PHYS 621 – Legendre Polynomials $P_l(x)$ – Useful formulas

Definition range: $(a, b) = (-1, 1)$.

Standard normalization: $P_l(1) = 1$.

Rodriguez formula:

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l \,.$$

Differential equation:

$$(1 - x^2) P_l''(x) - 2x P_l'(x) + l(l+1) P_l(x) = 0 \,,$$

$$[(1 - x^2) P_l'(x)]' + l(l+1) P_l(x) = 0 \,.$$

Recurrence relation:

$$(l+1) P_{l+1}(x) = (2l+1) x P_l(x) - l P_{l-1}(x) \,.$$

Derivative:

$$(1 - x^2) P_l'(x) = l[P_{l-1}(x) - x P_l(x)] = (l+1)[x P_l(x) - P_{l+1}(x)] \,.$$

Other relations with derivatives:

$$x P_l'(x) - P_{l-1}'(x) = l P_l(x) \,,$$

$$P_{l+1}'(x) - x P_l'(x) = (l+1) P_l(x) \,,$$

$$(2l+1) P_l(x) = P_{l+1}'(x) - P_{l-1}'(x) \,.$$

Generating function:

$$\sum_{l=0}^{\infty} P_l(x) z^l = \frac{1}{\sqrt{1 - 2xz + z^2}} \,, \qquad -1 < x < 1 \,, \quad |z| < 1 \,.$$

Series representation:

$$P_l(x) = 2^{-l} \sum_{k=0}^{[l/2]} (-1)^k \binom{l}{k} \binom{2l - 2k}{l} x^{l-2k} \,,$$

$$P_l(\cos\theta) = \sum_{k=0}^{l} g_k \, g_{l-k} \cos[(l - 2k)\theta] \,,$$

where $[l/2]$ = maximum integer smaller than $l/2$, and

$$g_k = \frac{(2k-1)!!}{2^k k!} \, .$$

Symmetry property:

$$P_l(-x) = (-1)^l P_l(x) \, .$$

Special values:

$$\begin{aligned}
&P_l(\pm 1) = (\pm 1)^l \, , \\
&P_{2l}(0) = (-1)^l g_l \, , \\
&P_{2l+1}(0) = 0 \, , \\
&P'_{2l}(0) = 0 \, , \\
&P'_{2l+1}(0) = (-1)^l (2l+1) g_l \, .
\end{aligned}$$

First polynomials:

$$\begin{aligned}
&P_0(x) = 1 \, , \\
&P_1(x) = x \, , \\
&P_2(x) = \frac{1}{2}(3x^2 - 1) \, , \\
&P_3(x) = \frac{1}{2}(5x^3 - 3x) \, , \\
&P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \, .
\end{aligned}$$

Orthonormality:

$$\int_{-1}^{1} dx \, P_l(x) P_{l'}(x) = \frac{2}{2l+1} \delta_{ll'} \, .$$

Other useful relations:

$$\int_{-1}^{1} dx \, x \, P_l(x) P_{l'}(x) = \begin{cases} \dfrac{2(l+1)}{(2l+1)(2l+3)} \delta_{l,l'-1} \\ \dfrac{2l}{(2l-1)(2l+1)} \delta_{l,l'+1} \end{cases} ,$$

$$\int_{-1}^{1} dx \, x^2 \, P_l(x) P_{l'}(x) = \begin{cases} \dfrac{2(l+1)(l+2)}{(2l+1)(2l+3)(2l+5)} \delta_{l,l'-2} \\ \dfrac{2(2l^2+2l-1)}{(2l-1)(2l+1)(2l+3)} \delta_{ll'} \end{cases} .$$