Chapter 6 HW

#1- (a) Prove that the probability current is zero for a bound state wave, eg. \( \psi = \sin(kx) \). Why is \( J=0 \) for any bound state?

\[
J = \frac{\hbar}{2i} \left( \psi \frac{d}{dx} \psi^{\ast} - \psi^{\ast} \frac{d}{dx} \psi \right) = \frac{\hbar}{m} \text{Im} \left( \psi \frac{d}{dx} \psi^{\ast} \right) = 0 \quad \text{because} \quad \psi = \text{real}
\]

\[
J = \frac{\hbar}{2i} \left( k \sin(kx) \cos(kx) - k \sin(kx) \cos(kx) \right) = 0
\]

Stationary state are real functions and necessarily \( J = 0 \)

(b) Find the probability current for a traveling plane wave \( \psi = e^{i(kx-\omega t)} \)

\[
J = \frac{\hbar}{2i} \left( \psi \frac{d}{dx} \psi^{\ast} - \psi^{\ast} \frac{d}{dx} \psi \right) = \frac{\hbar}{2i} \left( e^{i(kx-\omega t)}(e^{-i(kx-\omega t)}(e^{-i(kx-\omega t)} + e^{-i(kx-\omega t)}) \right)
\]

\[
= \frac{\hbar}{2i}(-2i) = \frac{\hbar k}{m}
\]

#2- From the continuity equation \( \frac{\partial \psi}{\partial t} + \vec{V} \cdot \vec{J} = 0 \)

Show that if the flow of current \( \vec{J} \) is continuous across a boundary at \( x=a \) or \( J_1(a) = J_2(a) \), then

\( \psi_1(a) = \psi_2(a) \) and \( \frac{d}{dx} \psi_1(a) = \frac{d}{dx} \psi_2(a) \)

\[
J = \frac{\hbar}{m} \text{Im} \left( \psi_1 \frac{d}{dx} \psi_1^{\ast} \right)
\]

\[
J_1(a) = J_2(a) \Rightarrow \text{Im} \left( \psi_1(a) \frac{d}{dx} \psi_1^{\ast}(a) \right) = \text{Im} \left( \psi_2(a) \frac{d}{dx} \psi_2^{\ast}(a) \right)
\]

\[
\psi_1(a) \frac{d}{dx} \psi_1^{\ast}(a) = \psi_2(a) \frac{d}{dx} \psi_2^{\ast}(a)
\]

\( \Rightarrow \psi_1(a) = \psi_2(a) \) and \( \psi_1^{\ast}(a) = \psi_2^{\ast}(a) \)

#3- Positronium is a bound state of an electron \( e^- \) and positron \( e^+ \). (a) Find the reduced mass \( \mu \) of the positronium atom. The average energy and radius of the electron in a hydrogen atom are given by

\[
E_n \propto \frac{m_e}{n^2} = -13.6 \text{ eV} \quad \text{and} \quad r_n \propto \frac{n^2}{m_e} = n^2 a_0
\]

(b) Write the similar expression for a positronium atom. Then determine the average energy and radius of the ground state of positronium.

Since \( \mu = \frac{m_e}{2} \) replace \( m_e \) by \( \frac{m_e}{2} \) \( \rightarrow E_n = \frac{-6.8 \text{ eV}}{n^2} \) and \( r_n = 2n^2 a_0 \)