Quantum Theory of Particles and Fields

Birthday volume dedicated to Jan Łopuszański

Edited by

Bernard Jancewicz

and

Jerzy Lukierski

Institute of Theoretical Physics
University of Wrocław, Poland

World Scientific

1983

NONLINEAR ELECTRODYNAMICS: VARIATIONS ON A THEME BY BORN AND INFELD

Iwo Bialynicki-Birula
Institute for Theoretical Physics, Polish Academy of Sciences
Lotnikow 32, 02-668 Warsaw, Poland

It is the embarrassment of metaphysics that it is able to accomplish so little with the many things that mathematics offers her.

Immanuel Kant

In 1983 Jan Łopuszański is turning 60 and the nonlinear electrodynamics of Born and Infeld (NEBI) is turning 50.

With great pleasure I dedicate my essay to both Jaś and NEBI.

In this essay I give a general review of nonlinear electrodynamics and I describe some old and new results of my own, which were inspired by the work of Born and Infeld.

1. INTRODUCTION

The origins of the nonlinear electrodynamics of Born and Infeld (NEBI) can be traced to the work of Gustav Mie[1], who made the first attempt to construct a purely electromagnetic theory of charged particles. The question, why a charged particle does not explode under the repulsion of the Coulomb forces acting between its constituents, had been asked by a number of physicists before Mie. With the advent of the theory of relativity, this question has balooned into a whole problem known under the name: the classical model of the electron. Many great physicists, including Einstein, Lorentz, Planck, Poincaré, Sommerfeld and von Laue have contributed to the discussion on this subject[2] and a faint echo of their debates is ringing even today in the physical literature[3].

In the classical models of the electron considered before Mie, the electron was not treated as a purely electromagnetic entity, but it was also "made of other stuff", like the Poincaré stresses and the mechanical mass. Mie wanted the electromagnetic field to be the only ingredient; everything else was to be derived from it. In particular, he wanted the electromagnetic current to be made of electromagnetism. In order to achieve this goal, Mie assumed that the Lagrangian depends on the four-potential A_μ not only through its four-dimensional curl $f_{\mu\nu}$,

$$f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ , \tag{1}$$

but also on the A_μ itself. The (minus) derivative of the Lagrangian with respect to the four-potential was interpreted as the electromagnetic current. The generation of the current from the field variables had been achieved in this manner, but at a very high price: the potentials acquired a direct physical meaning and the gauge invariance was totally lost. This has been found unacceptable and the theory of Mie was shelved for two decades.

In 1933 Max Born[4], joined shortly afterwards by Leopold Infeld, have revived Mie's theory in a somewhat different form. In a series of papers[5], Born and Infeld developed a general theory of nonlinear electrodynamics and also gave a new, more appealing variant of the original theory of Born.

In my article I begin with a general nonlinear theory of the electromagnetic field (Sec.2 and Sec.3). In Sec.4, I introduce the theory of Born and Infeld. In Sec.5, I show in what sense this theory is unique. Secs.6 and 7 are devoted to some interesting properties of NEBI, while Sec.8 contains a discussion of its limiting form. Finally, in Sec.9, I introduce an amusing mechanical model inspired by NEBI.

2. GENERAL STRUCTURE OF NONLINEAR ELECTRODYNAMICS

The electrodynamics of Born and Infeld is a special case of nonlinear electrodynamics. In order to appreciate its exceptional features, it is worthwhile to consider NEBI in a general framework of all relativistic, nonlinear and gauge invariant theories of the electromagnetic field (without higher derivatives). Every theory of that type is described by the well known set of source-free Maxwell equations (I use the notation of Ref.6),

$$\partial_t \vec{B} = -\nabla\times\vec{E} \ , \tag{2a}$$

$$\nabla\cdot\vec{B} = 0 \ , \tag{2b}$$

$$\partial_t \vec{D} = \nabla\times\vec{H} , \tag{2c}$$

$$\nabla\cdot\vec{D} = 0 , \tag{2d}$$

supplemented by the (nonlinear) constitutive relations,

$$\vec{E} = \vec{E}(\vec{D},\vec{B}) , \quad \vec{H} = \vec{H}(\vec{D},\vec{B}) . \tag{3}$$

The choice in these equations of $\vec{D}$ and $\vec{B}$ as the primary variables and of $\vec{E}$ and $\vec{H}$ as functions of those variables is suggested by the resemblance between Eqs.(2a) and (2c) and the Hamilton equations of classical mechanics. The assumption, made by Born and Infeld, that $\vec{D}$ and $\vec{B}$ form the canonical pair of variables leads to a consistent canonical formulation of the nonlinear theory.

The standard method to guarantee the inner consistency of Eqs.(2) and (3) and their relativistic covariance is to postulate the existence of an invariant action principle, from which these equations are derived. Following this line of reasoning, let us assume the existence of a scalar Lagrangian density L, which may be any function of the scalar invariant S and the pseudoscalar invariant of the electromagnetic field tensor P,

$$\begin{aligned} S &= -\tfrac{1}{4} f_{\mu\nu} f^{\mu\nu} = \tfrac{1}{2} (\vec{E}^2 - \vec{B}^2) , \\ P &= -\tfrac{1}{4} f_{\mu\nu} \hat{f}^{\mu\nu} = -1/8 \varepsilon^{\mu\nu\lambda\rho} f_{\mu\nu} f_{\lambda\rho} = \vec{E}\cdot\vec{B} . \end{aligned} \tag{4}$$

Eqs.(2a) and (2b) follow from the assumed existence of the potentials and Eqs.(2c) and (2d) follow from the variational principle employing the Lagrange function $L(S,P)$. In the relativistic tensor notation, these equations are:

$$\partial_\mu f_{\nu\lambda} + \partial_\lambda f_{\mu\nu} + \partial_\nu f_{\lambda\mu} = 0 , \tag{5a}$$

$$\partial_\mu h^{\mu\nu} = 0 , \tag{5b}$$

where

$$h^{\mu\nu} = -\frac{\partial L}{\partial f_{\mu\nu}} = \frac{\partial L}{\partial S} f^{\mu\nu} + \frac{\partial L}{\partial P} \hat{f}^{\mu\nu} . \tag{6}$$

The dynamical properties of the electromagnetic field are described by the energy-momentum tensor $T^{\mu\nu}$,

$$T^{\mu\nu} = f^{\mu}{}_{\lambda} h^{\lambda\nu} - g^{\mu\nu} L . \tag{7}$$

Its components

$$T^{00} = \vec{E}\cdot\vec{D} - L , \tag{8a}$$

$$T^{0i} = (\vec{E}\times\vec{H})^i , \tag{8b}$$

$$T^{i0} = (\vec{D}\times\vec{B})^i , \tag{8c}$$

$$T^{ij} = -E^i D^j - H^i B^j + \delta^{ij}(L + \vec{H}\cdot\vec{B}) \tag{8d}$$

have the interpretation of the energy density, the energy current, the momentum density and the Maxwell stresses, respectively. As a result of Eqs.(5) and (6), the energy momentum tensor is conserved,

$$\partial_\nu T^{\mu\nu} = 0 , \tag{9}$$

and symmetric,

$$T^{\mu\nu} = T^{\nu\mu}. \tag{10}$$

In the proof of the symmetry, the following identity is helpful,

$$f_{\mu\nu}\hat{f}^{\nu\lambda} = -\delta^\lambda_\mu P . \tag{11}$$

The properties (9) and (10) lead to the existence of 10 constants of motion: the energy-momentum vector P^μ and the relativistic angular momentum tensor $M^{\mu\nu}$,

$$P^\mu = \int d^3r \, T^{\mu 0} , \tag{12a}$$

$$M^{\mu\nu} = \int d^3r \, (x^\mu T^{\nu 0} - x^\nu T^{\mu 0}) . \tag{12b}$$

3. LEGENDRE TRANSFORMATIONS AND THE CANONICAL FORMALISM

Out of four field vectors $\vec{E},\vec{B},\vec{D}$ and $\vec{H}$ one may choose four different pairs, each consisting of one electric and one magnetic vector, and treat each pair as a set of independent variables, expressing the remaining two as functions of them. I will show now that the transitions between different choices of independent variables can be described in close analogy with Legendre transformations of classical mechanics.

As the initial choice, let us take the pair $\vec{E}$ and $\vec{B}$. These variables appear in the Lagrangian and they are the analogs of the mechanical variables $\dot{\vec{q}}$ and $\vec{q}$. The dependent variables $\vec{D}$ and $\vec{H}$ are to be determined from the constitutive relations, which I will write now in the vector form,

$$\vec{D} = \frac{\partial L}{\partial \vec{E}} , \qquad \vec{H} = -\frac{\partial L}{\partial \vec{B}} . \tag{13}$$

Thus, $\vec{D}$ is the analog of the canonical momentum $\vec{p}$ and $\vec{H}$ is the analog of the mechanical force.

Let us define now a new function ,

$$H(\vec{D},\vec{B}) = \vec{D}\cdot\vec{E}(\vec{D},\vec{B}) - L(\vec{E}(\vec{D},\vec{B}),\vec{B}) , \tag{14}$$

where I have explicitely written that H is to be treated as a function of the canonical variables. Upon differentiating H with respect to $\vec{D}$ and $\vec{B}$, one obtains with the help of (13):

$$\vec{E} = \partial H/\partial\vec{D} \ , \quad \vec{H} = \partial H/\partial\vec{B} \ . \tag{15}$$

Continuing in the same vein, one can obtain the remaining Legendre transformations. They are all shown in Table 1.

Table 1

Ind. var.	Legendre transformations	Dependent variables	
$\vec{E}\ \vec{B}$	$L(\vec{E},\vec{B})$	$\vec{D} = \partial L/\partial\vec{E}$	$\vec{H} = -\partial L/\partial\vec{B}$
$\vec{D}\ \vec{B}$	$H(\vec{D},\vec{B}) = \vec{E}\cdot\vec{D} - L$	$\vec{E} = \partial H/\partial\vec{D}$	$\vec{H} = \partial H/\partial\vec{B}$
$\vec{D}\ \vec{H}$	$\tilde{L}(\vec{D},\vec{H}) = \vec{E}\cdot\vec{D} - \vec{B}\cdot\vec{H} - L$	$\vec{E} = \partial\tilde{L}/\partial\vec{D}$	$\vec{B} = -\partial\tilde{L}/\partial\vec{H}$
$\vec{E}\ \vec{H}$	$\tilde{H}(\vec{E},\vec{H}) = \vec{B}\cdot\vec{H} + L$	$\vec{D} = \partial\tilde{H}/\partial\vec{E}$	$\vec{B} = \partial\tilde{H}/\partial\vec{H}$

The function $H(\vec{D},\vec{B})$ coincides with the energy density. With its use, the field equations (1a) and (1c) can be written in an almost Hamiltonian form:

$$\begin{aligned} \partial_t\vec{B} &= -\nabla\times\partial H/\partial\vec{D} \ , \\ \partial_t\vec{D} &= \nabla\times\partial H/\partial\vec{B} \ . \end{aligned} \tag{16}$$

One may also write the above equations with the use of the functional derivatives of the total energy expressed in terms of the canonical variables - the Hamiltonian,

$$\mathrm{H} = \int d^3r\ H(\vec{D},\vec{B}) \ , \tag{17}$$

$$\begin{aligned} \partial_t\vec{B} &= -\nabla\times\delta\mathrm{H}/\delta\vec{D} \ , \\ \partial_t\vec{D} &= \nabla\times\delta\mathrm{H}/\delta\vec{B} \ . \end{aligned} \tag{18}$$

This form of the field equations enables one to guess the correct form of the Poisson bracket between two arbitrary functionals of the canonical variables

$$\{F\,,G\} = \int d^3r\ \Big[\frac{\delta F}{\delta\vec{D}}\cdot\Big(\nabla\times\frac{\delta G}{\delta\vec{B}}\Big) - \frac{\delta G}{\delta\vec{D}}\cdot\Big(\nabla\times\frac{\delta F}{\delta\vec{B}}\Big)\Big] \ . \tag{19}$$

and, in particular, for the canonical variables themselves, one has

$$\begin{aligned} \{B_i(\vec{r})\,,D_j(\vec{r}\,')\} &= \varepsilon_{ijk}\,\partial_k\delta(\vec{r}-\vec{r}\,') \ , \\ \{D_i\,,D_j\} = 0 &= \{B_i\,,B_j\} \ . \end{aligned} \tag{20}$$

In spite of its nonrelativistic form, the Poisson brackets (19) define a relativistically invariant canonical structure, with the ten constants of motion (12) acting as the generators of the Poincaré transformations[6].

It is worth observing that the functions H and $\tilde{H}$, in contrast to L and $\tilde{L}$, are not relativistic scalars. Even so, one may formulate the criteria of relativistic invariance in terms of H and $\tilde{H}$.

The simplest method to find such criteria is based on the observation that the necessary and sufficient condition of relativistic invariance is the symmetry of the energy-momentum tensor. The necessity of this condition has already been shown and its sufficiency follows from the fact that any dependence of L on $\vec{E}^2 + \vec{B}^2$ will make T^{0i} different from T^{i0} (the rotational symmetry of L is, of course, assumed). The symmetry of $T^{\mu\nu}$ leads to a differential equation for H. To derive it, let me denote the three scalar combinations of the canonical field-vectors as follows,

$$\tau = \tfrac{1}{2}(\vec{D}^2 + \vec{B}^2) \quad ,$$
$$\xi = \tfrac{1}{2}(\vec{D}^2 - \vec{B}^2) \quad , \tag{21}$$
$$\eta = \vec{D}\cdot\vec{B} \quad .$$

The equality

$$\vec{E}\times\vec{H} = \vec{B}\times\vec{D} \quad , \tag{22}$$

expressed in terms of the derivatives of H reads

$$(\partial_\tau H)^2 - (\partial_\xi H)^2 - (\partial_\eta H)^2 = 1 \quad . \tag{23}$$

The Hamiltonian density of the linear Maxwell theory in free space, $H = \tau$, is the simplest solution of this equation.

Eq.(23) has an interesting O(2,1) symmetry, which will be explored in Sec.8. An equation of the form (23) also holds for $\tilde{H}$.

4. NONLINEAR ELECTRODYNAMICS OF BORN AND INFELD

The first model of nonlinear electrodynamics proposed by Born was based on the following choice of the Lagrange function,

$$L_B = -b^2((1 - (\vec{E}^2 - \vec{B}^2)/b^2)^{\frac{1}{2}} - 1) \quad , \tag{24}$$

where b is a constant having the dimension of the electromagnetic field. Born discovered his square-root Lagrangian, while searching for a theory which could give finite self-energy of a point charge. He noticed that the same mechanism, which restricts the particle's velocity in relativistic mechanics to values smaller than c, will restrict the electric field in the theory based on the Lagrangian (24) to values smaller than the critical field b (in the absence of the magnetic field).

The Lagrangian of Born and Infeld has the same square-root structure, but it contains also the pseudoscalar invariant of the e.m. field

$$L_{\text{Born-Infeld}} = -b^2((1 - (\vec{E}^2 - \vec{B}^2)/b^2 - (\vec{E}\cdot\vec{B})^2/b^4)^{\frac{1}{2}} - 1). \quad (25)$$

Its discovery represents one of those amusing cases of serendipity in theoretical physics. The reasoning which led Born and Infeld to their Lagrangian was fallacious; they thought that (25) is singled out by general relativistic covariance, because the expression under the square-root is the determinant of the tensor $g_{\mu\nu} + f_{\mu\nu}/b$. However, every function of S and P can be easily converted into a scalar under all coordinate transformations with the use of the metric tensor.

As has been shown much later, NEBI is a unique nonlinear theory of the electromagnetic field, but for an entirely different reason. It is the only nonlinear theory, which does not lead to the phenomenon of birefringence: the velocity of light in NEBI does not depend on its polarization. I will discuss this in detail in the next Section.

A natural correspondence, of course, exists between NEBI and the linear Maxwell theory. For fields much weaker than the critical field value b, L_{BI} and L_B reduce to

$$L_M = \tfrac{1}{2}(\vec{E}^2 - \vec{B}^2) \quad . \quad (26)$$

Due to the simple, algebraic form of the Lagrangian, all the Legendre transformations listed in Table 1 can be explicitely carried out and are shown in Table 2 (I set b = 1).

Table 2

NEBI: Legendre transforms and the constitutive relations

$$L = -(1 - \vec{E}^2 + \vec{B}^2 - (\vec{E}\cdot\vec{B})^2)^{\frac{1}{2}} + 1 = -\ell + 1$$

$$\vec{D} = (\vec{E} + (\vec{E}\cdot\vec{B})\vec{B})\,\ell^{-1} \qquad \vec{H} = (\vec{B} - (\vec{E}\cdot\vec{B})\vec{E})\,\ell^{-1}$$

$$H = (1 + \vec{D}^2 + \vec{B}^2 + (\vec{D}\times\vec{B})^2)^{\frac{1}{2}} - 1 = h - 1$$

$$\vec{E} = (\vec{D} - (\vec{D}\times\vec{B})\times\vec{B})\,h^{-1} \qquad \vec{H} = (\vec{B} + (\vec{D}\times\vec{B})\times\vec{D})\,h^{-1}$$

$$\tilde{L} = (1 + \vec{D}^2 - \vec{H}^2 - (\vec{D}\cdot\vec{H})^2)^{\frac{1}{2}} - 1 = \tilde{\ell} - 1$$

$$\vec{E} = (\vec{D} - (\vec{D}\cdot\vec{H})\,\vec{H})\,\tilde{\ell}^{-1} \qquad \vec{B} = (\vec{H} + (\vec{D}\cdot\vec{H})\,\vec{D})\,\tilde{\ell}^{-1}$$

$$\tilde{H} = (1 - \vec{E}^2 - \vec{\Pi}^2 + (\vec{E}\times\vec{\Pi})^2)^{\frac{1}{2}} - 1 = \tilde{h} - 1$$

$$\vec{D} = (\vec{E} + (\vec{E}\times\vec{\Pi})\times\vec{\Pi})\ \tilde{h}^{-1} \qquad \vec{B} = (\vec{\Pi} - (\vec{E}\times\vec{\Pi})\times\vec{E})\ \tilde{h}^{-1} .$$

The advantage of using the canonical pair of variables is clearly seen from this Table, since this is the only pair of independent field variables, which are not restricted in their values by the reality condition of the square-root.

5. PROPAGATION OF WEAK DISTURBANCES AND NEBI

I shall study in this Section the propagation of weak electromagnetic waves in the presence of a strong, constant electromagnetic field, in a general, relativistic nonlinear electrodynamics[7]. This study will lead us directly to NEBI. The weak, varying part of the field will be denoted by $f_{\mu\nu}$ and the strong, constant part by $F_{\mu\nu}$.

The strong field is assumed to be a given solution of the field equations, while the weak field will be determined from the field equations. Those can be obtained from the Lagrangian $L^{(2)}$, in which only the quadratic terms in the weak field are kept,

$$L^{(2)} = \gamma_S(-\tfrac{1}{4}\ f_{\mu\nu}f^{\mu\nu}) + \gamma_P(-\tfrac{1}{4}\ f_{\mu\nu}\hat{f}^{\mu\nu}) \tag{27}$$
$$+ \tfrac{1}{2}\gamma_{SS}(\tfrac{1}{2}F_{\mu\nu}f^{\mu\nu})^2 + \tfrac{1}{2}\gamma_{PP}(\tfrac{1}{2}\hat{F}_{\mu\nu}f^{\mu\nu})^2 + \tfrac{1}{4}\gamma_{SP}\ F_{\mu\nu}f^{\mu\nu}\hat{F}_{\lambda\rho}f^{\lambda\rho} ,$$

where the coefficients γ represent the first and the second derivatives of the Lagrangian with respect to S and P, evaluated at the constant field values of these invariants. The γ_P term in $L^{(2)}$ can be dropped, since it represents a four-divergence.

Varying the action with respect to the potentials, which are associated with $f_{\mu\nu}$, one obtains,

$$(\ \gamma_S(g^{\mu\nu}\partial_\lambda\partial^\lambda - \partial^\mu\partial^\nu) \tag{28}$$
$$- \gamma_{SS}\ F^{\mu\lambda}\partial_\lambda F^{\nu\rho}\partial_\rho - \gamma_{PP}\ \hat{F}^{\mu\lambda}\partial_\lambda\hat{F}^{\nu\rho}\partial_\rho - \gamma_{SP}(F^{\mu\lambda}\partial_\lambda\hat{F}^{\nu\rho}\partial_\rho + \mu\leftrightarrow\nu))A_\nu = 0 ,$$

The (complex) solution of this set of equations with constant coefficients can be written in the standard form,

$$A_\mu = \varepsilon_\mu(k)\ e^{-ik\cdot x} . \tag{29}$$

The polarization vector $\varepsilon_\mu(k)$ obeys the following set of linear algebraic equations:

$$(\ k^\mu k_\nu + \alpha_{SS}\ a^\mu a_\nu + \alpha_{PP}\ \hat{a}^\mu\hat{a}_\nu + \alpha_{SP}(\hat{a}^\mu a_\nu + a^\mu\hat{a}_\nu))\varepsilon^\nu = k^2\varepsilon^\mu , \tag{30}$$

where

$$\alpha_{SS} = \gamma_{SS}/\gamma_S \quad , \quad \alpha_{PP} = \gamma_{PP}/\gamma_S \quad , \quad \alpha_{SP} = \gamma_{SP}/\gamma_S \tag{31}$$

and

$$a^\mu = F^{\mu\nu}k_\nu \quad , \quad \hat{a}^\mu = \hat{F}^{\mu\nu}k_\nu \quad . \tag{32}$$

The connection between the frequency ω and the wave vector $\vec{k}$ is obtained from the condition that the determinant of the set of equations (30) vanishes. By looking at the l.h.s. of these equations, we can learn that ε^μ must lie in the subspace spanned by the three vectors a^μ, $\hat{a}^\mu$ and k^μ. In addition, the component of ε^μ in the k^μ direction is arbitrary (gauge invariance). In this manner we arrive at the eigenvalue problem in only two dimensions. Substituting

$$\varepsilon^\mu = \alpha a^\mu + \beta\hat{a}^\mu \tag{33}$$

into Eq.(30), we obtain

$$\begin{aligned} &(k^2 - \alpha_{SS}a^2 + \alpha_{SP}k^2P)\alpha + (\alpha_{SS}k^2P - \alpha_{SP}(a^2 + 2k^2S))\beta = 0 \quad , \\ &(\alpha_{PP}k^2P - \alpha_{SP}a^2)\alpha + (k^2 + \alpha_{SP}k^2P - \alpha_{PP}(a^2 + 2k^2S))\beta = 0 \quad , \end{aligned} \tag{34}$$

where the following algebraic relations were used,

$$k\cdot a = 0 = k\cdot\hat{a} \quad , \quad a\cdot\hat{a} = -k^2P \quad , \quad \hat{a}^2 = a^2 + 2k^2S \ . \tag{35}$$

The vanishing of the determinant of Eqs.(34) can be written as the following simple condition on the vector k^μ,

$$k^2 = a^2\lambda_\pm \quad , \tag{36}$$

where

$$\lambda_\pm = \tfrac{1}{2}(2S\Gamma - (\alpha_{SS} + \alpha_{PP}) \pm \Delta^{\frac{1}{2}})(P^2\Gamma + 2(S\alpha_{PP} - P\alpha_{SP}) - 1)^{-1} \quad , \tag{37}$$

$$\Delta = (\alpha_{SS} - \alpha_{PP} - 2S\Gamma)^2 + (2\alpha_{SP} - 2P\Gamma)^2 \quad , \tag{38}$$

$$\Gamma = \alpha_{SS}\alpha_{PP} - \alpha_{SP}^2 \quad . \tag{39}$$

The coefficient $\lambda_\pm$ depends on the external field. It has been called in Ref.7 the birefringence index, because its two values λ_+ and λ_- determine two forms of the dispersion law: for two different polarizations. The birefringence does not occur only for $\Delta = 0$, when both polarizations propagate according to the same dispersion law. Due to the double-quadratic form of Δ, its vanishing implies two partial differential equations for the Lagrange function:

$$\begin{aligned} \gamma_S(\gamma_{SS} - \gamma_{PP}) - 2S(\gamma_{SS}\gamma_{PP} - \gamma_{SP}^2) &= 0 \quad , \\ 2\gamma_S\gamma_{SP} - 2P(\gamma_{SS}\gamma_{PP} - \gamma_{SP}^2) &= 0 \quad . \end{aligned} \tag{40}$$

In 1970 Boillat[8] and Plebanski[9] have shown independently that the

Lagrangian of Born and Infeld is (almost) the only solution of these two equations. They derived Eqs.(40) by imposing the "no-birefringence condition" on the propagation of discontinuities of the electromagnetic field, not by studying weak disturbances in a given background.

Eqs.(40) are homogeneous in the Lagrange function and in the independent variables S and P and they also do not contain γ_P. Therefore, one may always multiply their solution by a constant, rescale the variables and add to the solution a constant and a linear term in P. Such changes , however, do not change the form of the field equations.

Plebanski also found an additional, rather pathological solution of the form: $L = S/P$.

Let me return now to the study of the propagation of weak disturbances and let us solve Eq.(36) with respect to ω:

$$\omega = \vec{k}\cdot\vec{\mathcal{S}} \pm [(\vec{k}\cdot\vec{\mathcal{S}})^2 + \vec{k}^2 - (\vec{k}\times\vec{\mathcal{E}})^2 - (\vec{k}\times\vec{\mathcal{B}})^2]^{1/2} \quad , \tag{41}$$

where the script letters denote the external fields "renormalized" as follows,

$$\vec{\mathcal{E}} = \vec{E}(\lambda^{-1} + \vec{E}^2)^{-1/2} \quad , \quad \vec{\mathcal{B}} = \vec{B}(\lambda^{-1} + \vec{E}^2)^{-1/2} \quad , \quad \vec{\mathcal{S}} = \vec{\mathcal{E}}\times\vec{\mathcal{B}} \quad . \tag{42}$$

The presence of the $\vec{k}\cdot\vec{\mathcal{S}}$ term in the formula (41) indicates the existence of the directive effect: the wave velocity changes when the wave vector is reversed. This is an analog of the Fresnel-Fizeau effect in moving media. The external electromagnetic field in a nonlinear theory is very much like a moving medium, as long as it has a non-vanishing Poynting vector. In the frame of reference, in which the $\vec{E}$ and $\vec{B}$ fields are parallel, the directive effect vanishes. It is the "rest" frame for the external field. I will calculate now the group velocity in this frame of reference in NEBI, in order to show that no violation of causality occurs.

First, let me notice that in NEBI

$$\lambda^{-1} = (1 - 2S) \quad , \tag{43}$$

which leads to the following simple expression for ω in the rest frame

$$\omega = [(\vec{k}^2 - (\vec{k}\times\vec{m})^2(\vec{E}^2 + \vec{B}^2)(1 + \vec{B}^2)^{-1}]^{+1/2} \quad , \tag{44}$$

where $\vec{m}$ is the unit vector in the direction of $\vec{E}$ and $\vec{B}$. The length of the group velocity vector squared, after straigtforward calculations, can be written in the form

$$(\partial\omega/\partial\vec{k})^2 = 1 + (1 + \vec{B}^2 + \vec{D}^2)^{-1} - (1 + (\vec{B}^2 + \vec{D}^2)\cos^2\theta)^{-1} \quad , \tag{45}$$

where θ is the angle between $\vec{m}$ and $\vec{k}$. The r.h.s. of (45) varies, as a function of the angle θ, from $(1 + \vec{B}^2 + \vec{D}^2)^{-1}$ to 1. The relativistic addition law for velocities guarantees that the group velocity in NEBI will not exceed the speed of light also in all other frames.

6. DUALITY ROTATIONS AND A NEW CONSERVATION LAW IN NEBI

Every relativistic theory of the e.m. field has 10 conservation laws, given by the formulas (12). In a linear field theory, every Fourier component of the field propagates independently and this leads to an infinite number of conserved quantities, bilinear in the Fourier amplitudes. In a nonlinear theory, additional conservation laws are not common, especially in four dimension, except for various charges associated with internal symmetries. Nonlinear electrodynamics has no internal symmetries, and therefore the existence of an additional conservation law for some choices of the Lagrangian is a pleasant surprise.

The symmetry, which leads to this new conservation law, is the invariance of the field equations under the duality rotations of the canonical fields $\vec{D}$ and $\vec{B}$,

$$\begin{aligned} '\vec{D} &= \vec{D}\cos\alpha - \vec{B}\sin\alpha \quad , \\ '\vec{B} &= \vec{D}\sin\alpha + \vec{B}\cos\alpha \quad . \end{aligned} \tag{46}$$

This is, clearly, not an internal symmetry, because it mixes objects of different tensorial character, but it closely resembles one. The transformation (46) is a canonical transformation, it leaves the Poisson brackets (20) unchanged. It will be a symmetry transformation if it also leaves the Hamiltonian unchanged. This indeed happens in NEBI (and also in some other theories), because both $\vec{D}^2 + \vec{B}^2$ and $\vec{D}\times\vec{B}$ are invariant under (46). The generator of the duality rotations can be easily guessed by analogy with the linear Maxwell theory, where the duality rotations of the $\vec{E}$ and $\vec{B}$ fields are well known[10]. It is the following nonlocal integral:

$$\Lambda = (1/8\pi)\int d^3r_1\int d^3r_2[\vec{D}_1 r_{12}^{-1}\cdot(\nabla\times\vec{D}_2) + \vec{B}_1 r_{12}^{-1}\cdot(\nabla\times\vec{B}_2)] \quad , \tag{47}$$

where $r_{12} = |\vec{r}_1 - \vec{r}_2|$ and the subscripts indicate that the fields are taken at the points $\vec{r}_1$ and $\vec{r}_2$. With the use of (20), one obtains

$$\{\vec{D}\,,\Lambda\} = \vec{B} \quad , \qquad \{\vec{B}\,,\Lambda\} = -\vec{D} \quad . \tag{48}$$

The time independence of Λ can be also established by a direct calculation, with the use of the field equations and with the help of the following identity which holds in NEBI:

$$\vec{E}\cdot\vec{B} = \vec{D}\cdot\vec{H} \quad . \tag{49}$$

This identity, expressed in terms of the Lagrangian variables,

$$\vec{E}\cdot\vec{B} = -(\partial L/\partial\vec{E})\cdot(\partial L/\partial\vec{B}) \quad , \tag{50}$$

or

$$P = P(\gamma_S^2 - \gamma_P^2) - 2S\gamma_S\gamma_P \quad , \tag{51}$$

serves as a test of the invariance of the Lagrangian under the duality rotations.

The physical interpretation of the generator Λ will be obtained in the next Section, where the formulation of NEBI in terms of a complex field vector is given.

7. FORMULATION OF NEBI IN TERMS OF COMPLEX FIELDS

In view of the symmetry between the first and the second pair of the Maxwell equations (2), it is natural to introduce two complex vectors,

$$\vec{F} = (\vec{D} + i\vec{B})/2^{\frac{1}{2}} \quad , \quad \vec{G} = (\vec{E} + i\vec{H})/2^{\frac{1}{2}} \tag{52}$$

In terms of $\vec{F}$ and $\vec{G}$, Eqs.(2) read:

$$i\partial_t\vec{F} = \nabla\times\vec{G} \quad , \tag{53a}$$

$$\nabla\cdot\vec{F} = 0 \quad . \tag{53b}$$

This complex notation becomes especially convenient when the theory is invariant under the duality rotations, since these rotations result in the changes of the phase of the vector $\vec{F}$,

$$'\vec{F} = e^{i\alpha}\,\vec{F} \quad . \tag{54}$$

The Hamiltonian will be invariant under these "gauge transformations of the second kind" if it contains only bilinear expressions in the field vector $\vec{F}$. The Hamiltonian of NEBI is clearly of this type,

$$H_{BI} = [1 + 2\vec{F}^*\cdot\vec{F} - (\vec{F}^*\times\vec{F})^2]^{\frac{1}{2}} - 1 \quad . \tag{55}$$

In this formulation, the connection between $\vec{F}$ and $\vec{G}$ has the form:

$$\vec{G} = \delta H/\delta\vec{F}^* \quad , \tag{56}$$

which leads to the following equation for $\vec{F}$:

$$i\partial_t \vec{F} = \nabla \times \frac{\vec{F} + (\vec{F}^* \times \vec{F}) \times \vec{F}}{[1 + 2\vec{F}^* \cdot \vec{F} - (\vec{F}^* \times \vec{F})^2]^{\frac{1}{2}}} \quad . \tag{57}$$

This equation is reminiscent of many nonlinear equations describing the propagation of charged fields, in which the nonlinearity is also a function of the bilinear combinations of the fields. In the case of charged fields, there is always an important constant of motion - the total charge, which is also the generator of the phase transformations of the field. The constant of motion Λ, defined in the previous Section is the exact analog of the total charge in NEBI. In terms of the complex field $\vec{F}$, it has the form:

$$\Lambda = (1/4\pi) \int d^3 r_1 \int d^3 r_2 \; \vec{F}_1^* r_{12}^{-1} \cdot (\nabla \times \vec{F}_2) \quad . \tag{58}$$

In order to extend this analogy even further, let me introduce the Fourier representation of $\vec{F}$,

$$\vec{F}(\vec{k}) = \int d^3 r \; e^{-i\vec{k}\cdot\vec{r}} \; \vec{F}(\vec{r}) \quad . \tag{59}$$

Since $\vec{F}$ lies in the plane perpendicular to $\vec{k}$, it can be expressed as a linear combination of any two orthonormal vectors, say $\vec{\ell}_1$ and $\vec{\ell}_2$, lying in that plane. It is convenient to form a complex combination $\vec{e}$ of $\vec{\ell}_1$ and $\vec{\ell}_2$

$$\vec{e} = (\vec{\ell}_1 + i\vec{\ell}_2)/2^{\frac{1}{2}} \quad , \tag{60}$$

which satisfies the following relations:

$$\vec{k} \times \vec{e} = -ik\,\vec{e} \quad , \quad \vec{k} \times \vec{e}^* = ik\,\vec{e}^* , \tag{61}$$

$$\vec{e}^* \cdot \vec{e} = 1 \quad , \quad \vec{e}^2 = 0 \quad . \tag{62}$$

Upon inserting the decomposition

$$\vec{F}(\vec{k}) = \vec{e} f_+ + \vec{e}^* f_- \tag{63}$$

into the formula (58), one obtains after doing a few Fourier integrals:

$$\Lambda = \int d^3 k \, k^{-1} (|f_+|^2 - |f_-|^2) \quad . \tag{64}$$

In the linear theory, this expression has an obvious interpretation (see for example Ref.6). It is the total intensity of radiation (the number of photons in the quantum theory) with the left-handed polarization minus the total intensity of radiation with the right-handed polarization. In a nonlinear theory, the superposition principle does not hold, so that the decomposition of a solution of the field equations into into a linear combination of circularly polarized waves can not, in general, be made. Therefore, it is quite remarkable that Λ is still a constant of motion.

8. THE LIMIT $b=0$ OF THE BORN-INFELD ELECTRODYNAMICS

It has been remarked in Sec.4 that in the limit, when the critical field value b tends to infinity, NEBI reduces to the linear Maxwell theory. The opposite limit, when b tends to zero, can not be performed in the Born-Infeld Lagrangian. However, this limit can easily be reached in the Hamiltonian formulation. To this end, let me insert back the constant b into the Hamiltonian density in NEBI:

$$H_{BI} = [b^4 + b^2(\vec{D}^2 + \vec{B}^2) + (\vec{D}\times\vec{B})^2]^{\frac{1}{2}} - b^2 \quad . \tag{65}$$

In the limit, when b=0, one obtains

$$H_{UBI} = |\vec{D}\times\vec{B}| \quad . \tag{66}$$

The energy density in this ultra-Born-Infeld electrodynamics (UNEBI) is equal to the length of the Poynting vector. Like the linear Maxwell theory, UNEBI does not contain any dimensional parameters. However, it is still a nonlinear theory, as can be seen from its field equations,

$$\partial_t\vec{B} = \nabla\times(\vec{n}\times\vec{B}) \quad ; \qquad \nabla\cdot\vec{B} = 0 \quad , \tag{67a}$$

$$\partial_t\vec{D} = \nabla\times(\vec{n}\times\vec{D}) \quad , \qquad \nabla\cdot\vec{D} = 0 \quad ,$$

where $\vec{n}$ is a unit vector in the direction of the Poynting vector,

$$\vec{n} = \vec{D}\times\vec{B}/|\vec{D}\times\vec{B}| \quad . \tag{68}$$

The vector $\vec{n}$ introduces a nonlinear element into Eqs.(67); only those solutions which share the same vector field $\vec{n}$ can be superposed. UNEBI has an interesting pathology: it can not be derived from a Lagrangian. The Legendre transformation

$$L = \vec{D}\cdot\vec{E} - H \quad , \tag{69}$$

leading from the Hamiltonian to the Lagrangian, gives identically zero. In the Hamiltonian formulation, however, the theory is well defined. I consider it interesting, because it posseses a rather large number of symmetries.

The list of the symmetries of UNEBI begins, of course, with the relativistic invariance. In the absence of the Lagrangian, the invariance under Lorentz transformations is most easily established with the help of Eq.(23) for the Hamiltonian density. In terms of our quadratic combinations τ, ξ and η, the expression (66) reads,

$$H_{UBI} = (\tau^2 - \xi^2 - \eta^2)^{\frac{1}{2}} \quad . \qquad (70)$$

Thus, UNEBI is the very theory which respects the full O(2,1) invariance of the condition (23). This symmetry must give rise to three constants of motions: the generators of the O(2,1) group. In order to find them, let us write the BI Hamiltonian density in terms of τ, ξ and η variables,

$$H_{BI} = ((1 + \tau)^2 - \xi^2 - \eta^2)^{\frac{1}{2}} - 1 \quad . \qquad (71)$$

This expression is only invariant under the subgroup of rotations in the ξ-η plane. One may easily check that these rotations are exactly the duality rotations (46), whose generator has been given by the formula (47). The remaining two linear transormations, which leave the expression (70) invariant, are the hyperbolic rotations in the τ-η and τ-ξ planes (boosts in the ξ and η directions). They correspond to the following canonical transformations of the $\vec{D}$ and $\vec{B}$ fields:

$$\begin{aligned} '\vec{D} &= \vec{D}\cosh\alpha + \vec{B}\sinh\alpha \quad , \\ '\vec{B} &= \vec{D}\sinh\alpha + \vec{B}\cosh\alpha \quad , \end{aligned} \qquad (72)$$

and

$$\begin{aligned} '\vec{D} &= e^{\lambda}\,\vec{D} \quad , \\ '\vec{B} &= e^{-\lambda}\,\vec{B} \quad . \end{aligned} \qquad (73)$$

The corresponding generators can be easily guessed by analogy with the expression (47) and they are of the form:

$$\Gamma_1 = (1/8\pi)\int d^3r_1 \int d^3r_2 \, [\, \vec{D}_1 r_{12}^{-1}\cdot(\nabla\times\vec{D}_2) - \vec{B}_1 r_{12}^{-1}\cdot(\nabla\times\vec{B}_2)\,] \quad , \qquad (74)$$

$$\Gamma_2 = (1/4\pi)\int d^3r_1 \int d^3r_2 \, \vec{B}_1 r_{12}^{-1}\cdot(\nabla\times\vec{D}_2) \quad . \qquad (75)$$

One may check by a direct calculation that both these integrals are time-independent. I do not have any clear physical interpretation of these two generators.

Most theories, which do not contain dimensional parameters, are invariant under the full, 15-parameter conformal group and UNEBI is no exception here. One may check that the trace of the energy-momentum tensor vanishes, which leads to additional 5 constants of motion (on top of the 10 generators of the Poincaré group): the generators of dilations D and 4-accelerations K^μ,

$$D = -\int d^3r \, x_\mu T^{\mu 0} \quad , \qquad K^\mu = \int d^3r \, (2x^\mu x_\nu - x^2\delta^\mu_\nu) T^{\nu 0} \quad . \qquad (76)$$

9. BORN_INFELD THEORY IN "ZERO DIMENSIONS"

The content of this Section is related to NEBI only by a purely formal analogy. I will describe here a quantum mechanical system, which is described by the Hamiltonian and by the equations of motion resembling very closely the Born-Infeld electrodynamics. This very close resemblance in form is the only excuse for including this material in an essay on NEBI.

The model in question describes a particle in three dimensions, whose motion is governed by the equations of motion derived from the following Lagrangian:

$$L(\dot{\vec{q}},\vec{q}) = -b^2([1-(\dot{\vec{q}}^2-\omega^2\vec{q}^2)/b^2-\omega^2(\dot{\vec{q}}\cdot\vec{q})^2/b^4]^{\frac{1}{2}}-1). \quad (77)$$

In what follows, I shall set both b and ω equal to 1. In the Hamiltonian formulation, the basic equations are:

$$H = [1+\vec{p}^2+\vec{q}^2+(\vec{q}\times\vec{p})^2]^{\frac{1}{2}}-1 = h-1 \quad , \quad (78)$$

$$\dot{\vec{q}} = (\vec{p}+(\vec{q}\times\vec{p})\times\vec{q})h^{-1} \quad ,$$
$$\dot{\vec{p}} = -(\vec{q}-(\vec{q}\times\vec{p})\times\vec{p})h^{-1} \quad . \quad (79)$$

The system under consideration can be classified as a nonlinear harmonic oscillator. In order to justify this name, let me write Eqs. (78) and (79) in the complex form, patterned after the formulation of NEBI described in Sec.7,

$$H = [1+2\vec{a}^*\cdot\vec{a}-(\vec{a}^*\times\vec{a})^2]^{\frac{1}{2}}-1 \quad , \quad (80)$$

$$\dot{\vec{a}} = (\vec{a}+(\vec{a}^*\times\vec{a})\times\vec{a})h^{-1} \quad , \quad (81)$$

where

$$\vec{a} = (\vec{p}+i\vec{q})/2^{\frac{1}{2}} \quad . \quad (82)$$

Since both the Hamiltonian and the angular momentum vector are constants of motion, Eq.(81), in spite of its nonlinear appearance, constitutes a set of linear differential equations for the complex vector $\vec{a}$ and its solution may be sought in the standard form,

$$\vec{a}(t) = \exp(i\hat{\Omega}t)\,\vec{a} \quad , \quad (83)$$

where $\hat{\Omega}$ is a time independent matrix. The matrix elements of $\hat{\Omega}$ depend on the initial values of the vector $\vec{a}$,

$$\Omega_{ij} = (\delta_{ij}+a_i a_j^*-a_i^* a_j) \quad . \quad (84)$$

The frequencies of the harmonic oscillations of the particle are given below in terms of the length of the angular momentum vector M and h,

$$\omega_0 = h^{-1} \quad , \quad \omega_\pm = (1 \pm M)h^{-1}. \tag{85}$$

The eigenvectors of $\hat{\Omega}$ can be expressed in terms of the unit vector $\vec{n}$ in the direction of the angular momentum and a pair of orthogonal vectors, $\vec{\ell}_1$ and $\vec{\ell}_2$, lying in the plane defined by $\vec{n}$,

$$\vec{e}_0 = \vec{n} \quad , \quad \vec{e}_\pm = (\vec{\ell}_1 \pm i\vec{\ell}_2)2^{\frac{1}{2}} \quad . \tag{86}$$

Therefore, the explicit form of the solution is:

$$\vec{a}(t) = e^{it/h}(a_0\vec{e}_0 + a_+\vec{e}_+e^{itM/h} + a_-\vec{e}_-e^{-itM/h}) \quad . \tag{87}$$

This equation justifies the name: nonlinear harmonic oscillator. The oscillations of $\vec{a}$(and of $\vec{q}$ and $\vec{p}$) are harmonic with their frequencies given by the formulas (85), but these frequencies are nonlinear functions of the initial conditions.

I shall close this Section with a brief discussion of the quantum version of this model. The best starting point is, of course, the Hamiltonian. The quantum Hamiltonian is obtained from (80) by the standard replacement of the classical variables with the quantum operators. Since the variables $\vec{a}$ and $\vec{a}^*$ are replaced by the annihilation and creation operators, the energy spectrum can be immediately found,

$$E_{n,1} = (1 + 2n + \ell(\ell + 1))^{\frac{1}{2}} - 1 \quad . \tag{88}$$

Also, the Schrödinger equation can be explicitely solved in terms of the wave functions (Hermite polynomials) of the harmonic oscillator.

10. AFTER FIFTY YEARS

I have given here a review of the nonlinear electrodynamics, intertwined with assorted pieces of my own research, which was inspired by the work of Born and Infeld. There is no evidence that their theory has any direct connection with the physical reality, but it stands out as a masterpiece of mathematical physics: an exceptional structure of many intricate theoretical features, as charming now as it appeared to its discoverers/creators 50 years ago.

REFERENCES

1. G.Mie, Grundlagen einer Theorie der Materie, Annalen der Physik 37, 511 (1912); 39, 1 (1912); 40, 1 (1913).

2. An excellent review of the history of the classical model of the electron can be found in: A.I.Miller, *Albert Einstein's Special Theory of Relativity* (Addison-Wesley, Reading 1981).

3. T.H.Boyer, Classical model of the electron and the definition of electromagnetic field momentum, Physical Review D25, 3246 (1982); F.Rohrlich, Comment on the preceding paper by T.H.Boyer, *ibid.* 3251; I.Bialynicki-Birula, Classical model of the electron. Exactly soluble example, *ibid.* (to be published).

4. M.Born, On the quantum theory of the electromagnetic field, Proceedings of the Royal Society A143, 410 (1934).

5. M.Born and L.Infeld, Foundations of the new field theory, Proceedings of the Royal Society A144, 425 (1934); On the quantization of the new field equations I, *ibid* . A147, 522 (1934); On the quantization of the new field theory II, *ibid.* A150, 141 (1935).

6. I.Bialynicki-Birula and Z.Bialynicka-Birula, *Quantum Electrodynamics* (Pergamon, Oxford 1975).

7. The analysis of this Section is based on our earlier paper: Z.Bialynicka-Birula and I.Bialynicki-Birula, Nonlinear effects in quantum electrodynamics. Photon propagation and photon splitting in an external field, Physical Review D2, 2341 (1970).

8. G.Boillat, Nonlinear electrodynamics: Lagrangians and equations of motion, Journal of Mathematical Physics 11, 941 (1970).

9. J.Plebanski, *Lecture Notes on Nonlinear Electrodynamics* (NORDITA 1970).

10. S.Deser and C.Teitelboim, Duality transformations of Abelian and non-Abelian gauge fields, Physical Review D13, 1592 (1976).