Phys 451  Homework Set 5   due Monday Oct 6, 2008

#1- An particle of mass m is in the ground state \( |\psi_1\rangle \) of an infinite square well of width \( a \). The well suddenly expands to width \( 2a \). Find the probability that the particle will transition to the \( n \)th level \( |\phi_n\rangle \) of the new system. In which state \( n \) do we have the maximum chance of finding the particle after expansion?

New eigenstates and energies
\[
|\psi_n\rangle = \frac{1}{\sqrt{a}} \sin\left(\frac{n\pi}{2a} x\right) \quad E_n = \frac{n^2 \pi^2}{2ma^2}
\]

\[
|\psi_i\rangle = \sum_n a_n |\phi_n\rangle \quad \text{expansion of ground state } |\psi_i\rangle \text{ in new states in terms new states } |\phi_n\rangle
\]

\[
a_n = <\psi_i|\phi_n> = \left[ \int_a^{-a} \sqrt{a} \sin\left(\frac{n\pi}{2a} x\right) \sin\left(\frac{m\pi}{2a} x\right) dx \right] + 0 \quad n = 0, 1, 2, 3, 4 \ldots
\]

\[
a_n = \frac{\sqrt{2}}{n} \left( \frac{n\pi}{2} \right) = 0 \quad n = 4, 6, 8, \ldots \quad n \neq 2
\]

\[
a_{11} = \frac{\sqrt{2}}{\pi} \left( \frac{\sin(\frac{\pi}{2})}{-1} - \frac{\sin(3\pi)}{2} \right) = \frac{\sqrt{2}}{\pi} (1 + \frac{2}{3}) = 0.6 \quad P_{11} = \frac{\sqrt{2}}{\pi} \frac{4}{3} = 0.360
\]

\[
a_{13} = \frac{\sqrt{2}}{\pi} \left( \frac{\sin(2\pi)}{-1} - \frac{\sin(5\pi)}{2} \right) = \frac{\sqrt{2}}{\pi} (1, -1) = \frac{\sqrt{2}}{\pi} \left( \frac{2}{1} - \frac{2}{5} \right) = P_{13} = \frac{8\sqrt{2}}{5\pi} = 0.519
\]

\[
a_{15} = \frac{\sqrt{2}}{\pi} \left( \frac{\sin(3\pi)}{-1} - \frac{\sin(7\pi)}{2} \right) = \frac{\sqrt{2}}{\pi} (2, -2) = 0.172 \quad P_{15} = \frac{\sqrt{2}}{\pi} \left( \frac{2}{3} - \frac{2}{7} \right) = 0.029
\]

\[
a_{17} = \frac{\sqrt{2}}{\pi} \left( \frac{\sin(5\pi)}{-1} - \frac{\sin(9\pi)}{2} \right) = \frac{\sqrt{2}}{\pi} (2, -2) = 0.080 \quad P_{17} = \frac{\sqrt{2}}{\pi} \left( \frac{2}{5} - \frac{2}{9} \right) = 0.006
\]

Some violation of probability density during this instantaneous perturbation!

#2- Given that the harmonic oscillator ground state wave function is given by
\[
|\psi_0\rangle = \pi^{-\frac{1}{4}} e^{\frac{1}{2}y^2}
\]

Use the raising operator \( a^+ \) to find
\[
|\psi_i\rangle = \frac{1}{\sqrt{2}} \left( \frac{d}{dy} + y \right) |\psi_0\rangle = \pi^{-\frac{1}{4}} \frac{1}{\sqrt{2}} \left( \frac{d}{dy} + y \right) e^{\frac{1}{2}y^2} = \frac{\sqrt{2}}{\pi^{-\frac{1}{4}}} y e^{\frac{1}{2}y^2}
\]
#3- A stream of particles of mass $m$ and energy $E$ move in the +x direction from $-\infty$ into a step barrier at $x=0$ of height $V_0$ with $(E>V_0)$. (a) Find the reflection and transmission coefficients $R$ and $T$ for the particle. (b) Determine the transmission current $J_T$ of particle moving beyond $x>0$. 

\[ \psi_1(x) = e^{ikx} + R e^{-ikx}, \quad \psi_2(x) = Te^{iqx} \]

\[ k = \sqrt{\frac{2mE}{\hbar}} \]

\[ q = \sqrt{\frac{2m(E-V_0)}{\hbar}} \]

Boundary conditions at $x=0$

\[ \psi_1(0) = \psi_2(0) \]

\[ i(k-q)R = i(k+q)T \]

(a) $R = \frac{k-q}{k+q}$, $T = \frac{k+q}{k+q}$

(b) $J_r = \frac{\hbar}{m} |\psi_1|^2$, $J_t = \frac{\hbar}{m} |\psi_2|^2$ \[ J_i = J_r + J_t \] with $J_i = J_i$ since $\frac{\partial}{\partial x} (\psi^* \psi) = 0$

\[ \frac{\hbar}{m} \left( \frac{k}{k+q} \right)^2 + \frac{\hbar}{m} \left( \frac{q}{k+q} \right)^2 = \frac{m}{k+q} \left( \frac{k+q}{k+q} \right)^2 + \frac{q}{k} \left( \frac{2k}{k+q} \right)^2 = 1 \]

#4- Alpha particles of energy $E=5$MeV and mass 3750 MeV/c^2 are trapped in a nucleus which we model as a simple square well of height 10MeV. The radius of the well is $r = 8$fm. The alpha particles are trapped by a 1fm barrier. Find the tunneling probability through the barrier for the alpha particles using the WKB approximation. What is half-life of the nucleus with respect to alpha decay? ($1fm = 1.0e-15m$)

(a) $P_r = e^{-2 \int Kdx} \int Kdx = \frac{\sqrt{2mE}}{\hbar} \int \sqrt{E-V}dx = \frac{\sqrt{2mE}}{\hbar} \sqrt{E-V} \Delta x$

\[ \int Kdx = \frac{7500 \text{ MeV} \text{ MeV}}{6.582e-15 \text{ eV} \text{ m}^2} \sqrt{5 \text{ MeV} \text{ eV} \text{ m}^2} \]

\[ = \frac{7500 \times 10^6 \text{ eV}^2 \sqrt{5 \times 10^6 \text{ eV}^2 \text{ m}^4}}{(6.582 \times 10^{-15} \text{ eV} \text{ m}^2)} \]

\[ = \frac{(7500)}{6.582} \times 10^6 \times 10^{-15} = 3 \times 10^0 = 0.981 \]

\[ P_r = e^{-2 \times 0.981} = 0.14 \]

(b) $V_a = \sqrt{\frac{2T}{m}} = \sqrt{\frac{2 \times 5 \text{ MeV}}{3750 \text{ MeV} / c^2}} = 0.0027 \times 3 \times 10^8 \text{ m/s} = 8 \times 10^8 \text{ m/s}$

\[ V = \frac{8 \times 10^5 \text{ m/s}}{8 \times 10^{-15} \text{ m}} = 1 \times 10^{10} \text{ s} \rightarrow \tau_{\alpha} = \frac{\ln 2}{\nu \tau_T} = 5.0 \times 10^{-10} \text{ s} \]