Using Relative Amplitude and Travel Times from Geometric Acoustics to Determine Nocturnal Effective Sound Speeds

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Background
Nocturnal Atmospheric Conditions
Impulse Propagation Data

Ray Theory Analysis
Mathematical Formulation
Solving the Eikonal Equation
Solving the Transport Equation

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Conclusions and Future Work
Impulse Propagation Through the Lower Atmosphere

**Atmosphere Data**
Atmospheric data: Locke Station, November 7, 2006, 18:15

- Downwind propagation in a temperature inversion has been studied previously using full wave and geometric modeling.
- Geometric analysis consisted of celerity (travel time) and arrival ranges with respect to launch angle of ray path.
- Current work includes additional analysis of the amplitude and waveforms predicted by geometric acoustics.
Impulse Propagation

**Full wave model predictions**

- The faster, distinct arrivals are predicted by geometric acoustics and the timing of the arrivals are in agreement with the full wave model, but the low frequency narrow-band tail corresponds with the lowest order mode and is predicted only in the full wave model.

- Analysis of caustic encounters following reflections combined with complex ground impedance were used to produce waveforms of the correct shape.

- Without a higher order solution, waveform shape can not be inferred from geometric acoustics (amplitude and exact caustic structure remain unknown).

**Acoustic Data**

1st Pulses, lowest mic

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The form of the complex pressure amplitude

Assume complex pressure amplitude can be written as a Debye series with a complex phase

\[ \hat{p}(\bar{x}, k_0) = e^{ik_0\psi(\bar{x})} \sum_{m=0}^{\infty} \frac{P_m(\bar{x})}{(ik_0)^m} = P_0(\bar{x}) e^{ik_0\psi(\bar{x})} + \ldots \]

High frequency approximation: solve in powers of \((k_0)^{-1}\).

\[ O(k_0^2) \rightarrow |\nabla \psi|^2 = \frac{c_0^2}{c_{\text{eff}}^2} = n^2 \]

\[ O(k_0) \rightarrow 2\nabla \psi \cdot \nabla P_0 - P_0 \nabla \psi \cdot \nabla \ln \rho_0 + P_0 \nabla^2 \psi = 0 \]
Ray paths and travel times

\[ \hat{p}(\bar{x}, k_0) = P_0(\bar{x}) e^{ik_0 \psi(\bar{x})} + \ldots \]

\[ |\nabla \psi|^2 = \frac{c_0^2}{c_{\text{eff}}(\bar{x})} = n^2(\bar{x}) \]

By defining \( \nabla \psi \equiv \bar{\nu} \), the Eikonal equation can be written as a Hamiltonian function \( H(\bar{x}, \bar{\nu}) = 0 \) solvable by Hamilton-Jacobi equations.

\[ \frac{\partial \bar{x}}{\partial \tau} = \bar{\nu}, \quad \frac{\partial \bar{\nu}}{\partial \tau} = \frac{1}{2} \nabla \frac{c_0^2}{c_{\text{eff}}(\bar{x})} \]

Change of variable to ray length:

\[ ds = |d\bar{x}| = \frac{c_0}{c_{\text{eff}}(\bar{x})} d\tau. \]

Travel time:

\[ t(s) = \frac{\psi(s)}{c_0}, \quad t(s) = \int_0^s \frac{1}{c_{\text{eff}}(\bar{x}')} ds' \]

Solution of the Eikonal allows calculation of both arrival \textit{geometry} and \textit{timing}. 
Ray Paths for the Nocturnal Boundary Layer

Eigenray geometry for **fast arrival** and **convergent zones**.

- Ray theory predicts wind jet induced structure.
- Multiple direct paths exist in the convergent zone.
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Ray Theory Analysis

Solving the Transport Equation

Acoustic pressure along a ray

\[ \hat{p}(\bar{x}, k_0) = P_0(\bar{x}) e^{ik_0\psi(\bar{x})} + \ldots \]

\[ 2\nabla \psi \cdot \nabla P_0 - P_0 \nabla \psi \cdot \nabla \ln \rho_0 + P_0 \nabla^2 \psi = 0 \quad \text{with} \quad \vec{v} \cdot \nabla = \frac{c_0}{c_{\text{eff}}} \frac{\partial}{\partial s} \]

\[ P_0(s, \theta) = \frac{1}{4\pi} \left\| \frac{\rho_0(s) c_{\text{eff}}(s) \cos \theta}{\rho_0(0) c_{\text{eff}}(0) D(s, \theta)} \right\|^\frac{1}{2} \]

So what is \( D(s, \theta) \)?

\[ D(s, \theta, \phi) = \frac{\partial (x, y, z)}{\partial (s, \theta, \phi)} = r(s, \theta) \left| \frac{\partial r}{\partial s} \frac{\partial z}{\partial \theta} - \frac{\partial r}{\partial \theta} \frac{\partial z}{\partial s} \right| \]

Ray Coordinates

Singularity of the transform - caustic

- Approximation fails at caustic
- Passage through caustic introduces Hilbert Transform
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- Ray Theory Analysis
- Solving the Transport Equation

### Acoustic Pressure Along a Ray

**Jacobian at Arrival**

- Additional information from the transport equation gives amplitude and caustic structure
- Turning points bounding the convergence zone are caustic structures at ground level

![Graph showing acoustic pressure along a ray with labels for range and height, as well as a color scale for Log[p(s)/p(10 m)].](image)
Choosing a Form for the Effective Sound Speed Profile

**Data Inversion Application:** the nocturnal boundary layer near the ground can be modeled by a Taylor series expansion

\[ c(z) = c_0 + \alpha z + \beta z^2 + \ldots \]

Variations in \( \alpha \) and \( \beta \) give a “space” of possible atmospheric states.

For a given receiver range, each pair of \((\alpha, \beta)\) produces a characteristic direct and single reflection phase which can be described by their relative travel times and amplitudes.
Minimizing the Error

Sample Case - Run 4 (18:32:30)
Delay = 31.7 ms, Ratio = 0.57

Error in delay (top left) shows nearly linear relation for minimum; ratio error (bottom left) shows constant $\beta$ value corresponding to minimum.

Errors added in quadrature to obtain total error (right). Resulting waveform shown below.
Comparing with Balloon Measurements

- Approximation matches well above the tower height (10 m)
- Near ground structure hints at possible logarithmic fit.
- Propagation region extends up to 70 meters, profile fit agrees well with data up to over 80 meters.
Fits to Other Arrivals

Run 8 (19:17:38): Delay = 42.0 ms, Ratio = 0.33

Run 14 (20:23:36): Delay = 39.7 ms, Ratio = 0.37
Conclusions/Summary

- Solving the geometric approximation beyond leading order yields additional information relevant to amplitude and caustic structure which can be used to generate waveform predictions.
- Assuming a fixed propagation distance, the direct and single reflection contributions to the waveform can be predicted for any atmospheric state.
- Using only the difference in travel times and the relative amplitudes of the direct and single reflection, a simple, second order approximation was sucessfully used to perform an inversion for the effective sound speed profile in the nocturnal boundary layer from acoustic data.
Current/Future Work

- The solution can be expanded to 3 dimensions to model propagation in complicated atmospheric models.
- In calculating the Jacobian, variations with $\theta$ and $\phi$ are relevant - maximum of 18 coupled equations.
- The same methods can be applied to propagation of infrasonics. Further, the methods of using acoustic data to check the accuracy of measured meteorological values is being considered (higher altitudes in infrasonics).