathbf{x} - \mathbf{y}\|}$$

is a Green's function.

Now let

$$L = \nabla^2 + k^2$$

in $\mathbf{R}^3$. Then

$$(\nabla^2 + k^2)G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y})$$

implies that

$$G(\mathbf{x},\mathbf{y}) = \frac{-1}{4\pi\|\mathbf{x}-\mathbf{y}\|}e^{ik\|\mathbf{x}-\mathbf{y}\|}$$

is a Green's function for L.

These last three examples are all called *free space Green's functions.* They can all be obtained using Fourier transforms. The functions

$$G(\mathbf{x},\mathbf{y}) = \frac{-1}{(2\pi)^n}\int_{\mathbf{R}^n}\frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}}{\|q\|^2 \pm i0^+}\,d^nq$$

are free space Green's functions for ∇^2 on $\mathbf{R}^n$ and

$$G(\mathbf{x},\mathbf{y}) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}}{k^2-\|q\|^2 \pm i0^+}\,d^nq$$

are free space Green's functions for $\nabla^2 + k^2$ on $\mathbf{R}^n$.

....

Now let G_1 and G_2 be different Green's functions for the same operator L. Then

$$L(G_1 - G_2) = \delta(\mathbf{x}-\mathbf{y})$$

so that G_1 and G_2 differ by a solution to $Lf = 0$. This allows one to choose a Green's function satisfying some boundary conditions. Given any Green's function G one must choose an f with $Lf = 0$ so that $G + f$ satisfies the desired boundary conditions.

Example 10.2: Let's construct a Green's function for

$$L = \nabla^2 + k^2$$

in $\{(x,y,z)\in\mathbf{R}^3 \big| x>0\}$ which satisfies Dirichlet boundary conditions at $x = 0$. Let

$$G(\mathbf{x},\mathbf{x}') = \frac{-1}{4\pi\|\mathbf{x}-\mathbf{x}'\|}e^{ik\|\mathbf{x}-\mathbf{x}'\|}$$

be the free space Green's function for L. Given

$$\mathbf{x}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

define

$$\tilde{\mathbf{x}}' = \begin{pmatrix} -x' \\ y' \\ z' \end{pmatrix}.$$

Note that if $\mathbf{x}'$ is in the domain then $\tilde{\mathbf{x}}'$ is not. Thus

$$LG(\mathbf{x}, \tilde{\mathbf{x}}') = 0$$

so that

$$G_D(\mathbf{x}, \mathbf{x}') = G(\mathbf{x}, \mathbf{x}') - G(\mathbf{x}, \tilde{\mathbf{x}}')$$

is a Green's function. A moments reflection shows that when $x = 0$ one has $G_D = 0$ so that G_D is the desired Green's function. The procedure used is known as the *method of images.* The Green's function for the Helmholtz equation which satisfies Neumann boundary conditions at $x = 0$ can be constructed similarly:

$$G_N(\mathbf{x}, \mathbf{x}') = G(\mathbf{x}, \mathbf{x}') + G(\mathbf{x}, \tilde{\mathbf{x}}').$$

This Dirichlet and Nemann Green's functions can also be constructed using eigenfunction expansions, a sine transform in the Dirichlet case:

$$G_D(\mathbf{x}, \mathbf{x}') = \frac{2}{\pi^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\sin(q_x x) \sin(q_x x')\, e^{iq_y(y-y')+iq_z(z-z')}}{k^2 - \|\mathbf{q}\|^2 + i0^+}\, dq_x\, dq_y\, dq_z,$$

and a cosine transform in the Neumann case:

$$G_N(\mathbf{x}, \mathbf{x}') = \frac{2}{\pi^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\cos(q_x x) \cos(q_x x')\, e^{iq_y(y-y')+iq_z(z-z')}}{k^2 - \|\mathbf{q}\|^2 + i0^+}\, dq_x\, dq_y\, dq_z.$$

....

There is a general technique for constructing Green's functions for ordinary differential operators of the form

$$L = \frac{d}{dx} P \frac{d}{dx} + Q, \tag{10.1}$$

for $x \in (a, b)$, satisfying any specified boundary conditions at a and b. To construct such a Green's function let ψ_1 and ψ_2 be any solutions of

$$\Big(\frac{d}{dx} P \frac{d}{dx} + Q\Big)\psi = 0$$

with ψ_1 satisfying the given boundary condition at a and ψ_2 satisfying the given boundary condition at b. Then

$$G(x,y) = c\,\psi_1(x_<)\psi_2(x_>)$$

with $x_< = \min(x,y)$ and $x_> = \max(x,y)$. The constant c is determined so that

$$\begin{aligned}
1 &= \lim_{\epsilon\downarrow 0}\int_{y-\epsilon}^{y+\epsilon} \delta(x-y) \\
&= \lim_{\epsilon\downarrow 0}\int_{y-\epsilon}^{y+\epsilon} \Big(\frac{d}{dx}P\frac{d}{dx}+Q\Big)G(x,y) \\
&= P(y)\lim_{\epsilon\downarrow 0}\frac{d}{dx}G(x,y)\big|_{y-\epsilon}^{y+\epsilon} \\
&= c\,P(y)\Big(\psi_1(y)\psi_2'(y)-\psi_1'(y)\psi_2(y)\Big).
\end{aligned}$$

Recalling the Wronskian,

$$c = \frac{-1}{P(y)\mathcal{W}(\psi_1,\psi_2;y)}$$

so that

$$G(x,y) = -\frac{\psi_1(x_<)\psi_2(x_>)}{P(y)\mathcal{W}(\psi_1,\psi_2;y)}. \tag{10.2}$$

Abel's formula shows that c is indeed a constant.

Often one is given an ordinary differential operator in standard form rather than the form (10.1),

$$\begin{aligned}
\tilde{L} &= \frac{d^2}{dx^2}+\frac{P'(x)}{P(x)}\frac{d}{dx}+\frac{Q(x)}{P(x)} \\
&= \frac{1}{P(x)}L.
\end{aligned}$$

Note that one has

$$\tilde{L}G(x,y) = \frac{1}{P(x)}\delta(x-y)$$

where $G(x,y)$ is given in (10.2) so that

$$\tilde{L}\tilde{G}(x,y) = \delta(x-y)$$

where

$$\tilde{G}(x,y) = P(y)G(x,y).$$

Example 10.3: The simplest example is

$$L = \frac{d^2}{dx^2}$$

on the interval $[a, b]$ with boundary conditions

$$\alpha_a f(a) + \beta_a f'(a) = 0$$

$$\alpha_b f(b) + \beta_b f'(b) = 0.$$

For this example the general solution of $l\psi = 0$ is

$$\psi(x) = A + Bx.$$

It follows that one may choose

$$\psi_1(x) = -\frac{\alpha_a a + \beta_a}{\alpha_a} + x$$

$$\psi_2(x) = -\frac{\alpha_b b + \beta_b}{\alpha_b} + x.$$

Thus

$$G(x, y) = \frac{1}{\frac{\alpha_a a+\beta_a}{\alpha_a} - \frac{\alpha_b b+\beta_b}{\alpha_b}}\left(-\frac{\alpha_a a + \beta_a}{\alpha_a} + x_<\right)\left(-\frac{\alpha_b b + \beta_b}{\alpha_b} + x_>\right).$$

....

Example 10.4: Now consider the 1-d Helmholtz equation on the line

$$L = \frac{d^2}{dx^2} + k^2.$$

The homogeneous solutions are all of the form

$$\psi(x) = Ae^{ikx} + Be^{-ikx}.$$

Note that if

$$\psi_1(x) = A_1 e^{ikx} + B_1 e^{-ikx}$$

$$\psi_2(x) = A_2 e^{ikx} + B_2 e^{-ikx}$$

then

$$\mathcal{W}(\psi_1, \psi_2) = 2ik(A_1 B_2 - B_1 A_2).$$

In this general case one has

$$G(x, y) = -\frac{1}{2ik(A_1 B_2 - B_1 A_2)} \left(A_1 e^{ikx_<} + B_1 e^{-ikx_<}\right)\left(A_2 e^{ikx_>} + B_2 e^{-ikx_>}\right)$$

The various Greens functions are determined by the homogeneous solutions they are asymptotic to as $x \to \pm\infty$. The "outgoing" Greens function is given by choosing $A_1 = 0 = B_2$. One finds in this case that

$$G(x, y) = \frac{1}{2ik} e^{ik|x-y|}.$$

If k is pure imaginary, $k = i\kappa$, then the bounded Greens function is given by

$$G(x, y) = -\frac{1}{2\kappa} e^{-\kappa|x-y|}.$$

....

Example 10.5: Consider the operators given by the differential equations of Euler type

$$\begin{aligned} L &= \frac{d^2}{dx^2} + \alpha \frac{1}{x}\frac{d}{dx} + \beta \frac{1}{x^2} \\ &= \frac{1}{x^\alpha}\frac{d}{dx} x^\alpha \frac{d}{dx} + \beta \frac{1}{x^2}. \end{aligned}$$

Solutions of the related ordinary differential equation may be obtained using the *Ansatz* $\psi(x) = x^r$. The exponent r satisfies the indicial equation

$$r(r-1) + \alpha r + \beta = 0$$

so that one obtains two solutions from the two roots

$$r_\pm = \frac{1-\alpha}{2} \pm \sqrt{\left(\frac{\alpha-1}{2}\right)^2 - \beta}.$$

Assuming that $r_+ \neq r_-$ one has

$$\mathcal{W}(x^{r_+}, x^{r_-}) = \frac{\sqrt{(\alpha-1)^2 - 4\beta}}{x^\alpha}.$$

It follows that the possible Greens functions of L are of the form

$$G(x,y) = \frac{x^{\alpha}(Ax_<^{r_+} + x_<^{r_-})(x_>^{r_+} + Bx_>^{r_-})}{(1-AB)\sqrt{(\alpha-1)^2 - 4\beta}}$$

....

Another method for constructing Greens functions is by eigenfunction expansion, if there is one available. Assume one has an eigenfunction expansion for L given by discrete spectrum ϵ_j, $\psi_j(x)$ and continuous spectrum Γ_n, $\psi_n(\epsilon, x)$. Let $\eta \in \mathbf{C}$ not be in the spectrum of L. Then the Greens function for $L - \eta$ which satisfies the boundary conditions used in defining L may be given by

$$G_\eta(x,y) = \sum_j \frac{\psi_j(x)\overline{\psi_j(y)}}{\epsilon_j - \eta} + \sum_n \int_{\Gamma_n} \frac{\psi_n(\epsilon,x)\overline{\psi_n(\epsilon,y)}}{\epsilon - \eta}\, d\epsilon. \tag{10.3}$$

Examples are provided by the Fourier, sine and cosine transform representations of the Greens functions for the Helmholtz equation given in Example 10.1 and Example 10.2. In these cases the desired value for η, say $-k^2$ is in the continuous spectrum, which is a branch cut for $G_\eta(x,y)$. Greens functions are obtained by allowing $\eta = -k^2 \pm i0^+$ to approach $-k^2$ from above or below in the complex plane. Note that different Greens functions are obtained depending on whether $-k^2$ is approached from above or below.

Note that, as functions of η, these Greens functions are analytic in η as long as η is not in the spectrum of L. The discrete eigenvalues of L are simple poles of $G_\eta(x,y)$; the residues at the eigenvalues are the projection operators given as integral operators by

$$\psi_j(x)\overline{\psi_j(y)}.$$

The components of the continuous spectrum can be shown to be branch cuts of $G_\eta(x,y)$. Using the formula

$$\operatorname{Im} \frac{1}{x + i0^+} = \pi\delta(x)$$

one finds that

$$G_{\eta+i0^+}(x,y) - G_{\eta-i0^+}(x,y) = -2\pi i \sum_{m \text{ with } \eta\in\Gamma_m} \psi_m(\eta,x)\overline{\psi_m(\eta,y)}. \tag{10.4}$$

Note that for second order ordinary differential operators (10.4), in conjunction with (10.2), provides a general method for finding the normalization of the continuous spectrum eigenfunctions. If L is given by (10.1) on $(-\infty,\infty)$ and the Greens function $G_z(x,y)$ for $L-z$ by (10.2) then one has

$$\begin{aligned}
G_{\eta+i0^+}(x,y)-&G_{\eta-i0^+}(x,y)\\
&=\int_a^b\Big[G_{\eta+i0^+}(x,w)\big(\frac{d}{dw}P(w)\frac{d}{dw}+Q(w)-\eta\big)G_{\eta-i0^+}(w,y)\\
&\quad-G_{\eta-i0^+}(w,y)\big(\frac{d}{dw}P(w)\frac{d}{dw}+Q(w)-\eta\big)G_{\eta+i0^+}(x,w)\Big]\,dw\\
&=\Big[G_{\eta+i0^+}(x,w)\frac{d}{dw}P(w)G_{\eta-i0^+}(w,y)\\
&\quad-G_{\eta-i0^+}(w,y)\frac{d}{dw}P(w)G_{\eta+i0^+}(x,w)\Big]\Big|_{w=-\infty}^{\infty}.
\end{aligned}\tag{10.5}$$

Example 10.6: Consider

$$L=\frac{d^2}{dx^2}$$

on $(-\infty,\infty)$. Note that the spectrum of L is continuous, is given by $(-\infty,0)$ and is two-fold degenerate. Let $\eta\in\mathbf{C}$ with $\eta\notin(-\infty,0)$ and consider the bounded Greens function for $L-\eta$ given by

$$\begin{aligned}
G_\eta(x,y)&=-\frac{1}{2\sqrt{\eta}}e^{-\sqrt{\eta}\,|x-y|}\\
&=-\frac{1}{2\sqrt{\eta}}e^{\sqrt{\eta}\,x_<}e^{-\sqrt{\eta}\,x_>}
\end{aligned}$$

with the square root above chosen to have positive real part, in other words with branch cut $(-\infty,0)$. Then (10.4) and (10.5) give for $\epsilon\in(-\infty,0)$

$$2\pi i\sum_{n=\pm}\psi_n(\epsilon,x)\overline{\psi_n(\epsilon,y)}=-\frac{1}{2i\sqrt{-\epsilon}}\left(e^{i\sqrt{-\epsilon}\,(x-y)}+e^{-i\sqrt{-\epsilon}\,(x-y)}\right)$$

from which one obtains the normalization

$$N_\pm(-\epsilon)=\sqrt{\frac{1}{4\pi\sqrt{-\epsilon}}}$$

of the continuum eigenfunctions obtained previously from comparison to Fourier transformation.

....

Example 10.7: Consider

$$L = \nabla^2$$

in free space in cylindrical coordinates. In cylindrical coordinates the Greens function for $L - \eta$ satisfies

$$\Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} - \eta\Big) G_\eta(r,\theta,z,r',\theta',z') = \frac{\delta(r-r')}{r}\delta(\theta-\theta')\delta(z-z').$$

It may be obtained by eigenfunction expansion using Fourier transform in z, Fourier serires in θ and Hankel transform in r. One obtains

$$G_\eta(r,\theta,z,r',\theta',z') = \frac{1}{4\pi^2}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{e^{im(\theta-\theta')+ik(z-z')}J_m(qr)J_m(qr')}{-k^2-q^2-\eta} q\,dk\,dq.$$

Alternatively, one may use a mixed approach. Use eigenfunction expansion, Fourier transform in z and Fourier serires in θ, for z and θ and (10.2) for r. The two eigenfunction expansions give

$$G_\eta(r,\theta,z,r',\theta',z') = \frac{1}{4\pi^2}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{im(\theta-\theta')+ik(z-z')} g_m(k,r,r')\,dk$$

with

$$\Big(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} - k^2 - \eta\Big) g_m(k,r,r') = \frac{\delta(r-r')}{r}.$$

Using (10.2) one finds

$$\begin{aligned} g_m(k,r,r') &= -\frac{J_m(\sqrt{-k^2-\eta}\,r_<)H_m^{(1)}(\sqrt{-k^2-\eta}\,r_>)}{\mathcal{W}\big(J_m(\sqrt{-k^2-\eta}\,r), H_m^{(1)}(\sqrt{-k^2-\eta}\,r)\big)} \\ &= \frac{\pi}{2}J_m(\sqrt{-k^2-\eta}\,r_<)H_m^{(1)}(\sqrt{-k^2-\eta}\,r_>) \end{aligned}$$

so that

$$G_\eta(r,\theta,z,r',\theta',z') = \frac{1}{8\pi}\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{im(\theta-\theta')+ik(z-z')} J_m(\sqrt{k^2-\eta}\,r_<)H_m^{(1)}(\sqrt{k^2-\eta}\,r_>)\,dk.$$

....

Example 10.8: Consider

$$L = \nabla^2$$

in free space in spherical coordinates. Assume an outgoing radiation condition at $r = \infty$:

$$\psi(\mathbf{x}) \sim \frac{e^{ikr}}{r}$$

as $r \to \infty$. The Greens function for $L - \eta$ satisfies

$$\begin{aligned}\Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\big(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\big) - \eta\Big)G_\eta(r,\theta,\phi,r',\theta',\phi')& \\ = \frac{\delta(r-r')}{r^2}\frac{\delta(\theta-\theta')}{\sin\theta}\delta(\phi-\phi').&\end{aligned}$$

Expanding in spherical harmonics one has

$$G_\eta(r,\theta,\phi,r',\theta',\phi') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}(\theta,\phi)\overline{Y_{lm}(\theta',\phi')}g_{\eta;l,m}(r,r')$$

with

$$\Big(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2} - \eta\Big)g_{\eta;l,m}(r,r') = \frac{\delta(r-r')}{r^2}.$$

One has

$$g_{\eta;l,m}(r,r') = -\frac{j_l(\sqrt{-\eta}\,r_<)h_l^{(+)}(\sqrt{-\eta}\,r_>)}{r^2\mathcal{W}\big(j_l(\sqrt{-\eta}\,r), h_l^{(+)}(\sqrt{-\eta}\,r)\big)}.$$

Using

$$\begin{aligned}\mathcal{W}\big(j_l(\sqrt{-\eta}\,r), h_l^{(+)}(\sqrt{-\eta}\,r)\big) &= i\mathcal{W}\big(j_l(\sqrt{-\eta}\,r), y_l(\sqrt{-\eta}\,r)\big) \\ &= -\frac{i(2l+1)}{\sqrt{-\eta}\,r^2}\end{aligned}$$

one finds

$$g_{\eta;l,m}(r,r') = -i\frac{\sqrt{-\eta}}{2l+1}\,j_l(\sqrt{-\eta}\,r_<)h_l^{(+)}(\sqrt{-\eta}\,r_>).$$

....

We end by mentioning some applications. The basic application of a Green's function is to solving inhomogeneous problems of the form

$$Lf = g.$$

If a Greens function G for L is known then

$$f(\mathbf{x}) = f_0(\mathbf{x}) + \int G(\mathbf{x},\mathbf{y})g(\mathbf{y})d^n y$$

where $Lf_0 = 0$.

Example 10.9: A classic example is Poisson's equation

$$\nabla^2\phi(x) = -4\pi\rho(x).$$

By Example 10.1 the solution is given by in free space by Coulomb's law

$$\phi(x) = \int \frac{\rho(\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|}\, d^3y.$$

....

Green's functions can also be used in a subtle way to solve Helmholtz equations with inhomogeneous boundary conditions in general domains using the Kirchof-Helmholtz integral theorem which is derived as follows. Let

$$\left(\nabla^2 + k^2\right)\psi = 0$$

in some domain V with some inhomogeneous boundary conditions on ∂V. Let $G(\mathbf{x},\mathbf{y})$ be any Green's function of $\nabla^2 + k^2$ which is symmetric with respect to interchange of $\mathbf{x}$ and $\mathbf{y}$ (for example, the free speace Green's function). Then

$$\begin{aligned}
\psi(\mathbf{x}) &= (\nabla_x^2 + k^2)\int_V G(\mathbf{x},\mathbf{y})\psi(\mathbf{y})\, d^ny \\
&= \int_V \left[\Big((\nabla_y^2 + k^2)G(\mathbf{x},\mathbf{y})\Big)\psi(\mathbf{y})\right] d^ny \\
&= \int_V \left[\Big((\nabla_y^2 + k^2)G(\mathbf{x},\mathbf{y})\Big)\psi(\mathbf{y}) - G(\mathbf{x},\mathbf{y})\Big((\nabla_y^2 + k^2)\psi(\mathbf{y})\Big)\right] d^ny \\
&= \int_V \nabla_y \cdot \left[\Big(\nabla_y G(\mathbf{x},\mathbf{y})\Big)\psi(\mathbf{y}) - G(\mathbf{x},\mathbf{y})\Big(\nabla_y\psi(\mathbf{y})\Big)\right] d^ny \\
&= \int_{\partial V} \left[\Big(\nabla_y G(\mathbf{x},\mathbf{y})\Big)\psi(\mathbf{y}) - G(\mathbf{x},\mathbf{y})\Big(\nabla_y\psi(\mathbf{y})\Big)\right] \cdot \hat{\mathbf{n}}\, d\sigma(\mathbf{y}).
\end{aligned}$$

The resulting formula,

$$\psi(\mathbf{x}) = \int_{\partial V} \left[\Big(\hat{\mathbf{n}}\cdot\nabla_y G(\mathbf{x},\mathbf{y})\Big)\psi(\mathbf{y}) - G(\mathbf{x},\mathbf{y})\Big(\hat{\mathbf{n}}\cdot\nabla_y\psi(\mathbf{y})\Big)\right] d\sigma(\mathbf{y}), \tag{10.6}$$

is known as the Helmholtz-Kirchoff integral theorem (or vica-versa).

One use of the Helmholtz-Kirchoff integral theorem is to problems in which an inhomogeneous boundary condition of the form

$$\alpha(\mathbf{x})\,\psi + \beta(\mathbf{x})\,\hat{\mathbf{n}}\cdot\nabla\psi(\mathbf{x}) = \gamma(\mathbf{x}) \tag{10.7}$$

is specified on ∂V. Assume (without loss of generality) that

$$\alpha^2 + \beta^2 = 1.$$

Eq. (10.6) can be written

$$\begin{aligned}
\psi(\mathbf{x}) &= \int_{\partial V} \Big[\Big(\beta\, G(\mathbf{x},\mathbf{y}) - \alpha\,\hat{\mathbf{n}}\cdot\nabla_y G(\mathbf{x},\mathbf{y})\Big)\Big(\alpha\,\psi(\mathbf{y}) + \beta\,\hat{\mathbf{n}}\cdot\nabla\psi(\mathbf{y})\Big) \\
&\qquad - \Big(\alpha\, G(\mathbf{x},\mathbf{y}) + \beta\,\hat{\mathbf{n}}\cdot\nabla_y G(\mathbf{x},\mathbf{y}))\Big)\Big(\beta\,\psi(\mathbf{y}) - \alpha\,\hat{\mathbf{n}}\cdot\nabla_y\psi(\mathbf{y})\Big)\Big]\, d\sigma(\mathbf{y}) \\
&= \int_{\partial V} \Big[\Big(\beta\, G(\mathbf{x},\mathbf{y}) - \alpha\,\hat{\mathbf{n}}\cdot\nabla_y G(\mathbf{x},\mathbf{y})\Big)\gamma(\mathbf{y}) \\
&\qquad - \Big(\alpha\, G(\mathbf{x},\mathbf{y}) + \beta\,\hat{\mathbf{n}}\cdot\nabla_y G(\mathbf{x},\mathbf{y}))\Big)\Big(\beta\,\psi(\mathbf{y}) - \alpha\,\hat{\mathbf{n}}\cdot\nabla_y\psi(\mathbf{y})\Big)\Big]\, d\sigma(\mathbf{y}).
\end{aligned}$$

Thus if $G(\mathbf{x},\mathbf{y})$ is chosen to satisfy the homogeneous version of (10.7),

$$\alpha\, G(\mathbf{x},\mathbf{y}) + \beta\,\hat{\mathbf{n}}\cdot\nabla_y G(\mathbf{x},\mathbf{y})) = 0$$

then one has the explicit result

$$\psi(\mathbf{x}) = \int_{\partial V} \Big(\beta(\mathbf{y})\, G(\mathbf{x},\mathbf{y}) - \alpha(\mathbf{y})\,\hat{\mathbf{n}}\cdot\nabla_y G(\mathbf{x},\mathbf{y})\Big)\gamma(\mathbf{y})\, d\sigma(\mathbf{y}).$$

Note the special cases: the Neumann problem is given by $\alpha = 0$ and $\beta = 1$, the Dirichlet problem by $\beta = 0$ and $\alpha = 1$.

Example 10.10: Let

$$\Big(\nabla^2 + k^2\Big)\psi(\mathbf{x}) = 0$$

for $r = |\mathbf{x}| > R$ (that is, outside of the sphere of radius R). Let the normal derivative of ϕ be specified on the surface of the sphere (the so-called Neumann problem):

$$\frac{\partial\phi}{\partial r}\Big|_{r=R} = \gamma(\theta,\phi)$$

with γ known. Assume an outgoing radiation condition at $r = \infty$:

$$\psi(\mathbf{x}) \sim \frac{e^{ikr}}{r}$$

as $r \to \infty$.

Let $G(r, \theta, \phi, r', \theta', \phi')$ be the Greens function in spherical coordinates satisfying Neumann boundary conditions at $r = R$

$$\frac{\partial G}{\partial r}\Big|_{r=R} = 0$$

and the radiation condition

$$G(r, \theta, \phi, r', \theta', \phi') \sim \frac{e^{ikr}}{r}$$

as $r \to \infty$. Then

$$\psi(r, \theta, \phi) = R^2 \int_0^{2\pi} \int_0^{\pi} G(r, \theta, \phi, R, \theta', \phi')\gamma(\theta', \phi') \sin\theta' \, d\theta' \, d\phi'.$$

One may use

$$G_\eta(r, \theta, \phi, r', \theta', \phi') = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y_{lm}(\theta, \phi)\overline{Y_{lm}(\theta', \phi')} g_{\eta;l,m}(r, r')$$

with

$$g_{\eta;l,m}(r, r') = -\frac{\left(j_l(\sqrt{-\eta}\, r_<) - \frac{j_l'(\sqrt{-\eta}\, R)}{y_l'(\sqrt{-\eta}\, R)} y_l(\sqrt{-\eta}\, r_<)\right) h_l^{(+)}(\sqrt{-\eta}\, r_>)}{r^2 \mathcal{W}\Big(\big(j_l(\sqrt{-\eta}\, r) - \frac{j_l'(\sqrt{-\eta}\, R)}{y_l'(\sqrt{-\eta}\, R)} y_l(\sqrt{-\eta}\, r)\big), h_l^{(+)}(\sqrt{-\eta}\, r)\Big)}.$$

Note that $g_{\eta;l,m}(r, r')$ is independent of m (we will drop the index for m). Using

$$\begin{aligned}
\mathcal{W}\Big(\big(j_l(\sqrt{-\eta}\, r) &- \frac{j_l'(\sqrt{-\eta}\, R)}{y_l'(\sqrt{-\eta}\, R)} y_l(\sqrt{-\eta}\, r)\big), h_l^{(+)}(\sqrt{-\eta}\, r)\Big) \\
&= \mathcal{W}\Big(\big(j_l(\sqrt{-\eta}\, r) - \frac{j_l'(\sqrt{-\eta}\, R)}{y_l'(\sqrt{-\eta}\, R)} y_l(\sqrt{-\eta}\, r)\big), j_l(\sqrt{-\eta}\, r) + i y_l(\sqrt{-\eta}\, r)\Big) \\
&= \Big(i + \frac{j_l'(\sqrt{-\eta}\, R)}{y_l'(\sqrt{-\eta}\, R)}\Big) \mathcal{W}\big(j_l(\sqrt{-\eta}\, r), y_l(\sqrt{-\eta}\, r)\big) \\
&= -\Big(i + \frac{j_l'(\sqrt{-\eta}\, R)}{y_l'(\sqrt{-\eta}\, R)}\Big) \frac{2l+1}{\sqrt{-\eta}\, r^2}
\end{aligned}$$

one has

$$g_{\eta;l}(r, r') = \frac{\sqrt{-\eta}}{(2l+1)\Big(i + \frac{j_l'(\sqrt{-\eta}\, R)}{y_l'(\sqrt{-\eta}\, R)}\Big)} \Big(j_l(\sqrt{-\eta}\, r_<) - \frac{j_l'(\sqrt{-\eta}\, R)}{y_l'(\sqrt{-\eta}\, R)} y_l(\sqrt{-\eta}\, r_<)\Big) h_l^{(+)}(\sqrt{-\eta}\, r_>).$$

If

$$\gamma_{lm} = R^2 \int_0^{2\pi} \int_0^{\pi} \overline{Y_{lm}(\theta', \phi')} \gamma(\theta', \phi') \sin\theta' \, d\theta' \, d\phi'$$

Then

$$\psi(r, \theta, \phi) = \sum_{l=0}^{\infty} g_{\eta;l}(r, r') \sum_{m=-l}^{l} \gamma_{lm} Y_{lm}(\theta, \phi).$$

....

Problems

1) Consider the 1-dimensional wave equation

$$(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})p = 0.$$

Here c is a constant. Show that

$$\begin{aligned}\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} &= (\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t})(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}) \\ &= (\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t})(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t}).\end{aligned}$$

Use this to show that for any differentiable function f

$$p(x,t) = f(x \pm ct)$$

are two solutions. Show that the plane wave solutions are of this form and determine the appropriate function f.

2) Find the plane wave solutions to Maxwell's equations.

3) Consider the vector space V equal to the set of all complex linear combinations

$$a\cos(kx) + b\sin(kx).$$

Here $a, b \in \mathbf{C}$. What is the dimension of V over $\mathbf{C}$? Show that V is precisely the set of solutions of the 1-dimensional Helmholtz equation

$$(\frac{d^2}{dx^2} + k^2)p = 0.$$

Express $\frac{d}{dx}$ and $\frac{d^2}{dx^2}$ as a matrix with respect to the basis $\sin(kx)$, $\cos(kx)$. Repeat for the basis e^{ikx}, e^{-ikx}.

4) Find the exponential solutions to the plate equation. Assume a time dependence of the form $e^{i\omega t}$.

5) Show that in the linear approximation to fluid dynamics ρ' and p' both satisfy the wave equation. (Showing that this is true for the traveling wave solutions is not enough.)

6) Let H be an n by n self-adjoint matrix. Assume the eigenvalues of H are ϵ_j and the eigenvectors, chosen to be orthonormal, are v_j.

a) Find the solution of the associated Shrödinger equation

$$i\frac{d\psi}{dt} = H\psi$$

with initial condition $\psi(0) = \phi_0$.

b) Find the solution of

$$\frac{d^2\mathbf{y}}{dt^2} = H\mathbf{y}$$

with initial conditions $\mathbf{y}(0) = \mathbf{p}$ and $\mathbf{y}'(0) = \mathbf{v}$.

c) Carry out (a) and (b) for the matrix

$$H = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

7) Consider a linear array of 5 microphones. Let $P_j(\theta)$ be the response of the j microphone to a plane wave incident on the array at an angle of θ to the normal to array. Let

$$P_I(\theta) = e^{-10(\theta-\theta_0)^2}$$

be an "ideal" beam pattern. Given coefficients c_j, let

$$P(\theta) = \sum_{j=-2}^{2} c_j P_j(\theta).$$

Determine the coefficients c_j making is as close as possible to $P_I(\theta)$ in the least squares sense: by minimizing

$$\|P - P_I\| = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |P(\theta) - P_I(\theta)|^2 \, d\theta.$$

Use any convenient computer software to do the computations and plot the resulting beam pattern P.

8) Let $f, g : \mathbf{R}^2 \to \mathbf{R}^2$ be given by

$$f(x,y) = \begin{pmatrix} e^{xy} \\ \sin x + \cos y \end{pmatrix}$$

and

$$g(x,y) = \begin{pmatrix} x^2 y \\ y - x \end{pmatrix}.$$

Find $Df(x,y)$, $Dg(x,y)$, $(Df \circ g)(x,y)$,

$$\frac{\partial}{\partial x} f(g(x,y))$$

and

$$\frac{\partial}{\partial y} f(g(x,y)).$$

Find the linear approximation to $f \circ g$ near $(x,y) = (0, \frac{\pi}{4})$.

9) Using $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ show that $a\,dx + b\,dy$ is exact only if

$$\frac{\partial a}{\partial y} = \frac{\partial b}{\partial x}.$$

Use this to show that $-y\,dx + x\,dy$ is not exact. Find $d \arctan \frac{y}{x}$ and show that $\frac{1}{x^2+y^2}$ is an integrating factor for $-y\,dx + x\,dy$.

10) Transform the Cartesian gradient in $\mathbf{R}^2$ into polar coordinates and find the Laplace operator in these coordinates. Repeat the exercise for cylindrical coordinates in $\mathbf{R}^3$.

11) Consider the parabolic coordinates $w,\ \tau$

$$x = \tau$$

$$y = w\tau^2.$$

Transform the Cartesian gradient and find the Laplace operator in these coordinates.

12) Let $f(x, y) = e^{xy}$. Find all critical points for f (points where $\nabla f = \mathbf{0}$) and determine if they are maxima, minima or saddles. If any are saddles determine the directions through which the saddles are mins and through which they are maxs.

13) Consider an ideal gas in which $P = \rho RT$. Let $P = P_0 + P'$, $\rho = \rho_0 + \rho'$ and $T = T_0 + T'$ where the primed variables are small disturbances about some quiescent state given by P_0, ρ_0 and T_0. It can be shown that if the specific heat c_P is constant then the difference in the entropy of the disturbed state S and the quiescent state S_0 is

$$S - S_0 = c_P \ln \frac{T}{T_0} - R \ln \frac{P}{P_0}.$$

Assuming that the processes are isentropic ($S = S_0$) express ρ as a function of P and find the speed of sound

$$c = \left(\left[\left(\frac{\partial \rho}{\partial P} \right)_S \right] \Big|_{S=S_0,\, P=P_0} \right)^{-\frac{1}{2}}.$$

14) What is the surface element on the surface of a cylinder of radius R whose axis is the z axis?

15) Use the fact that a sphere of radius r is the set of solutions to

$$x^2 + y^2 + z^2 = r^2$$

to find its unit normal $\mathbf{n}$ in spherical coordinates. Use this result to show that

$$\mathbf{n} \cdot \nabla = \frac{\partial}{\partial r}.$$

16) Let $D = \{\mathbf{x} \big| \|\mathbf{x}\| \leq R \text{ and } x > 0,\ y > 0,\ z > 0\}$ be the restriction of the ball of radius R to the first quadrant. Compute

$$\int_D xz \, dx \, dy \, dz.$$

17) Show that, in 3 dimensions,

$$\left(\Delta + k^2\right)\frac{1}{\|\mathbf{x}-\mathbf{y}\|}e^{ik\|\mathbf{x}-\mathbf{y}\|} = -4\pi\delta(\mathbf{x}-\mathbf{y}).$$

18) Show that given a function $F(\mathbf{x})$ any solution ϕ to the equation

$$\left(\Delta + k^2\right)\phi = F$$

can be written

$$\phi(\mathbf{x}) = \phi_0(\mathbf{x}) - \frac{1}{4\pi}\int \frac{1}{\|\mathbf{x}-\mathbf{y}\|}e^{ik\|\mathbf{x}-\mathbf{y}\|}F(\mathbf{y})\,d^3y$$

where ϕ_0 is a solution of the homogeneous Helmholtz equation

$$\left(\Delta + k^2\right)\phi_0 = 0.$$

19) Let $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ and $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$. Find the poles of $\tan z$ and find the first three terms in the Laurent expansions about each of these poles.

20) The function z^p for arbitrary p is usually defined by

$$z^p = e^{p\ln z}.$$

Show that z^p is single valued only when p is an integer. What phase does z^p pick up when z circles 0 counterclockwise once? Make a choice for z^p and compute

$$\int_{C_\epsilon(0)} z^p\,dz.$$

For which p is this integral 0?

21) Compute

$$\int_{-\infty}^{\infty}\frac{1}{x^2+1}\,dx$$

and

$$\int_{-\infty}^{\infty}\frac{e^{ix}}{(x^2+1)^2}\,dx.$$

22) Find the general solution of

$$\left(\frac{d^2}{dx^2} + 4\frac{d}{dx} + 4\right) f = 0.$$

23) Consider the model introduced in Example 5.3. Express the solutions as linear combinations of exponentials $e^{\pm ikx}$ rather than sines and cosines. Find the general solution which obeys the condition that for $x > 0$

$$f(x) = \text{const } e^{ik_+ x}.$$

24) Find the solution of

$$\left(\frac{d^4}{dx^4} - k^2\right) f = 0$$

which satisfies

$$f(0) = -f''(0) = \alpha_0$$

and

$$f'(0) = f^{(3)}(0) = 0.$$

Instead, impose the condition that as $x \to \infty$

$$f(x) \sim \text{const } e^{-i\sqrt{k}\, x}.$$

What constraint does this impose on the allowed values of $f(0)$, $f'(0)$, $f''(0)$ and $f^{(3)}(0)$?

25) Find all the solutions of

$$\left(x^2 \frac{d^2}{dx^2} + 2x\frac{d}{dx} - 1\right) f(x) = 0$$

which are finite at $x = 0$.

26) Recall that

$$J_m(x) = \frac{1}{m!}\left(\frac{x}{2}\right)^m\Big(1 + c_1 x + c_2 x^2 + \dots\Big).$$

Find c_1 and c_2. Recall further that

$$Y_m(x) = \frac{2}{\pi}J_m(x)\ln x - \frac{(m-1)!}{\pi}\left(\frac{x}{2}\right)^{-m}\Big(1 + d_1 x + d_2 x^2 + \dots\Big).$$

What choice for Y_m has to be made so that all the odd coefficients $d_{2j+1} = 0$? Make this choice and then determine d_2 for $m \neq 1$. What can be said about d_2 for $m = 1$?

27) Using Abel's formula and the small x asymptotics calculate $\mathcal{W}(J_m, Y_m)$. Similarly, using the large x asymptotics compute $\mathcal{W}(H_m^{(+)}, H_m^{(-)})$. Finally, assuming that

$$H_m^{(\pm)} = c(J_m \pm iY_m)$$

use these two Wronskians to determine c.

28) Consider (5.5). Find functions u and Q so that setting $f(x) = u(x)g(x)$ the new unknown g satisfies

$$\Big(\frac{d^2}{dx^2} + Q\Big)g = 0.$$

Thus, to study second order linear equations it is sufficient to study equations with no first derivative term.

29) Using the technique developed in the previous problem find the general solution to the spherical Bessel equation with $l = 0$.

30) Find the first correction to the large x asymptotic forms for J_m and Y_m.

31) In (5.18) find δ and the two possible values for a_1.

32) Find the large x asymptotic form for solutions to

$$\Big(\frac{d^2}{dx^2} + \frac{\lambda}{x} + 1\Big) f(x) = 0.$$

Choose a form

$$f(x) = x^p e^{\kappa x^\alpha} \Big(1 + \dots\Big)$$

and determine p, κ and α.

33) Show that the eigenvalues of a self adjoint operator are real.

34) Let $L = \frac{d^2}{dx^2} + q$ have self adjoint boundary conditions on (a, b). Let $\psi_j(x)$ be the normalized eigenfunctions for L corresponding to the eigenvalues λ_j. Let $\delta_{(a,b)}$ be the delta function restricted to (a, b): for any $y \in (a, b)$

$$f(y) = \int_a^b \delta_{(a,b)}(y - x)\, f(x)\, dx.$$

Show that for any $y \in (a, b)$

$$\sum_j \psi_j(y)\, \overline{\psi_j(x)} = \delta_{(a,b)}(y - x).$$

Now consider the equation

$$(L - \epsilon) f = g$$

for some ϵ not in the spectrum of L. Show that the general solution may be written in the form

$$f(x) = f_0(x) + \sum_j \int_a^b \frac{\psi_j(x)\, \overline{\psi_j(y)}}{\lambda_j - \epsilon}\, g(y)\, dy$$

for some f_0 satisfying $(L - \epsilon) f_0 = 0$. Show that if f satisfies the imposed boundary conditions then $f_0 = 0$. If ϵ is in the spectrum of L What condition on g must be satisfied if $(L - \epsilon) f = g$ is to have a solution?

35) Consider one dimensional lossless acoustics,

$$\Big(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big) P(x,t) = 0$$

$$\rho_0 \frac{\partial v(x,t)}{\partial t} = -\frac{\partial P(x,t)}{\partial x},$$

in the interval $(0, L)$. Assume that in the steady state,

$$\begin{pmatrix} P(x,t) \\ v(x,t) \end{pmatrix} = \begin{pmatrix} P_A(x) \\ v_A(x) \end{pmatrix} e^{-i\omega t},$$

impedances $Z_0(\omega)$ and $Z_L(\omega)$ are specified at $x = 0$ and $x = L$ respectively,

$$\frac{P_A(0)}{v_A(0)} = Z_0$$

and

$$\frac{P_A(L)}{v_A(L)} = Z_L.$$

Find an equation whose solutions are the resonant frequencies of the system. Let ω_j and ω_k be two distinct resonant frequencies. Let P_{Aj} and P_{Ak} be the corresponding resonant pressure amplitudes. Find

$$\langle P_{Aj}, P_{Ak} \rangle.$$

Under what conditions on Z_0 and Z_L is the problem of finding the resonant frequencies and amplitudes self adjoint?

36) Consider a vibrating string of length L, clamped at both ends. Let $u(x,t)$ be the displacement of the string at x and t. Imagine striking the string at $x = \frac{L}{3}$ and model this striking by the initial condition

$$\frac{\partial u}{\partial t}\Big|_{t=0} = A\,\delta(x - \frac{L}{3}).$$

Assume as well that the initial displacement of the string is 0. Find the subsequent motion of the string. Which modes don't get excited?

37) Solve the time dependent Schrödinger equation for a particle of mass m confined to a rigid cylindrical container of radius R and height H.

38) Consider a rigid rectangular box of dimensions L, W and H. Find the resonant frequencies and modes for the Laplacian in this box with Neumann boundary conditions. Under what conditions are there no degenerate resonant frequencies?

39) Consider a cube A with sides of length 1. Find the lowest eigenvalue of $-\nabla^2$ in this cube with boundary conditions

$$\hat{\mathbf{n}} \cdot \nabla\psi\big|_{\partial A} = -\alpha\psi\big|_{\partial A}$$

for $0 \leq \alpha \leq \infty$.

40) Consider a semi-infinite duct whose cross section A is constant and has area $|A|$. Let the lowest non-zero Neumann eigenvalue of $-\nabla^2$ in A be λ_1. Imagine that a piston is mounted in a baffle at $z = 0$ somewhere in A. Let the area of the piston be $|\tilde{A}|$. If the piston has velocity $u_0 \cos(\omega t)$ with $\omega < c\sqrt{\lambda_1}$ find the large z asymptotic form for the pressure $P(x, y, z, t)$. What length scale must z be much larger than in order for this asymptotic form to be valid?

41) Consider a duct in air ($c = 343$ m/sec) with square cross section 5 cm by 5 cm. For what range of frequencies are there precisely 4 propagating modes? Write out the velocity dispersion for these modes.

42) Estimate the eigenvalues of a vibrating bar of length L free at both ends. Plot the first 5 eigen modes.

43) Consider an annular membrane of inner radius r_1 and outer radius r_2. Find the Dirichlet eigenfunctions and eigenvalue condition. For $r_2 = 1$ and $r_1 = 0.1$ find the first three resonant frequencies.

44) Show that, given a_{1m} for $m \in \{-1, 0, 1\}$, there is a constant vector $\mathbf{A}$ with

$$\frac{1}{r^2} \sum_{m=-1}^{1} a_{1m} Y_{1m}(\theta, \phi) = \mathbf{A} \cdot \nabla \frac{1}{r}.$$

Give an explicit expression for $\mathbf{A}$ in terms of the a_{1m}.

45) Consider the acoustic resonant frequencies and modes of a spherical resonator of radius R filled with a gas in which the speed of sound is c. Find equations whose solutions are the resonant frequencies. Given the resonant frequencies find the resonant modes. For the first three spherically symmetric modes estimate the resonant frequencies and find the normalization constants for the modes.

46) Consider a sphere of radius R whose surface is vibrating with velocity

$$u(\theta, \phi) = \mathrm{Re} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \phi).$$

Find the acoustic pressure of the sound radiating from the sphere into empty space.

47) Find the generalized eigenfunction expansion for

$$L = \frac{d^2}{dx^2}$$

on $(1, \infty)$ with boundary conditions $\psi(1) = 0$.

48) Find the generalized eigenfunction expansion for

$$L = -\frac{d^2}{dx^2} + v(x)$$

for

$$v(x) = \begin{cases} -10 & \text{if } x < 1 \\ 0 & \text{otherwise} \end{cases}$$

on $(0, \infty)$ with boundary conditions $\psi'(0) = 0$.

49) Find the Fourier transforms of the following functions:

$$f(t) = \begin{cases} 0 & \text{if } t < t_0 \\ e^{-(a+bi)t} & \text{if } t \geq t_0 \end{cases}$$

$$f(x) = \frac{1}{x - x_0 + i\alpha}$$

$$f(x) = \frac{\cos(\kappa x)}{(x - x_0)^2 + \alpha^2}$$

50) Show that a necessary and sufficient condition for a function $f(x)$ to be real is that it's Fourier transform $\hat{f}(k)$ satisfy

$$\overline{\hat{f}(k)} = \hat{f}(-k).$$

51) Consider a simple closed circuit with a voltage source $V(t)$ and one element connected in series.

a) Show that if the element is a capacitor, $Z(\omega) = i\omega C$, the current $I(t)$ is $\frac{1}{C}$ times the integral of V.

b) Show that if the element is an inductor, $Z(\omega) = \frac{1}{i\omega L}$, the current $I(t)$ is L times the derivative of V.

c) If the element is a linear combination of a capacitor and a resistor, $Z(\omega) = R + i\omega C$, and if the voltage source provides an impulse at $t = t_0$, $V(t) = v_0 \delta(t - t_0)$ find the current $I(t)$.

52) Consider the three dimensional heat equation

$$\left(\frac{\partial}{\partial t} - \nabla^2\right) T(\mathbf{x}, t) = 0.$$

If one has the initial condition

$$T(\mathbf{x}, 0) = \delta(\mathbf{x})$$

find $T(\mathbf{x}, t)$ for $t > 0$.

53) Let

$$f_{\pm}(x) = \frac{1}{2\pi} \lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{iq(x-y)}}{k^2 - q^2 \pm i\epsilon} g(y)\, dy\, dq.$$

Show that both $f_{\pm}$ satisfy

$$\Big(\frac{d^2}{dx^2} + k^2\Big) f_{\pm}(x) = g(x).$$

Assuming that

$$\hat{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\, g(x)\, dx$$

is analytic with $|\hat{g}(k)| \leq \text{const }(1+k^2)^m$ for some m, calculate $f_{\pm}$.

54) Consider a cube with sides of length L. Two opposing sides of the cube are vibrating, one with velocity $u(t)$, the other with velocity $-u(t)$. Find the acoustic pressure $P(r,\theta,\phi,t)$ for large r if

a) the Fourier transform of u is

$$\hat{u}(\omega) = Ae^{-\lambda(\omega-\omega_0)^2},$$

b) the Fourier transform of u is

$$\hat{u}(\omega) = A\Big(\frac{1}{\omega - \omega_0 + \alpha i} - \frac{2}{\omega - 2\omega_0 + \alpha i}\Big).$$

Assume that $L \ll \frac{c}{\omega_0}$, $r \gg \frac{c}{\omega_0}$, $\lambda \gg \frac{1}{\omega_0^2}$ and $\alpha \ll \omega_0$.

55) Find the generalized eigenfunction expansion for $\frac{d^2}{dx^2}$ on $(0,\infty)$ with the boundary condition $f(0) + f'(0) = 0$.

56) Consider a semi-infinite pipe with square cross section of length L. A piston that is mounted at the open end of the pipe has area A and velocity $u(t)$.

a) Find, but don't attempt to evaluate, an expression for the pressure on the face of the piston.

b) If $A = L^2$ evaluate the pressure on the face of the piston.

57) Let

$$L = \frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{m^2}{r^2}$$

on $(1, \infty)$ with the boundary condition $\psi'(1) = 0$. Find the outgoing Green's function for $L - \eta$ and the generalized eigenfunction expansion for L.

58) Consider the wedge given in cylindrical coordinates by

$$\{r, \theta, z \big| 0 < r < \infty,\ 0 < \theta < \theta_0,\ 0 < z < \infty\}.$$

Let

$$\Big(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Big)P(r, \theta, z, t) = 0.$$

Let $P(r, \theta, z, t)$ satisfy a radiation condition at spatial infinity and Neumann conditions on the surfaces of the wedge

$$\frac{\partial P}{\partial \theta}\Big|_{\theta=0} = 0 = \frac{\partial P}{\partial \theta}\Big|_{\theta=\theta_0}.$$

Assume that the value of $P(r, \theta, 0, t) = f(r, \theta, t)$ is known on the bottom of the wedge. Use the Helmholtz-Kirchoff integral theorem (10.6) to find $P(r, \theta, z, t)$ in terms of sums and integrals over known quantities.

59) Let

$$L = \nabla^2 + k^2$$

in three dimensions where

$$k = \begin{cases} k_- & \text{if } 0 \leq |\mathbf{x}| < R \\ k_+ & \text{if } |\mathbf{x}| \geq R. \end{cases}$$

Assuming that $k_- > k_+$ find the eigenfunction expansion for L.