The Liouville Theorem

• *Statement:* The value of the distribution function along a trajectory in phase space is a constant of the motion, or ρ evolves like an incompressible fluid. Technically, the statement that

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial\rho}{\partial t} + \sum_{\mathrm{dof}} \left(\frac{\partial\rho}{\partial q^i}\,\dot q^i + \frac{\partial\rho}{\partial p_i}\,\dot p_i \right) = 0\;.$$

• *Proof:* The time rate of change of the probability of a region R in Γ must equal the flux of probability through its surface, which means that ρ must satisfy an equation of continuity,

$$0 = \frac{\partial\rho}{\partial t} + \vec\nabla\cdot(\vec v\,\rho) = \frac{\partial\rho}{\partial t} + \left(\frac{\partial\dot q^i}{\partial q^i} + \frac{\partial\dot p_i}{\partial p_i} \right)\rho + \left(\frac{\partial\rho}{\partial q^i}\,\dot q^i + \frac{\partial\rho}{\partial p_i}\,\dot p_i \right),$$

where $\vec v = (\dot q^1, \dot q^2, ..., \dot q^{3N}; \dot p_1, \dot p_2, ..., \dot p_{3N})$, and the terms in the first set of parentheses cancel by virtue of the Hamilton equations of motion.

General Considerations on Equilibrium Distribution Functions

• *Recall:* The distribution function $\rho(q,p;t)$ is a constant of the motion.
• *Goal:* Find a general form for $\rho(q,p)$ for an equilibrium system.
• *Possible situations:* Two situations in which a system can reach equilibrium are if it is isolated (in which case we will assume the value of the energy is known), and if it is in thermal equilibrium with a bath at fixed temperature, with which it can exchange energy and possibly particles.

The Microcanonical Distribution

• *Idea:* The distribution function describing an isolated system with fixed value for the energy.
• *Distribution function:* The energy of the system is conserved, so if it known to have a certain value E,

$$\rho(q,p) = \text{constant}\times\delta(H(q,p)-E)\;,\quad \text{the microcanonical distribution}\;,$$

which incorporates the principle of equal a priori probabilities (otherwise the "constant" could have been any function of all $6N-1$ constants of the motion. [To avoid the use of this axiom, we could also have used the large-system approximation as in the thermal equilibrium case, but that would have had an intrinsic contradiction, since the log of this ρ is not linear in H.]

Relevant Sections: Phys 731; Chandler; Halley, second half of Ch 1; Reif; Schwabl.

Related Topics

• *Additive constants of the motion:* @ J M Amigo & H Reeh, *Annalen der Physik* **36** (1988) 929-937.
• *Non-extensive statistics:* What changes with respect to the treatment here?
• *Other possible distribution functions:* Which ones have been used?