General Considerations

- *Setting:* We will apply the renormalization group approach to the Ising model, as a way to find out information about it without directly calculating Z, which in some cases may be too difficult to do.
- *Idea:* Starting from a theory with N degrees of freedom (spins), coupling constants $K = \{K_i\}_{i=1,2,3,...}$, and partition function $Z(K,N)$, find a (Kadanoff) transformation to a theory with fewer degrees of freedom

$$N' = r\,N\ ,\ \ K_i \mapsto K'_i(K,N)\ , \quad \text{with} \quad Z(K,N) = f(K)^{N'}\,Z(K',N')\ .$$

Then, iterating the transformation, we can hope to either eventually obtain an f and Z that are easy to calculate, or find that the iterated map has different asymptotic behaviors depending on the initial values of K and N, with critical values leading to fixed points. Notice that a good quantity to look for is the intensive

$$g(K) := \frac{1}{N}\ln Z(K,N) = \frac{N'}{N}\big[\ln f(K) + g(K')\big]\ .$$

- *Remark:* Technically, discontinuities and phase transitions can only occur for $N\to\infty$.

The 1D Ising Model Example

- *Idea:* Although in 1D we can calculate everything exactly, we apply the method as a warmup for 2D.
- *Setup:* Start with N lattice sites, with periodic boundary conditions, and a partition function

$$Z(K,N) = \sum\nolimits_{\{s_i\}} \mathrm{e}^{K\Sigma'_{i,j}s_is_j} = \sum\nolimits_{\{s_i\}} \mathrm{e}^{K\Sigma_i s_is_{i+1}}\ ,$$

where $s_i = \pm1$, $K = \beta J$ and a prime denotes a summation over nearest neighbors. We now define the decimation process as summing over all possible values of half of the spins, for example the even ones. Then

$$Z(K,N) = \sum\nolimits_{\text{odd }s_i} \big(\mathrm{e}^{K(s_1+s_3)} + \mathrm{e}^{-K(s_1+s_3)}\big)\big(\mathrm{e}^{K(s_3+s_5)} + \mathrm{e}^{-K(s_3+s_5)}\big)\,...$$

- *Kadanoff transformation:* Now we look for a K' such that Z has the form of an Ising model partition function. Imposing that $\mathrm{e}^{K(s+s')} + \mathrm{e}^{-K(s+s')} = f(K)\,\mathrm{e}^{K'ss'}$ for all s and s', we get

$$K' = \tfrac12 \ln\cosh(2K)\ , \qquad f(K) = 2\,(\cosh 2K)^{1/2}\ .$$

Therefore (using $K' = \frac12\cosh^{-1}\mathrm{e}^{2K'}$),

$$g(K') = 2\,g(K) - \ln\big[2\,(\cosh 2K)^{1/2}\big]\ , \qquad g(K) = \tfrac12\,g(K') + \tfrac12\,K' + \tfrac12\ln 2\ .$$

- *Result:* If we plot $K(K')$ or $K'(K)$ and iterations of $g(K')$ and $g(K)$, we see that there are no fixed points except for $K = 0$ and ∞, where the system (respectively, totally disordered and ordered) is scale-free.

The 2D Ising Model Example

- *Setup:* Again start with N lattice sites, with periodic boundary conditions, and a partition function

$$Z_{(1)} = \sum\nolimits_{\{s_i\}} \mathrm{e}^{K\Sigma'_{i,j}s_is_j}\ ,$$

where $K_1 = \beta J$. Again define the decimation process as summing over values of half of the spins, chosen to form a checkerboard pattern.

- *Kadanoff transformation:* The partially summed partition function cannot be seen as a Z of the same form as the original one, but if we introduce

$$Z_{(3)}(K_1,K_2,K_3,N) = f(K)^N \sum\nolimits_{\{s_i\}} \mathrm{e}^{K_1\Sigma'_{i,j}s_is_j + K_2\Sigma''_{l,m}s_ls_m + K_3\Sigma'''_{p,q,r,t}s_ps_qs_rs_t}\ ,$$

where a double prime denotes a sum over next-nearest neighbors, and a triple prime one over squares, then...
- *Result:*

Relevant Sections: Chandler, §5.6-5.7; Halley, 167-189; Schwabl, §7.1-7.3.