Using Research to Investigate and Enhance Learning in Upper-division Mechanics

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Supported by NSF grants DUE-0441426 and DUE-0442388

Special acknowledgements

- Michael Wittmann *(U. Maine) [UME],*
  Co-PI, *Intermediate Mechanics Tutorials* project

- Lillian C. McDermott, Peter Shaffer, Paula Heron *(U. Washington);*
  Stamatis Vokos, John Lindberg *(Seattle Pacific University)*

- Juliet Brosing *(Pacific University),*
  Maja Krcmar *(Grand Valley State University)*
  Dawn Meredith *(U. New Hampshire),*
  Carolyn Sealfon *(West Chester University),*
  Carrie Swift *(U. Michigan-Dearborn)*

- National Science Foundation

Physics education research (PER)
is research in the teaching and learning of physics

Two general approaches:

- *Empirical approach*

  Emphasis on student learning of specific topics
  (*e.g.*, mechanics, optics, relativity)

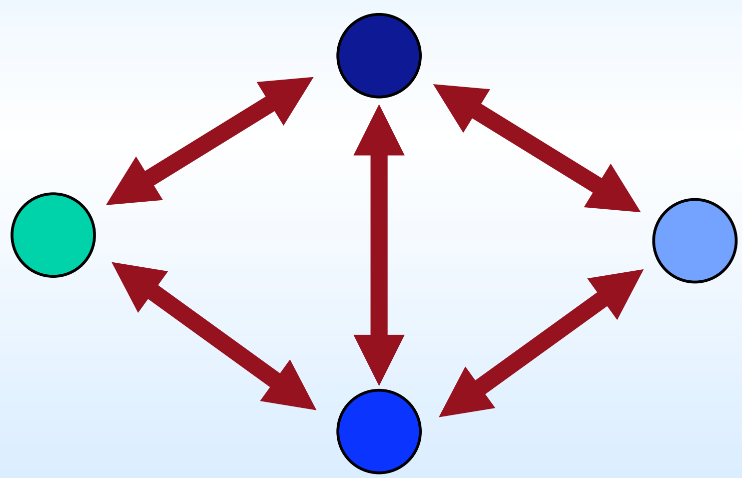
- *Theoretical approach*

  Emphasis on predictive models of student cognition

Physics education research (PER)
is research in the teaching and learning of physics

Investigation of student understanding

Development of models of student thinking



Examination of effect on student learning

Development of instructional strategies

Outline of presentation

- Background and motivation for investigation
- Research to **probe student thinking** in intermediate mechanics
  - Selected lessons learned from PER at introductory level
  - Identification of need for conceptual emphasis beyond introductory course
- Research to **enhance student learning** in intermediate mechanics
  - Development and assessment of teaching strategies
- Summary and discussion
  - Some **new** lessons learned from PER **beyond** introductory level

Context of investigation and curriculum development

Primary student populations: Intermediate mechanics

- Grand Valley State University (GVSU)
- University of Maine (U. Maine)
- Seattle Pacific University (SPU)

Primary research methods

- Ungraded quizzes (pretests)
- Written examinations

"Explain your reasoning."

- Formal and informal observations in classroom
- Individual and group student clinical interviews

Typical content in upper-division mechanics

Foundational topics
(introductory level)

- Vectors
- Kinematics
- Newton's laws
- Work, energy, energy conservation
- Linear and angular momentum

New applications and extensions

- Velocity-dependent forces
- Linear and non-linear oscillations
- Conservative force fields
- Non-inertial reference frames
- Central forces, orbital mechanics

New formalism and representations

- Scalar and vector fields; del operator; gradient, curl
- Variational methods; Lagrangian mechanics
- Phase space diagrams

As an *instructor* of intermediate mechanics

One might expect students to have already developed:

- *functional understanding* of physical concepts covered at the introductory level

- mathematical and reasoning skills necessary to extend those concepts in solving more sophisticated problems, *both qualitative and quantitative*

As a *physics education researcher* teaching intermediate mechanics

The following research questions arise:

- To what extent have students developed a functional understanding of basic concepts in mechanics?
- What prevalent conceptual and reasoning difficulties do students encounter, and to what extent are they based upon:
  - basic concepts?
  - more advanced concepts?
  - connections between physics and mathematical formalism?

Take-home message: Conceptual understanding and reasoning skills must form an <u>essential focus</u> in upper-level mechanics.

Reason #1: Many conceptual and reasoning difficulties *persist* beyond introductory level

At the introductory level, students have difficulty discriminating between a **quantity** and its **rate of change:**

- position *vs.* velocity*
- velocity *vs.* acceleration*
- height *vs.* slope of a graph**
- electric field *vs.* electric potential †
- electric (or magnetic) flux *vs.* change in flux
- …*and many other examples*

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* Trowbridge and McDermott, Am. J. Phys. **48** (1980) and **49** (1981);
Flores and Kanim, Am. J. Phys. **72** (2004); Shaffer and McDermott, Am. J. Phys. **73** (2005).

** McDermott, Rosenquist, and van Zee, Am. J. Phys. **55** (1987).

† Allain, Ph.D. dissertation, NCSU, 2001; Maloney *et al.,* Am. J. Phys. Suppl. **69** (2001).

“Curved ramp” task (2D kinematics)

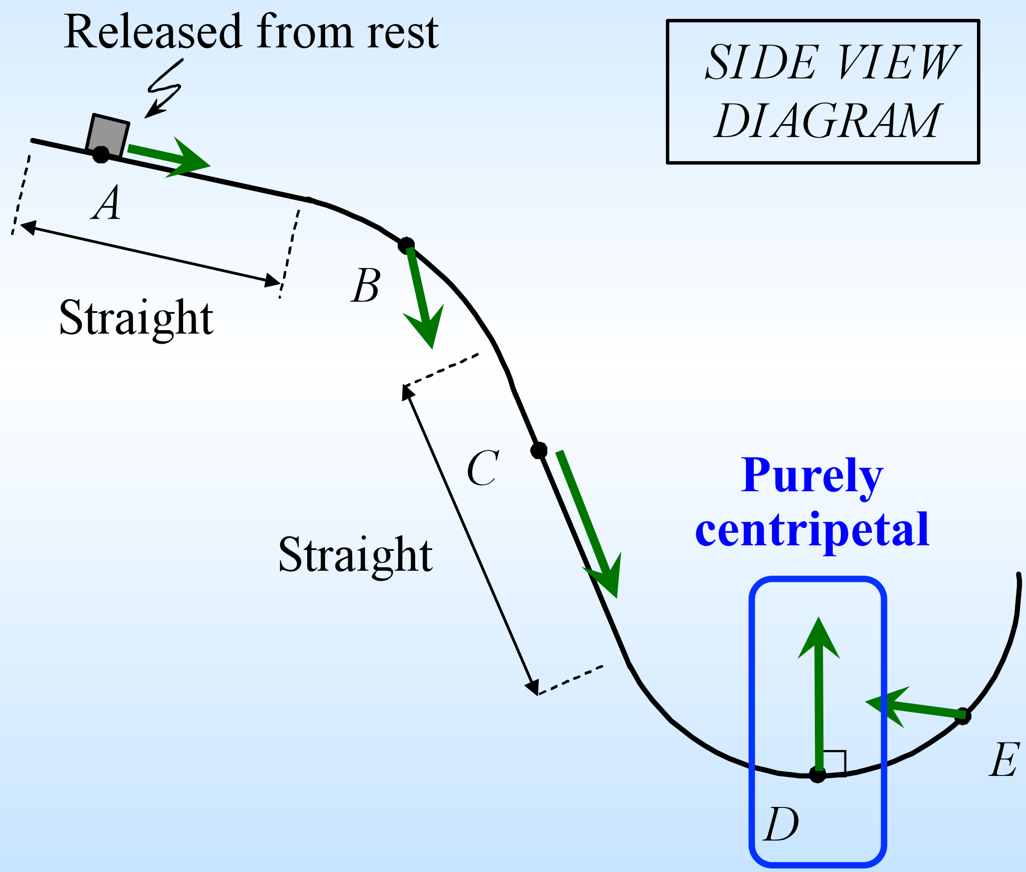
Intermediate mechanics, GVSU

A long, frictionless ramp, consisting of straight and curved portions, is shown. A block is released from rest at point *A*.

At each labeled point, draw arrows to indicate the directions of the (i) ***velocity*** and (ii) ***acceleration*** of the block at that point.

If the acceleration is zero at any point, indicate so explicitly.

Explain your reasoning.

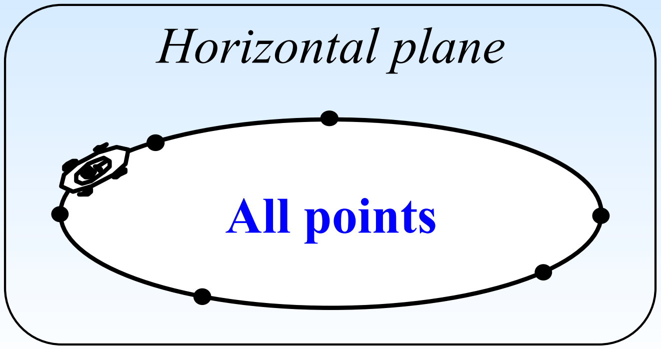


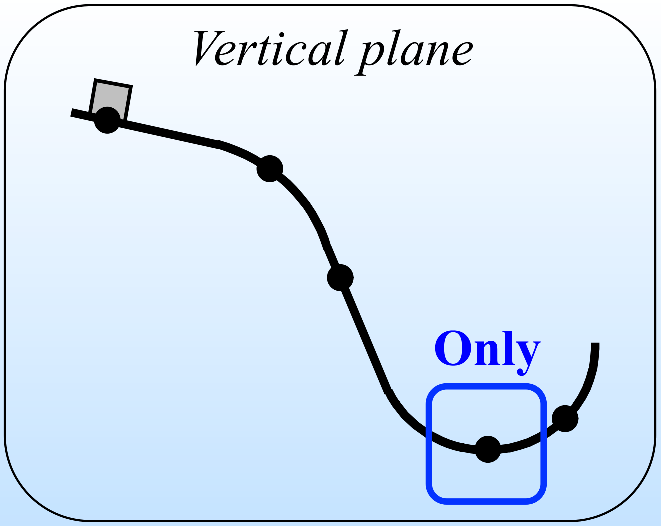
Results from questions after traditional instruction

Acceleration in a horizontal *or* vertical plane

Centripetal cases only

Correct responses with correct reasoning *(to nearest 5%)*	
Algebra-based mechanics, GVSU (*N* ~ 200)	**5%**
Calculus-based mechanics,* UW (*N* ~ 7,000)	**20%**
Intermediate mechanics, GVSU (*N* = 22)	**25%**
Physics graduate students,* UW (*N* ~ 75)	**65%**





* Shaffer and McDermott, Am. J. Phys. **73** (2005).

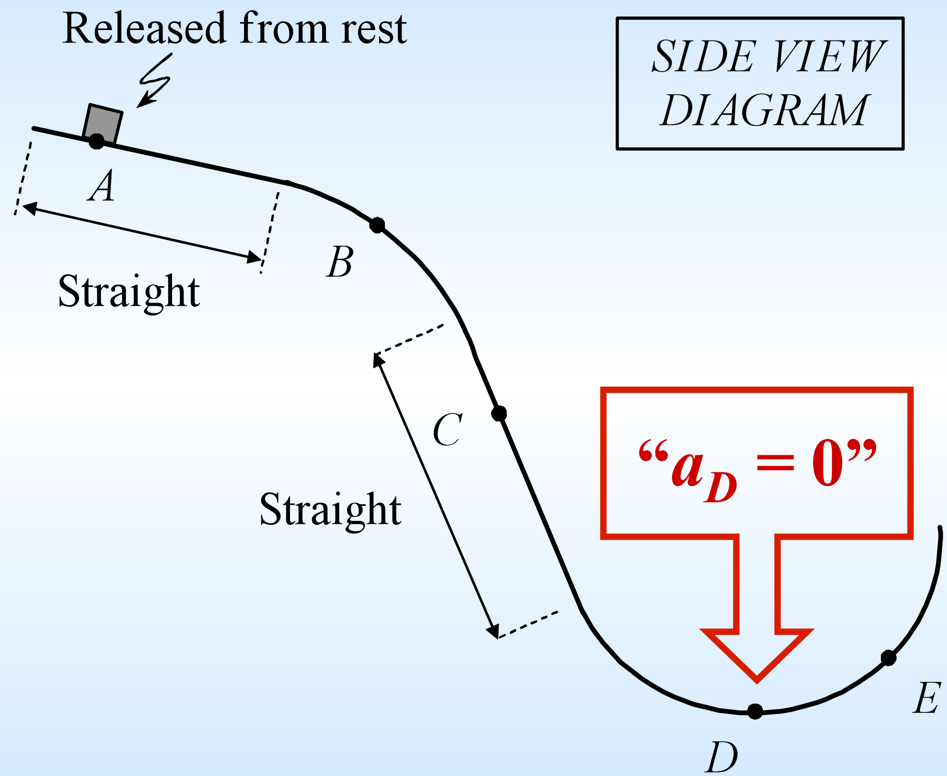
Curved ramp pretest: Student difficulties

Intermediate mechanics, GVSU

Most students incorrectly stated that acceleration is ***zero*** at point *D*.

Example:

"The block is not falling [at point *D*]."



Failure to distinguish between a quantity *(vertical velocity)* ***and its rate of change*** *(vertical acceleration)*

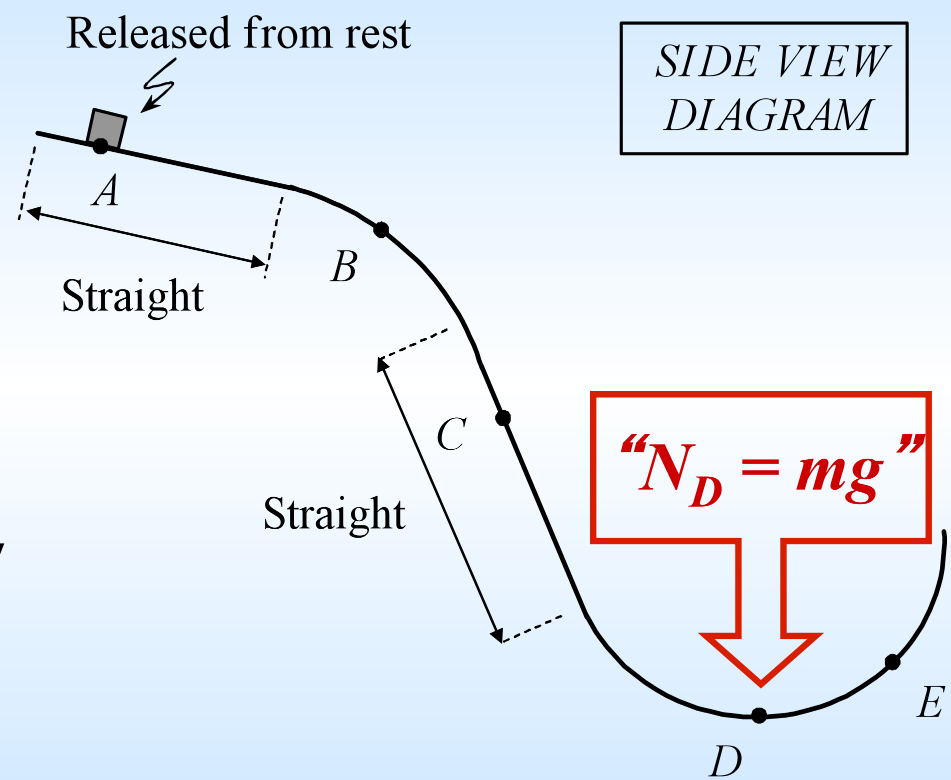
Curved ramp pretest: Student difficulties

Intermediate mechanics, GVSU

Most students incorrectly stated that normal force at point *D* is ***equal to*** weight.

Example:

"[The] block would not stay on the track if these forces were not equal."



Essentially all students who gave incorrect responses for <u>acceleration</u> gave consistent (but incorrect) responses regarding <u>net force.</u>

(See also Flores, Kanim, and Kautz, *Am. J. Phys.* **72** (4), 460-468.)

What we teach **about conservative forces**

A force $\vec{F}(\vec{r})$ is conservative if and only if:

- the work by that force around any closed path is zero
- $\vec{\nabla} \times \vec{F} = 0$ at all locations
- a potential energy function $U(\vec{r})$ exists so that $\vec{F} = -\vec{\nabla} U$

(generalization of $\vec{E} = -\vec{\nabla} V$ from electrostatics)

Research question: What difficulties do students have in understanding and applying this relationship?

"Equipotential map" pretest

Intermediate mechanics

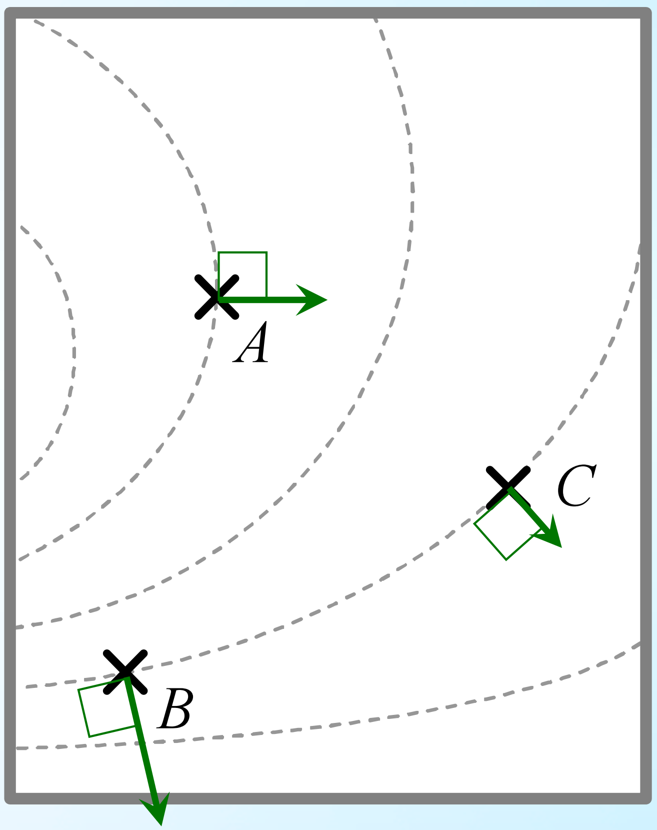
After all lecture instruction in introductory E&M

In the region of space depicted at right, the dashed curves indicate locations of *equal potential energy* for a test charge $+q_{\text{test}}$ placed within this region.

It is known that the potential energy at location *A* is *greater than* that at *B* and *C*.

A. At each location, draw an arrow to indicate the direction in which the test charge $+q_{\text{test}}$ would move when released from that location. Explain.

B. Rank the locations *A, B,* and *C* according to the magnitude of the force exerted on the test charge $+q_{\text{test}}$. Explain your reasoning.



(Qualitatively correct force vectors are shown.)

Equipotential map pretest: Results

Intermediate mechanics, GVSU (N = 73, 8 classes)

After all lecture instruction in introductory E&M

Percent correct *with correct reasoning:*
(rounded to nearest 5%)

Part A (Directions of force vectors)	50%	(35/73)
Part B (Ranking force magnitudes)	20%	(14/73)
Both parts correct	**15%**	**(9/73)**

Similar results have been found after lecture instruction at U. Maine and pilot test sites (N = 115, 11 classes).

Equipotential map pretest: Results

Intermediate mechanics

After all lecture instruction in introductory E&M

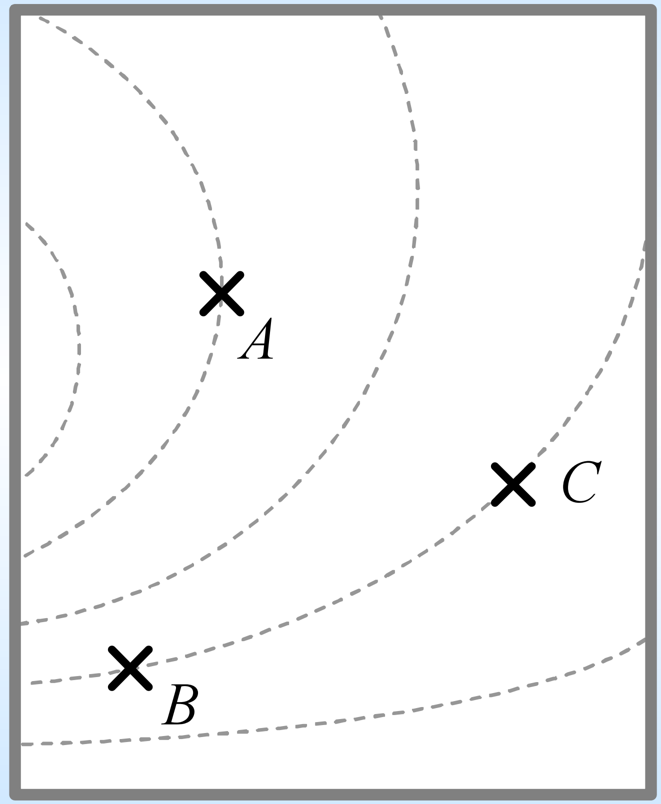
Most common *incorrect* ranking: $F_A > F_B = F_C$

Example: "A has the highest potential, so it can exert a larger force on a test charge. B and C are on the same potential curve and thus have equal abilities to exert force."

Example: "Since F is proportional to V, higher V means higher F."

Example: $[V_A > V_B = V_C]$ … $F(x) = -\,dV/dx$ $\Rightarrow$ $F_C = F_B$ in magnitude and $F_A > F_C$ in magnitude."

Example: "A has the most potential pushing the charge fastest. B & C are on the same level."



Failure to discriminate between a quantity *(potential energy U)* ***and its rate of change*** *(force $\vec{F} = -\vec{\nabla} U$)*

Reason #2: Conceptual and mathematical difficulties are often intertwined

What we teach about harmonic oscillators:

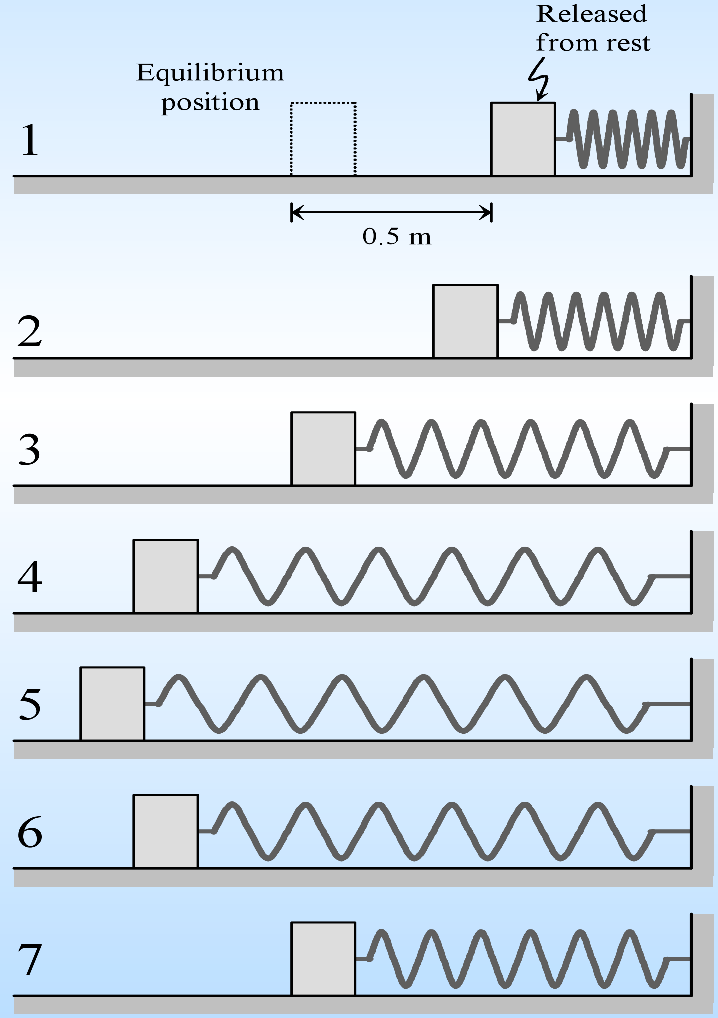
	Equation of motion	Solution for *x(t)*
Simple harmonic motion	$m\ddot{x} = -kx$	$x(t) = A_o \cos(\omega_o t + \varphi)$ where $\boxed{\omega_o = \sqrt{k/m}}$
Underdamped motion $(g < w_o)$	$m\ddot{x} = -kx - c\dot{x}$ $\left(\ddot{x} = -\omega_o^2 x - 2\gamma\dot{x}\right)$	$x(t) = A_o\, e^{-\gamma t} \cos(\omega_d t + \varphi)$ where $\omega_d = \sqrt{\omega_o^2 - \gamma^2}$

Research question: How well do students understand the factors that affect oscillation frequency?

"Simple harmonic oscillator" pretest

(excerpt)

A block is connected to a spring and placed on a frictionless surface. A student releases the block <u>0.5 m</u> to the <u>right</u> of equilibrium.



For each change listed below, how (if at all) would that change affect the **period** of motion? Explain your reasoning.

- The block is released <u>0.7 m</u> to the <u>left</u> of equilibrium.
- The spring is replaced with a stiffer spring.
- The block is replaced with another block four times the mass as the original one.

Predicting effect on oscillation frequency

After lecture instruction (GVSU, 6 classes, $N \sim 50$)

The good news…

Parts ii & iii (changing ***spring*** or ***mass***):

Most students **(~ 65%)** gave correct answers with acceptable explanations.

The bad news…

Part i (increasing ***amplitude***):

Most students answered correctly (~ 65%) but very few gave acceptable explanations.

Most common incorrect* **(~25%):** "Larger [period], because the block travels farther during each period."

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* See also: B. Frank, S. Kanim, and L. Gomez, *Phys. Rev. ST Phys. Educ. Res.* **4** (2008), "Accounting for the variability in student responses to motion questions."

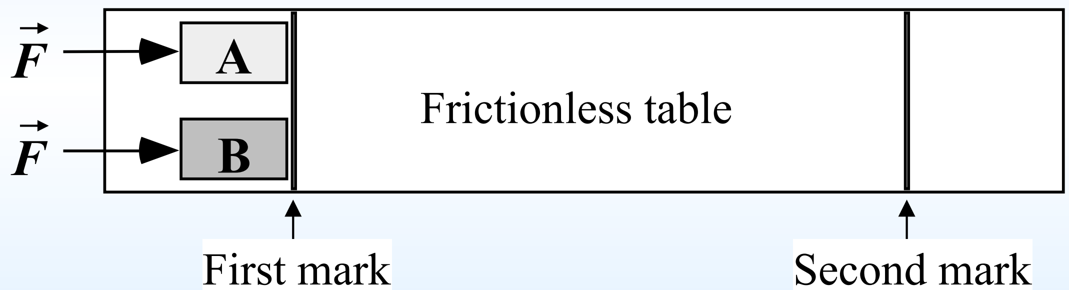
From previous research at the introductory level

Students use inappropriate **“compensation arguments”** when comparing quantities that involve two or more variables.

Example: Two carts, $m_A < m_B$, are at rest on a level, frictionless table.

Equal forces are exerted on the carts as they move between the two marks.*

Top view diagram



After instruction, introductory physics students often incorrectly predict:

“$p_A = p_B$” because larger mass of B “compensates for” smaller speed ($p = mv$)

“$KE_A > KE_B$” because faster speed of A “matters more” than mass ($KE = ½mv^2$)

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* R.A. Lawson and L.C. McDermott, *Am. J. Phys.* **55** (1987); O’Brien Pride, Vokos, and McDermott, *Am. J. Phys.* **66** (1998).

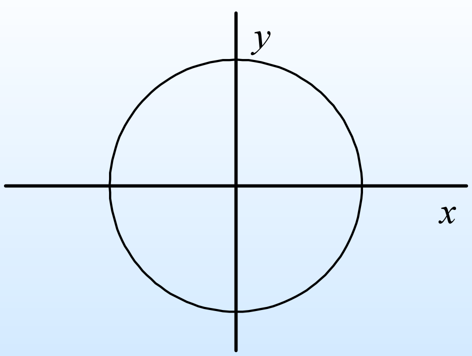
2D oscillator pretest

Consider the motion of a 2D oscillator, with $U(x, y) = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 y^2$, or equivalently, $U(x, y) = \frac{1}{2} m\omega_1^2 x^2 + \frac{1}{2} m\omega_2^2 y^2$.

Q: For each *x*-*y* trajectory shown, could the oscillator follow that trajectory?

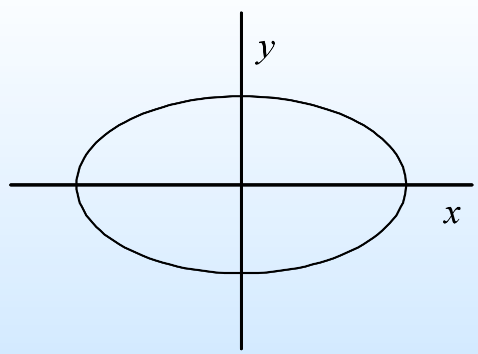
If so: Is ω_1 *greater than, less than,* or *equal to* ω_2*?* Explain.*

If not: Explain why not.



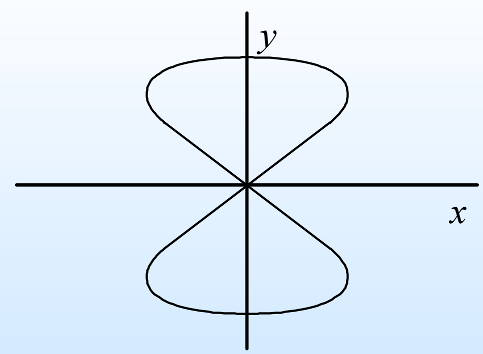
Case #1

(Ans: $\omega_1 = \omega_2$)



Case #2

(Ans: $\omega_1 = \omega_2$)



Case #3

(Ans: $\omega_1 > \omega_2$)

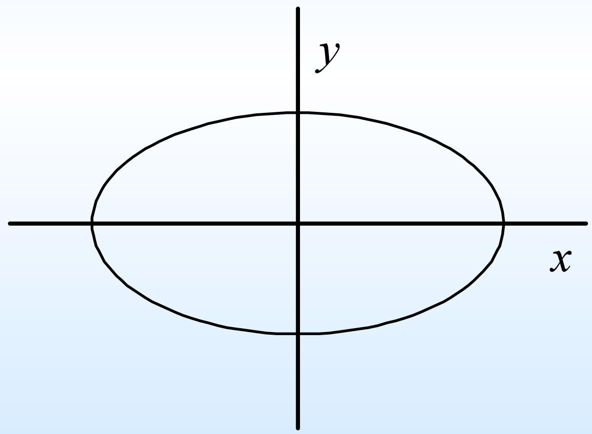
* Original phrasing asked for a comparison between k_1 and k_2.

2D oscillator pretest: Results

Intermediate mechanics, GVSU (4 classes) and UME (1 class)

After relevant lecture instruction

- Few students **(0% - 15%)** answered all cases correctly.
- Most incorrect responses based on compensation arguments* involving relative amplitudes along *x*- and *y*-axes:



Case #2

Example responses for Case #2:

"$k_1 < k_2$, the spring goes farther in the *x*-direction, so spring must be less stiff in that direction."

"$w_2 > w_1$. Since we now have an oval curve with the *x*-axis longer, w_2 must be greater to compensate."

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* R.A. Lawson and L.C. McDermott, *Am. J. Phys.* **55** (1987); O'Brien Pride, Vokos, and McDermott, *Am. J. Phys.* **66** (1998).

Alternate version of 2D oscillator pretest

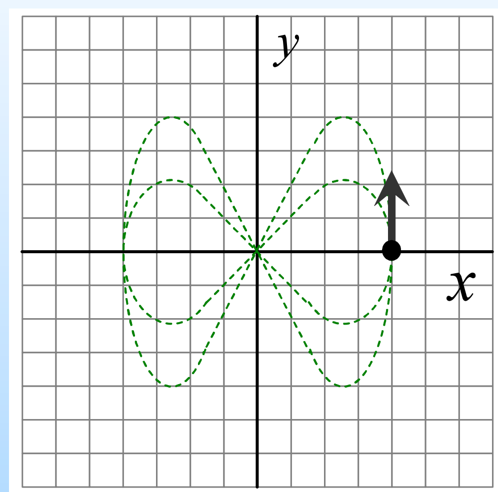
Consider an object that moves along a horizontal frictionless surface (*e.g.,* an air hockey puck on a level air table). Suppose that the object moves under the influence of a net force expressed as follows:

$$\mathbf{F}_{\mathbf{net}}(x,y) = \left(-k_x x\,\hat{i}\right) + \left(-k_y y\,\hat{j}\right)$$

Note: The above net force can be modeled by two long, mutually perpendicular springs with force constants k_x and k_y.

Q: For each case, carefully sketch a qualitatively correct *x-y* trajectory for the object. Explain your reasoning.

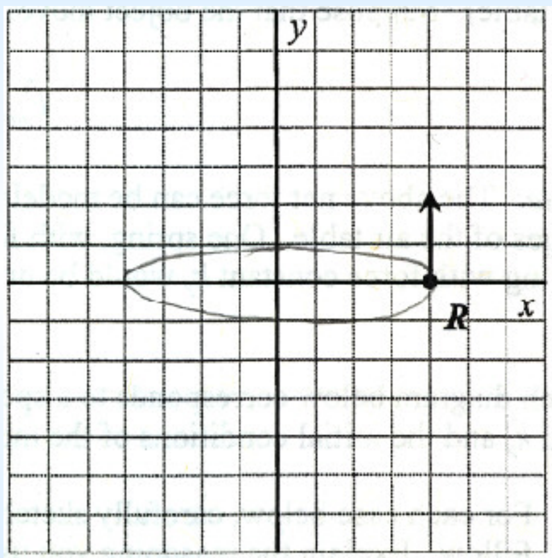
Example non-isotropic case, $\boldsymbol{k_y = 4k_x}$**:**



Alternate 2D oscillator pretest: Results

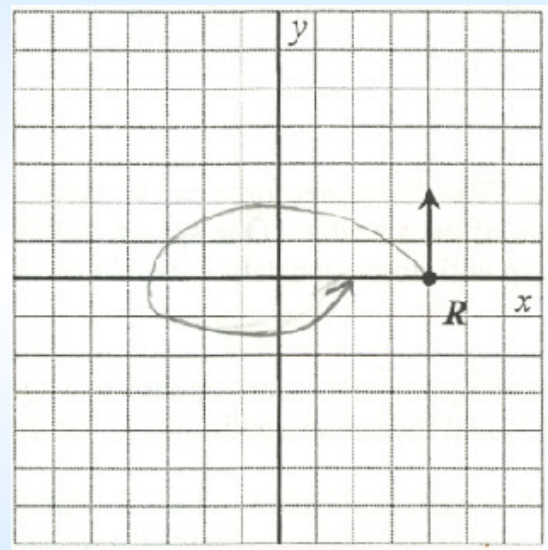
GVSU (2 classes) and pilot site (1 class)

"Compensation arguments" with amplitudes and force constants:



$k_y = 4k_x$

"An ellipse rather than a circle because the spring forces are different."



$k_y = 4k_x$

"The object travels less in the *y*-direction because of the stiffer spring. The springs attempt to return the object to equilibrium."

Reason #2: Conceptual and mathematical difficulties are often intertwined

What we teach about harmonic oscillators:

	Equation of motion	Solution for *x(t)*
Simple harmonic motion	$m\ddot{x} = -kx$	$x(t) = A_o \cos(\omega_o t + \varphi)$ where $\omega_o = \sqrt{k/m}$
Underdamped motion $(g < w_o)$	$m\ddot{x} = -kx - c\dot{x}$ $\left(\ddot{x} = -\omega_o^2 x - 2\gamma\dot{x}\right)$	$x(t) = \boxed{A_o\, e^{-\gamma t}} \cos(\omega_d t + \varphi)$ where $\boxed{\omega_d = \sqrt{\omega_o^2 - \gamma^2}}$

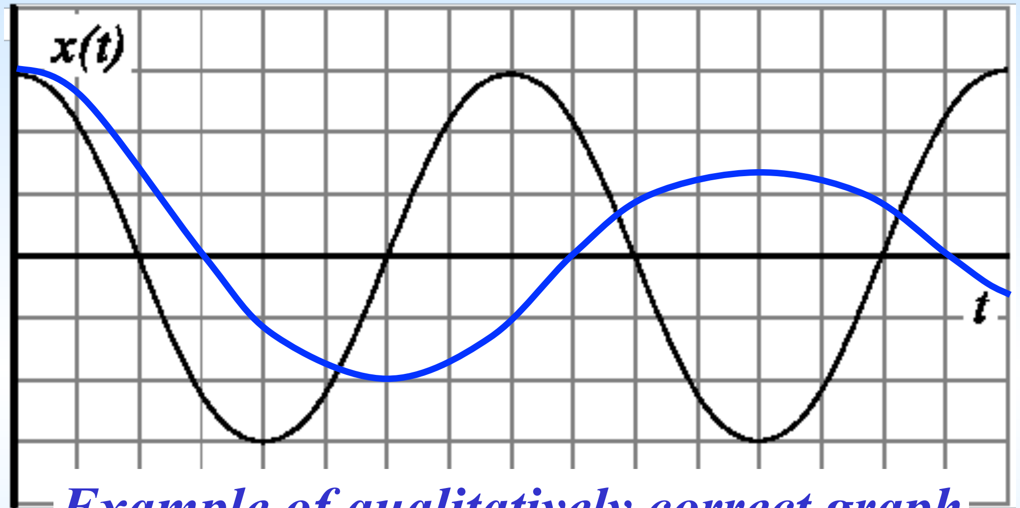
- Damping force lowers oscillation frequency ($\boldsymbol{w_d < w_o}$)
- Damping force causes amplitude to decrease over time, with *constant* ratio between successive maxima

"Underdamped oscillator" pretest

(excerpt)

The x vs. t graph represents the motion of a simple harmonic oscillator that is released from rest at $t = 0$.

A. Clearly indicate and label (i) amplitude, (ii) period. Explain your reasoning.



Example of qualitatively correct graph

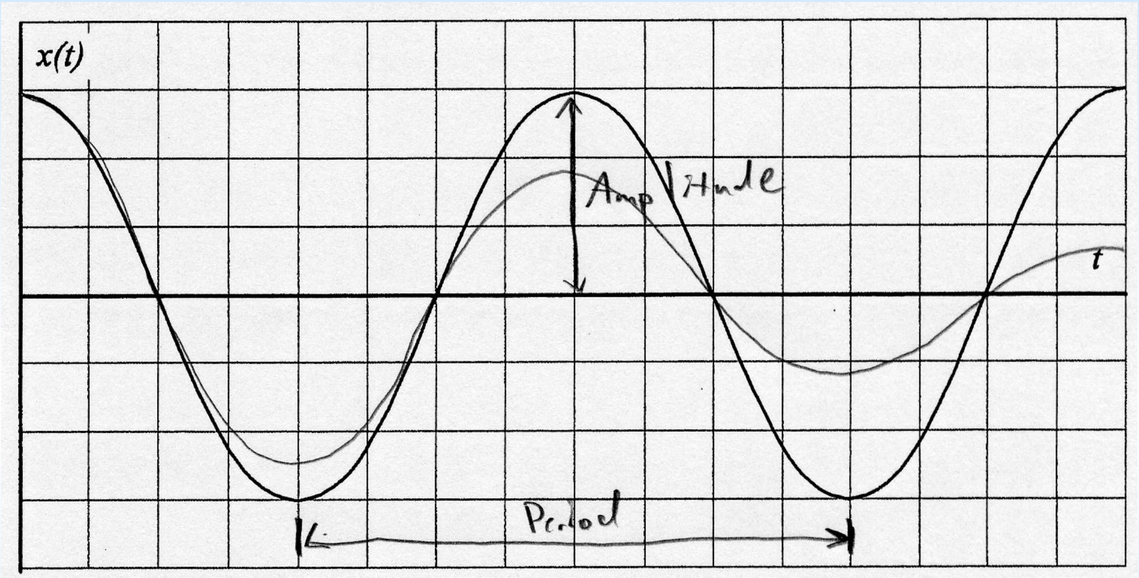
B. Suppose that a retarding force were applied to cause the oscillator to become underdamped.

On the axes above, sketch a qualitatively correct x vs. t graph for the oscillator when it is released *from rest* at the *same initial position as before.* Explain how you decided to draw the graph the way you did.

Underdamped oscillator pretest: Results

After lecture instruction, GVSU (5 classes) and pilot sites (9 classes)

Most common *incorrect* response (60% - 70%):

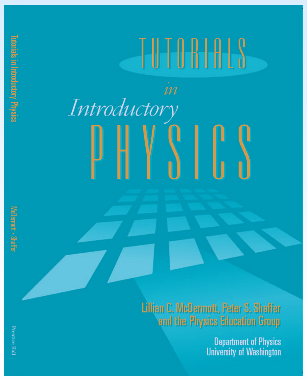


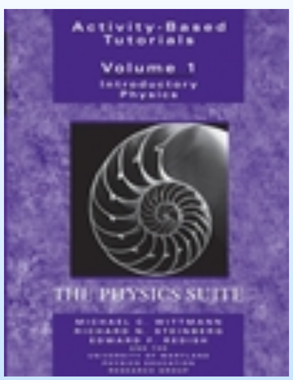
"The amplitude of the underdamped oscillator decreases exponentially, but the period of motion remains the same.

"The amplitude will shrink with time but the period shouldn't change since they are independent of each other."

Failure to recognize that damping force affects frequency

Reason #3: Specific conceptual and reasoning difficulties must be *directly* addressed





A research-tested guided-inquiry approach for supplementing lectures in *introductory physics:*

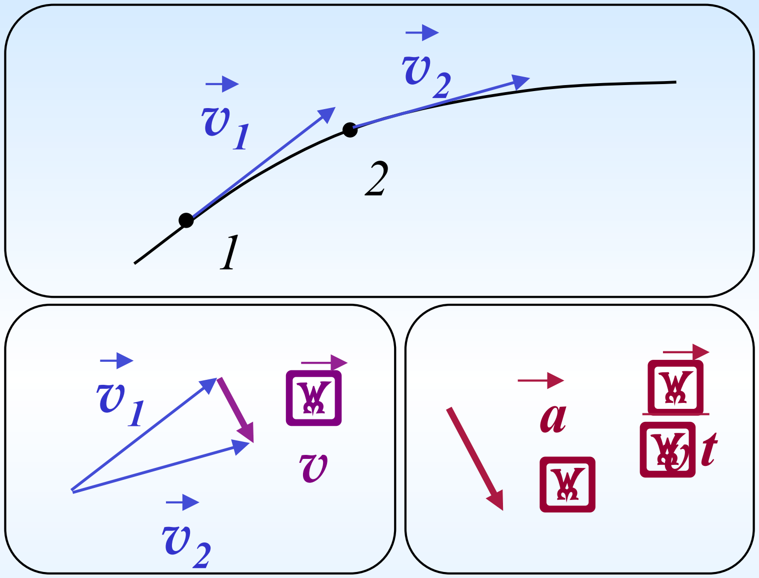
- Teaching-by-questioning strategies designed to:
  - address specific conceptual and reasoning difficulties
  - help students connect the mathematics to physics

- Tutorial components:
  - pretests (ungraded quizzes, ~10 min)
  - tutorial worksheets (small-group activities, ~50 min)
  - tutorial homework
  - examination questions (post-tests)

Conceptual review in intermediate mechanics*

GVSU, 2001 – present

- ***Motion in two dimensions***
  - Operational definition of acceleration
  - Homework includes examples in horizontal plane and in vertical plane



- ***Newton's 2nd and 3rd laws***
  - New homework questions added to reinforce and extend results from *Motion in two dimensions*

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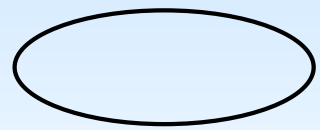
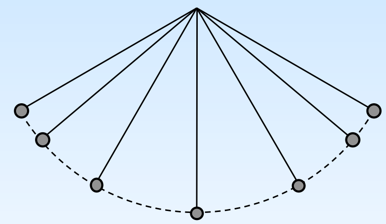
* Adapted from *Tutorials in Introductory Physics* (McDermott, Shaffer, and PEG at U. Wash.)

Results from questions before and after tutorial instruction

		
	Pretest	**Post-test**
Correct responses with correct reasoning:	Centripetal cases only	Centripetal & tangential cases (ignoring endpoints)
Calculus-based mechanics, UW ($N \sim 700$)	**20%**	**60%**
Intermediate mechanics, GVSU ($N = 22$)	**25%**	**90%**
Physics graduate students, UW ($N \sim 75$)	**65%**	**----**

Prevalent questions in physics education research *(and not just in advanced topics)*

- When conceptual or reasoning difficulties arise, do student ideas tend to be:
  - robust and deeply-seated?
  - based on naïve intuitions?

- Which instructional strategies seem to be productive in addressing such difficulties, and under what circumstances?
  - *elicit-confront-resolve*[1]
  - building and/or refining intuitions[2]

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[1] McDermott, *Am. J. Phys.* **61** (1993), 295 – 298.

[2] Elby, *Am. J. Phys.* Phys. Ed. Res. Suppl. **69** (2001), S54 – S64.

*Intermediate Mechanics Tutorials**

Collaboration between GVSU (Ambrose) and UME (Wittmann)

- Newton's laws and velocity-dependent forces
- Simple harmonic motion
- Damped harmonic motion
- Driven harmonic motion
- Phase space diagrams
- Conservative force fields
- Harmonic motion in two dimensions
- Accelerating reference frames
- Orbital mechanics
- Generalized coordinates and Lagrangian mechanics

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* Development and dissemination support by NSF grants DUE-0441426 and DUE-0442388

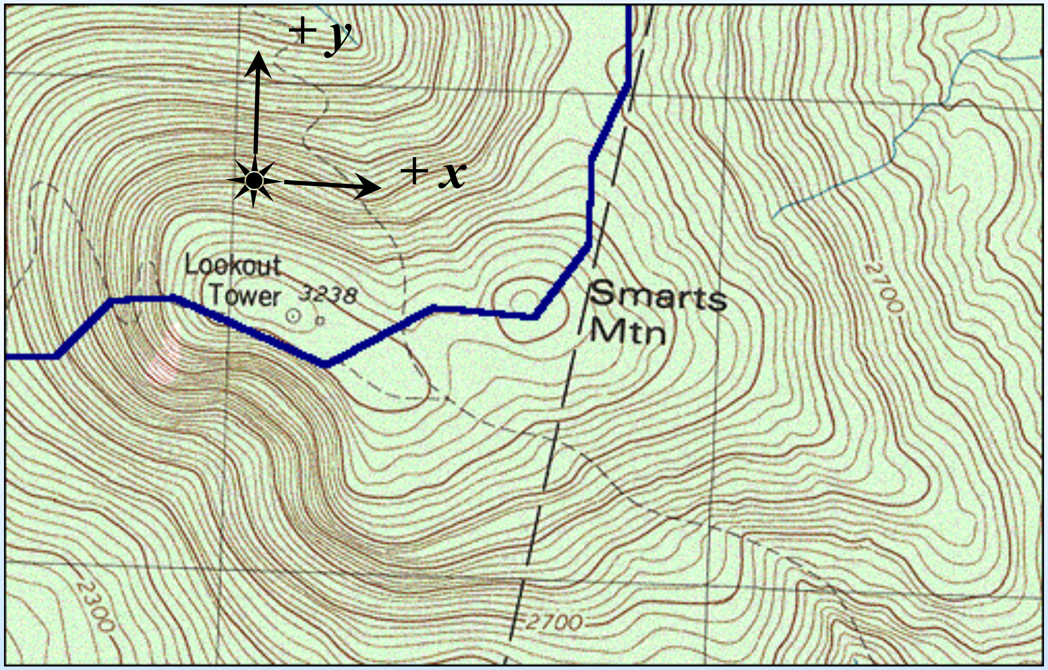
Helping students connect meaning between the physics and the mathematics

In the tutorial *Conservative forces and equipotential diagrams:*

Students develop a qualitative relationship between **force vectors** and local **equipotential contours**…

…and construct an **operational definition of the gradient** of potential energy:

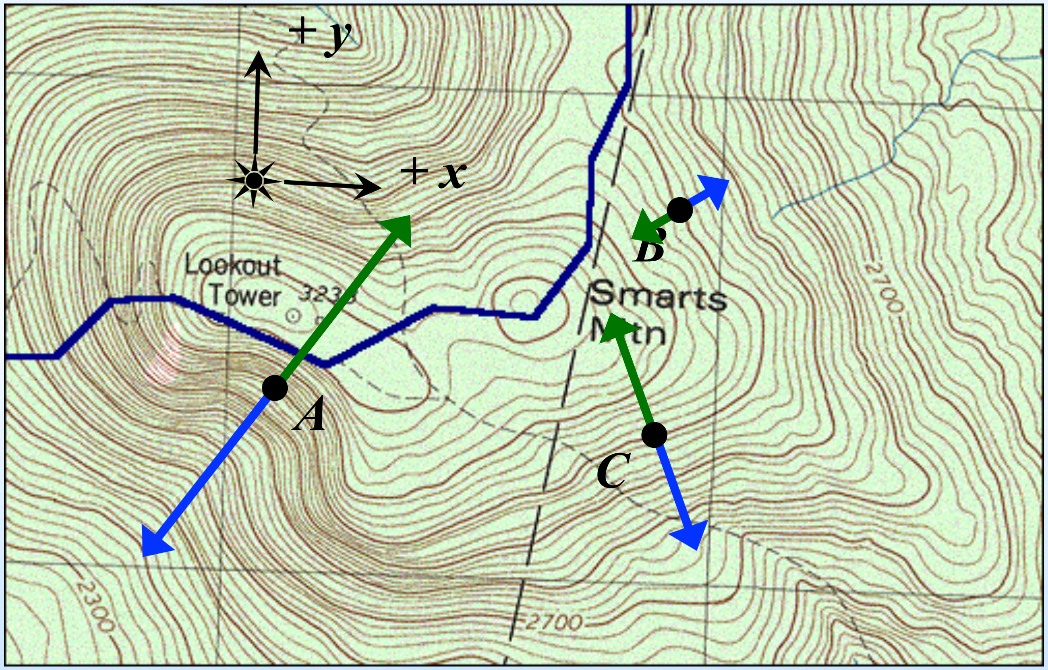
$$\vec{\nabla} U = \left(\frac{\partial U}{\partial x} \hat{i} + \frac{\partial U}{\partial y} \hat{j} \right)$$



Helping students connect meaning between the physics and the mathematics

Students construct operational definition of *gradient:*

- *In words,* how would you calculate $\frac{\partial U}{\partial x}$ and $\frac{\partial U}{\partial y}$?
- Is $\frac{\partial U}{\partial x}$ pos, neg, or zero?
- Is $\frac{\partial U}{\partial y}$ pos, neg, or zero?
- Compare $\left|\frac{\partial U}{\partial x}\right|$ and $\left|\frac{\partial U}{\partial y}\right|$.
- Draw $\vec{\nabla} U = \left(\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j}\right)$.



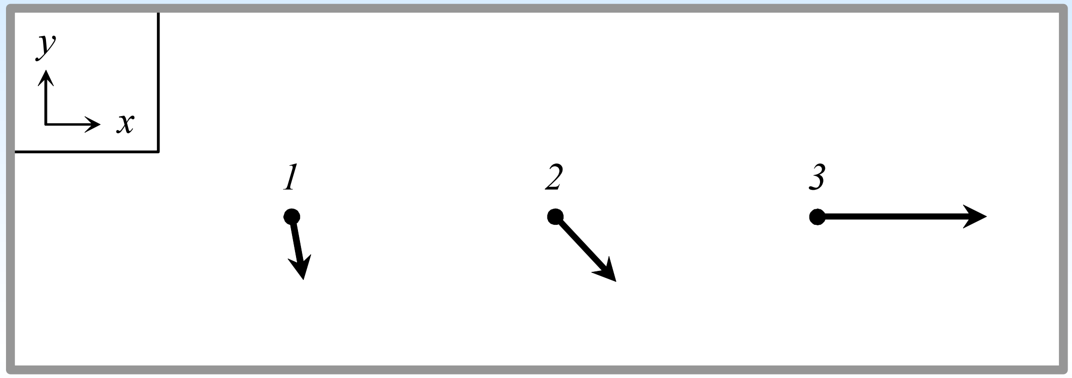
$\vec{F} \longrightarrow$ $\vec{\nabla} U \longrightarrow$

"Unknown equipotentials" post-test

Exam after tutorial, GVSU 2003 ($N = 7$)

Three identical particles are located at the labeled locations *(1, 2, and 3)*.

Each vector represents the force ***F****(x, y)* exerted at that location, with:



$$F_3 > F_2 > F_1$$

A. In the space above, *carefully sketch an equipotential diagram* for the region shown. Make sure your equipotential lines are consistent with the force vectors shown. Explain the reasoning you used to make your sketch.

B. On the basis of your results in part A, rank the labeled locations according to the *potential energy* of the particle at that location. Explain how you can tell.

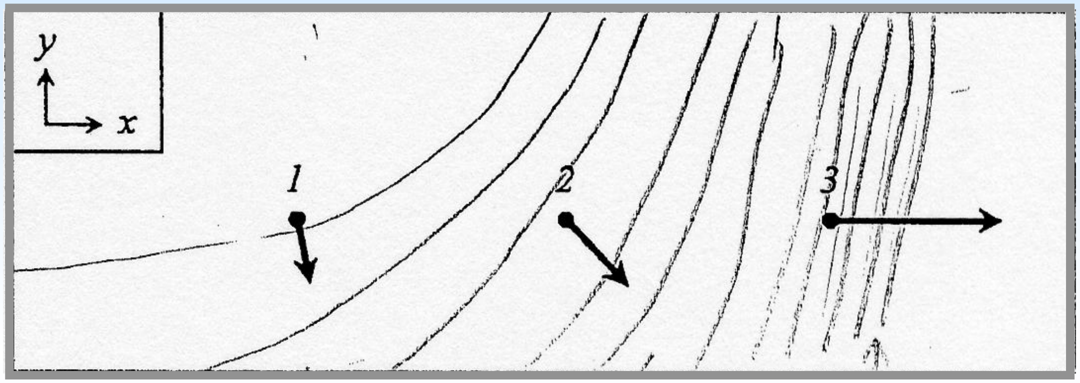
“Unknown equipotentials” post-test: Results

Exam after tutorial, GVSU 2003 (N = 7)

Three identical particles are located at the labeled locations *(1, 2, and 3)*.

Each vector represents the force ***F****(x, y)* exerted at that location, with:

$$F_3 > F_2 > F_1$$



Acceptable student diagram *(part A)*

Part A: Relative spacing of equipotentials: **4/7 correct**

Orientation of equipotentials: **5/7 correct**

Part B: Rank points by potential energy: **1/7 correct**

"Unknown equipotentials" post-test: Results

Exam after tutorial, GVSU 2003 ($N = 7$)

Example of a partially correct response:

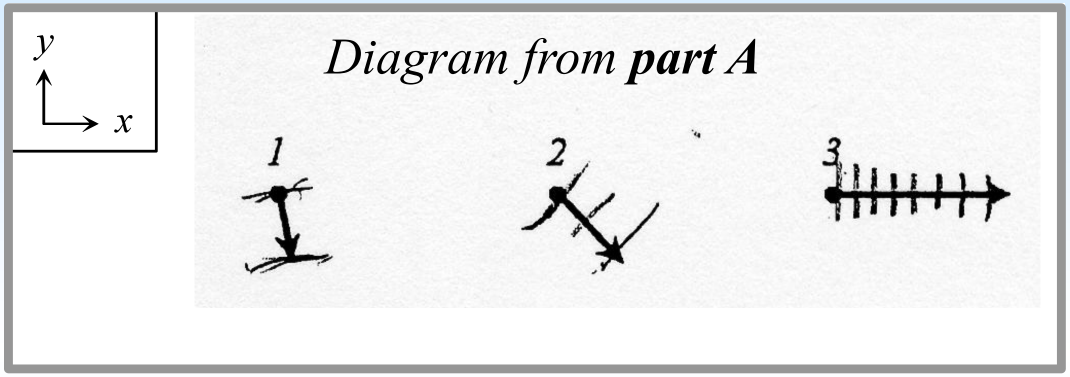


Diagram from ***part A***

Part B *(rank points by potential energy):* 3 > 2 > 1

The greater the force, the higher potential energy $\vec{F} = -\nabla V$

Persistent *confusion between a quantity (potential energy U) and its rate of change (force $\vec{F} = -\vec{\nabla} U$)*

Helping students connect meaning between the physics and the mathematics

Tutorial concludes with students reflecting upon what gradient *means* **<u>and</u>** what it *does not mean:*

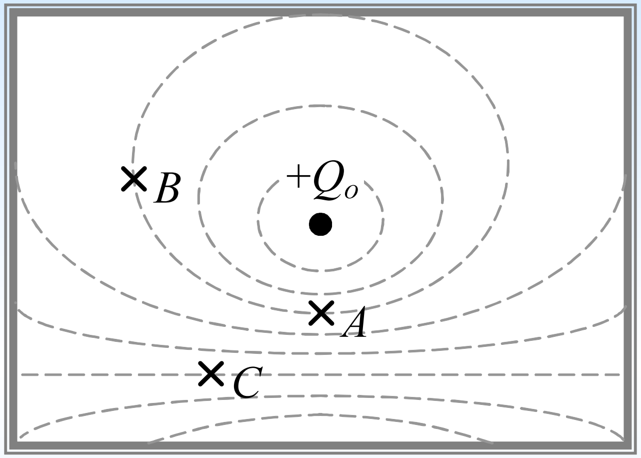
Summarize your results: Does $\vec{\nabla} U$ ...

- point in the direction of *increasing* or *decreasing* potential energy?
- point in the direction in which potential energy changes the *most* or the *least* with respect to position?
- **have the *same magnitude* at all locations having the *same potential energy?* Explain why or why not.**

Examples of assessment questions

On written exams after modified instruction

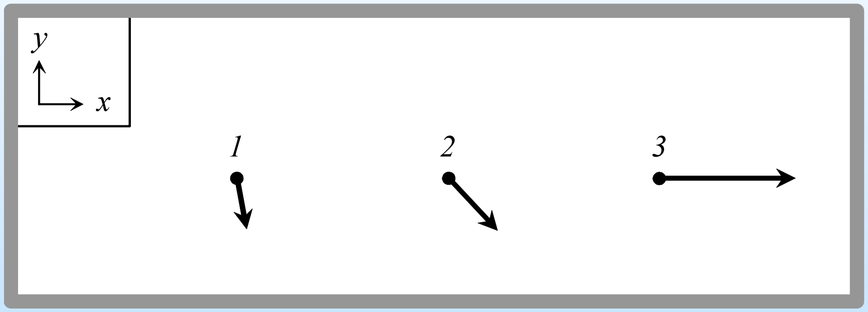
Task: Given equipotential map, predict directions and relative magnitudes of forces.



GVSU: **85% correct** (2 classes)

SPU: **75% correct** (1 class)

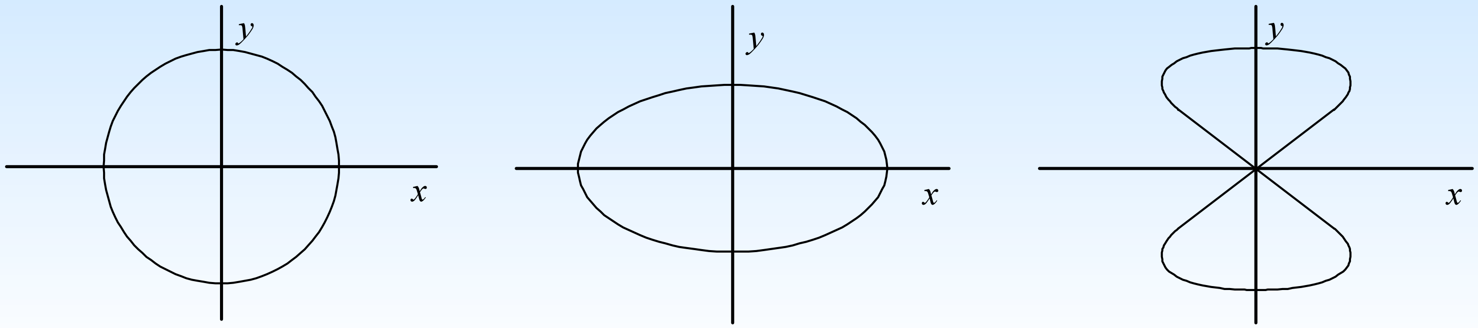
Task: Given several force vectors, sketch possible equipotential map and rank points by potential energy.



GVSU: **50% correct** (3 classes)

Pilot sites: **50 - 65% correct** (4 classes)

Helping students build and refine productive intuitions about the physics



Case #1 Case #2 Case #3

In the tutorial *Harmonic motion in two dimensions,* students are guided to recognize:

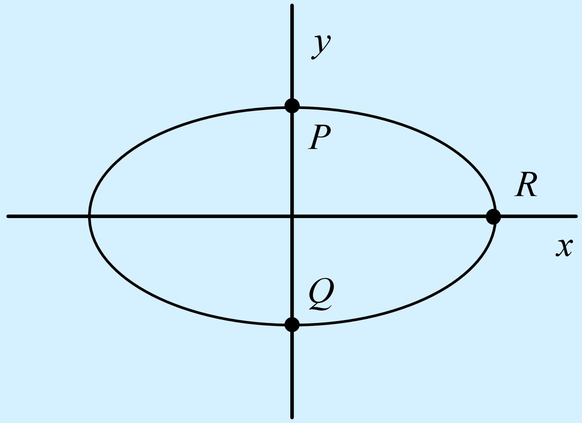
- how many oscillations occur along the *y*-axis for each oscillation along the *x*-axis
- how differences in force constants affect periods and frequencies
- how phase difference between *x*- and *y*-motions affect trajectories of isotropic oscillators

Students are guided to connect amplitude to potential energy (<u>not</u> frequency)

Excerpt from tutorial homework—revised in 2003—from *Harmonic motion in two dimensions:*

A. Critique the following statement. Explain.

"The oscillator goes farther in the *x*-direction than in the *y*-direction. That means the spring in the *y*-direction must be stiffer than the spring in the *x*-direction."



B. Rank points *P, Q,* and *R* according to (i) total energy, (ii) potential energy, (iii) kinetic energy.

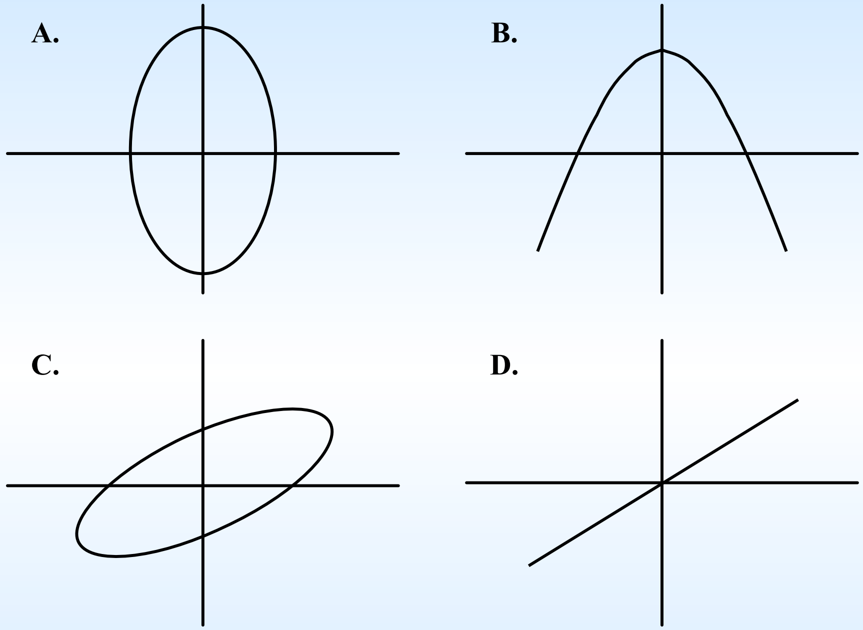
Explain how the difference in the *x*- and *y*-amplitudes, used *incorrectly* in the statement in part A, can help justify a *correct* answer here in part B.

Examples of assessment questions

On written exams <u>after</u> modified instruction (GVSU)

Qualitative: "Is k_x *greater than, less than,* or *equal to* k_y? Explain."

Quantitative: "Evaluate the ratio k_y/k_x. Show all work."

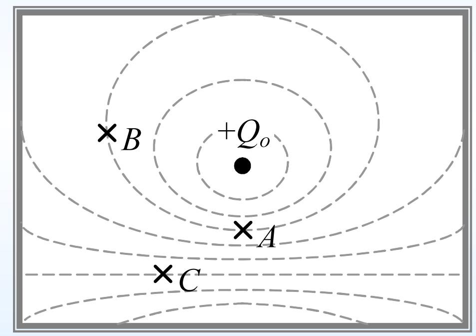


<u>Before</u> revised tutorial HW ('01 – '02): ≈ 50% correct

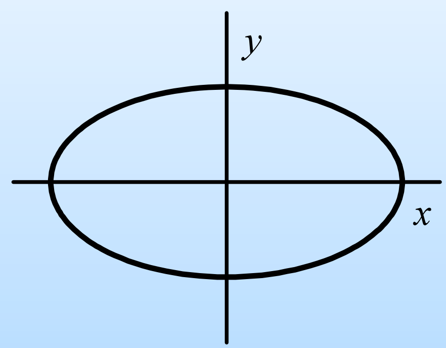
<u>After</u> revised tutorial HW ('03 – present): **≈ 90% correct**

Summary and reflections

- Physics majors in *advanced* courses can and do experience conceptual and reasoning difficulties similar in nature to those already identified at the *introductory* level.
  - Difficulty discriminating between a **quantity** and its **rate of change**



  - Reliance on inappropriate **“compensation arguments”** with two or more variables



Summary and reflections

- Students need guidance to extract physical meaning from the mathematics.
  - **Guided sense-making** seems more important than derivations.
  - Students need practice articulating **in their own words** the physical meaning expressed in the *graphical representations* and in the *mathematics* they use.
- Specific difficulties must be addressed *explicitly* and *repeatedly* for meaningful learning to occur.
  - Assessments of **conceptual underpinnings** should be done explicitly and repeatedly.

Summary and reflections

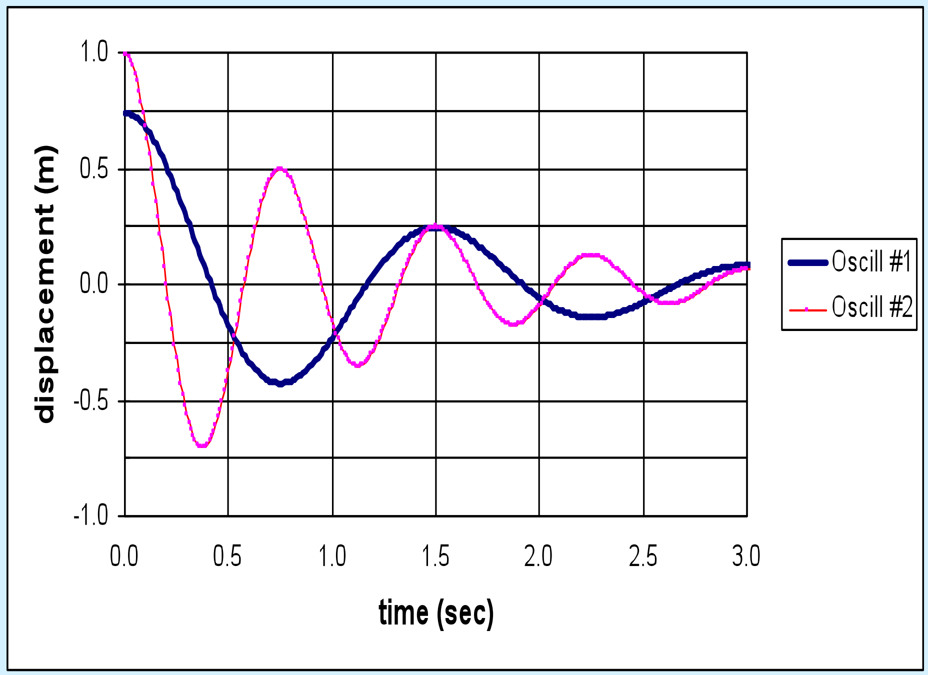
- Intermediate mechanics offers rich opportunities for exploring how students navigate the interplay between math and physics.

Q: Which oscillator, if any, has:

- the larger *damping constant (g)?*
- the larger *quality factor?*

Q: Use the graph for oscillator #1 (blue) to deduce values of *a* and *b:*

$$\ddot{x} + a\dot{x} + b = 0$$

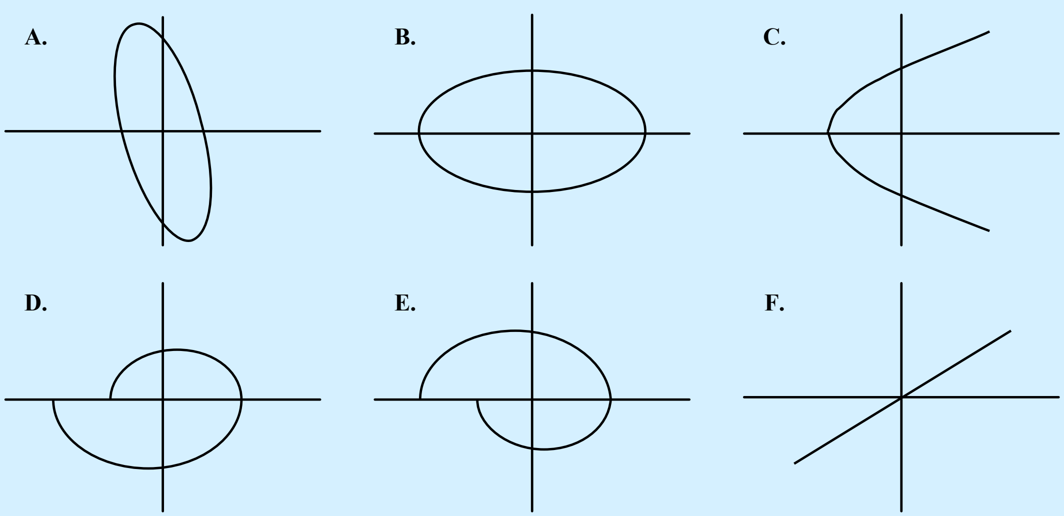


Summary and reflections

- Intermediate mechanics also offers context in which to assess coherence and organization of student knowledge.

Identify which diagram(s), if any, could be:

i. phase space plot of a simple harmonic oscillator
ii. phase space plot of an underdamped oscillator
iii. trajectory of a 2-D oscillator for which $k_y > k_x$
iv. trajectory of a 2-D oscillator for which $k_y = k_x$



Intermediate Mechanics Tutorials

Project website:

http://faculty.gvsu.edu/ambroseb/research/IMT.html

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Supported by NSF grants DUE-0441426 and DUE-0442388

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