Chapter 3  Wave Particle Duality

3-2 Classical E&M Light Waves
In classical electromagnetic theory we express the electric and magnetic fields in a complex form. eg.

\[ E(x,t) = E_0 e^{i(kx - \omega t)} \quad B(x,t) = B_0 e^{i(kx - \omega t)} \]

\[ c = \lambda \nu = \left( \frac{2\pi}{k} \right) \left( \frac{\omega}{2\pi} \right) = \frac{\omega}{k} = \text{speed of light} \]

Plane Wave
\[ E \perp B \]

\[ S = \frac{1}{\mu_0} \times \vec{E} \times \vec{B} = \frac{1}{\mu_0} E_0 B_0 = \frac{1}{2c\mu_0} E_0^2 \]
\[ E_0 = cB_0 \]

The physically measureable electric field would be the absolute magnitude of this complex field
\[ E(x,t) = \text{Re} \{ E(x,t) \} = E_0 \cos(kx - \omega t) \]

The phase \( \phi \) of the E&M field is defined as the argument of the exponential or cosine. We determine the velocity of the wave by sitting at a point of constant phase.
\[ \phi = kx - \omega t = \text{const} \tan t \]
\[ d\phi = kdx - \omega dt = 0 \quad \Rightarrow \quad v_{\text{phase}} = \frac{dx}{dt} = \frac{\omega}{k} = c \]

\( v_{\text{phase}} \) = phase velocity of the wave = \( c \) for light wave

Energy density
\[ u = \frac{1}{\varepsilon_0} |E(x,t)|^2 = \text{Energy density} \left( \frac{\text{energy}}{\text{volume}} \right) \]
\[ p = \frac{u}{c} = \text{momentum density along direction of motion} \]

3-4 Photoelectric Effect and Einstein Equations
From the photoelectric effect Einstein predicted that light consisted of energy packets called photons.
\[ E = \hbar \omega \quad \text{for each photon} \]
\[ E = N\hbar \omega \quad \text{for} \ N \ \text{photons} \]

Einstein’s theory of special relativity states that
\[ E^2 = p^2 c^2 + m^2 c^4 \quad \text{and} \quad E = pc \quad \text{for massless particles – photons} \ (m = 0) \]
\[ p = \frac{E}{c} = \frac{\hbar}{\lambda} \quad \text{or} \quad p = \hbar k \quad \text{momentum of a photon} \]
3-5 Classical Interference of Light Waves
Young’s double slit experiment displayed the classical wave interference of light rays. The quantum physicists saw a connection.

\[ I(y) = \left| I_0 e^{ikx_1} + I_0 e^{ikx_2} \right|^2 = \left( I_0 e^{ikx_1} + I_0 e^{ikx_2} \right) \left( I_0 e^{-ikx_1} + I_0 e^{-ikx_2} \right) \]

\[ = 2I_0^2 + 2I_0^2 \cos(k \Delta x) = 2I_0^2 (1 + \cos(k \Delta x)) \quad \theta = k \Delta x \]

3-6 Quantum Interference of Light
If the incident light source is attenuated to allow single photons to approach the slits the same intensity pattern is seen over a long exposure. We do not know which slit the photon passes through! But the observable pattern is one of a double slit. We conclude that the photon field can be represented by a quantum state \( \Psi_{12} = \psi_1 e^{ikx_1} + \psi_2 e^{ikx_2} \), with probability

\[ P(y) = \left| \psi_{12}(y) \right|^2 = \left| \psi_1 e^{ikx_1} + \psi_2 e^{ikx_2} \right|^2 = 2 |\psi|^2 (1 + \cos(k \Delta x)) \]

Feynman Path Approach (Feynman Lectures vol3)

The photon may pass thru either of the slits, thus there are two probability amplitudes contributing to the final state \( |\Psi_F\rangle \) (position on screen).
3-7.8 Quantum and Classical Particles

Classical
The momentum and energy of a non-relativistic classical particle is given by

\[ E = \frac{p^2}{2m} + V \quad \text{and} \quad p = mv \]

Quantum- Matter waves
DeBroglie suggested that all matter could be thought of as quantum in nature and suggested a characteristic wavelength for matter waves:

\[ p = \frac{\hbar}{\lambda} = \hbar k \quad \text{and} \quad E = \frac{\hbar^2 k^2}{2m} + V \]

The phase velocity of the matter wave is be defined as before

\[ V_{\text{phase}} = \frac{\omega}{k} = \frac{\hbar \omega}{\hbar k} = \frac{E}{p} = \frac{p^2/2m}{p} = \frac{p}{2m} = \frac{1}{2} V \]

How can this be understood that the quantum velocity of a matter wave is one-half its classical velocity?

3-9 Wave Packets and Recipe for Finding \( \Psi(x,t) \)

Generalized particle wave packets can be expressed as a series of plane waves of varying wave numbers. There will exist a relationship of \( \omega \) on \( k \), \( \omega(k) \), called the dispersion relationship.

The series can be expressed as a Fourier integral also.

\[ \Psi(x,t) = \phi_1 e^{i(k_1 x - \omega_1 t)} + \phi_2 e^{i(k_2 x - \omega_2 t)} + \phi_3 e^{i(k_3 x - \omega_3 t)} + \ldots \quad \text{Fourier Series} \]

\[ \Psi(x,t) = \int \phi(k) e^{i(k x - \omega t)} dk \quad \text{Fourier Integral Expansion} \]

\[ \Psi(x,0) = \int \phi(k) e^{i k x} dk \quad t = 0 \]

We can determine \( \Psi(x,t) \) if \( \Psi(x,0) \) is known! First we must take the inverse Fourier transform of \( \Psi(x,0) \)

\[ \phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx} dx \]

Then insert \( \phi(k) \) back in to the Fourier Integral Expansion above and walla!

\[ \Psi(x,t) = \int \phi(k) e^{i(k x - \omega t)} dk \]

Group and Phase Velocity of Simple Two Wave Packet

\[ \psi = A \cos(k_1 x - \omega_1 t) + A \cos(k_2 x - \omega_2 t) \]

\[ \cos(\alpha) + \cos(\beta) = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \]

\[ = 2A \cos\left(\frac{1}{2} \left[ (k_2 + k_1)x - (\omega_2 + \omega_1)t \right] \right) \cos\left(\frac{1}{2} \left[ (k_2 - k_1)x - (\omega_2 - \omega_1)t \right] \right) \]
3-9 Gaussian Wave Packet

Let us assume a particle at \( t=0 \) is represented by a wave packet centered at \( x=x_0 \) and with initial wave number \( k_0 \). The packet (particle) has a Gaussian spread of \( \Delta x \) around \( x=x_0 \).

\[
\Psi(x,0) = A e^{ik_0x} e^{-(x-x_0)^2/2\Delta x^2}
\]

Gaussian Packet

\[
\rho_0 = \hbar k_0 \\
x_0 \\
\Delta x =
\]

initial momentum
initial position
r.m.s. spread around \( x = x_0 \)

After Fourier inversion an amazing thing happens, the Fourier distribution of momenta (\( k \) vectors) are spread about \( k_0 \) with a Gaussian spread \( \Delta k \) also! Only the Gaussian wave packet has this unique property!
Additionally, we have an inverse relation between $\Delta k$ and $\Delta x$

$$\Delta k \Delta x = 1/2$$

The inverse relation is only true for an ideal gaussian shaped packets. In general

$$\Delta k \Delta x \geq 1/2$$

3-11 Heisenberg Uncertainty Relationship
If we interpret the wave packet as representing a quantum particle then $p = h k$ and

$$\Delta p \Delta x \geq h / 2$$

Heisenberg first showed this relationship in his matrix approach to quantum theory.

3-10 Evolution of the Wave Packet- finding $\Psi(x,t)$
After taking the Fourier integral of $\phi(k)$ we can obtain the time evolving wave packet.

$$\Psi(x,t) = \int \phi(k) e^{i(kx-x_0)t} dk$$

$$\Psi(x,t) \approx \frac{1}{\Delta x(t)} e^{i(x-x_0-y_0)/2\Delta x(t)^2}$$  \text{(See Fitzpatrick!)}$$

$$\nu_g = \frac{d\omega}{dk}$$

$\Delta x^2(t) = \Delta x(0)^2 + a \frac{t^2}{\Delta x^2}$

$\Psi(x,t)$ spreading in time
Group and Phase Velocity Revisited
Particles in quantum theory can be described by wave packets. The group velocity of the packet corresponds to the particle velocity.

\[ \frac{E}{p} = \frac{1}{2m} = \frac{1}{2} v_{\text{classical}} \]

\[ v_{\text{g}} = \frac{d\omega}{dk} = \frac{dE}{dp} = \frac{p}{m} = v_{\text{classical}} \]

3-12 Heuristic Derivation of the Time Dependent Schrödinger Equation

\[ \Psi(x,t) = A e^{i \left( \frac{p}{\hbar} x - \frac{E}{\hbar} t \right)} \]

Quantum Plane Wave

\[ \frac{d\Psi}{dx} = A i \left( \frac{p}{\hbar} \right) e^{i \left( \frac{p}{\hbar} x - \frac{E}{\hbar} t \right)} = i \left( \frac{p}{\hbar} \right) \Psi \]

\[ \frac{d^2\Psi}{dx^2} = -\left( \frac{p}{\hbar} \right)^2 \Psi \to E = \frac{p^2}{2m} = \frac{\hbar^2}{2m} \frac{d^2\Psi}{dx^2} \]

\[ \frac{\partial\Psi}{\partial t} = -\frac{E}{\hbar} \Psi \to E = i\hbar \frac{\partial\Psi}{\partial t} \]

\[ -\frac{\hbar^2}{2m} \frac{d^2\Psi}{dx^2} + V\Psi = i\hbar \frac{\partial\Psi}{\partial t} \]

Time Dependent Schrödinger Equation

3-13 Collapse of the Wave Function and Quantum Decoherence
After making a measurement of a particle's state (position, momentum, spin, etc.) it sits in this state till decoherence occurs (memory loss). A second measurement very soon after \( t < t_{\text{decoherence}} \) will yield the same result! This is called Collapse of the Wave Function.

(Copenhagen Interpretation!)