PHYS 621 – Bessel functions – Useful formulas

BESSEL FUNCTIONS

Differential equation:

$$\left[z^2\frac{d^2}{dz^2} + z\frac{d}{dz} + (z^2 - \nu^2)\right]u_\nu(z) = 0\,,$$

Independent solutions:

$$J_\nu(z) \qquad \text{and} \qquad J_{-\nu}(z)\,.$$

Usually, instead of $J_{-\nu}(z)$ the following function is defined:

$$Y_\nu(z) = \frac{1}{\sin(\nu\pi)}\left[J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)\right].$$

J_ν and Y_ν are called Bessel functions of first and second kind.

Series representation:

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{k!\Gamma(\nu+k+1)}$$

Hankel functions of first and second kind:

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z)\,, \qquad H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z)\,.$$

Properties:

$$H_{-\nu}^{(1)}(z) = e^{i\nu\pi}H_\nu^{(1)}(z)\,, \qquad H_{-\nu}^{(2)}(z) = e^{-i\nu\pi}H_\nu^{(2)}(z)\,.$$

If $\nu = n$ is an integer:

$$J_{-n}(z) = (-1)^n J_n(z)\,, \qquad Y_{-n}(z) = (-1)^n Y_n(z)\,.$$

$$Y_n(z) = -\frac{(z/2)^{-n}}{\pi}\sum_{k=0}^{\infty}\frac{(n-k-1)!}{k!}(z^2/4)^k + (2/\pi)\ln(z/2)J_n(z) + \\ -\frac{(z/2)^n}{\pi}\sum_{k=0}^{\infty}[\psi(k+1)+\psi(n+k+1)]\frac{(-z^2/4)^k}{k!(n+k)!}\,,$$

where $\psi(n) = -\gamma + \sum_{k=1}^{n-1} k^{-1}$.

Integral representation:

$$J_n(z) = \frac{1}{\pi}\int_0^\pi d\theta\,\cos(z\sin\theta - n\theta)\,,$$

Small arguments:

$$J_\nu(z) \sim \frac{(z/2)^\nu}{\Gamma(\nu+1)}\,, \quad \nu \neq -1, -2, -3\ldots$$

$$Y_\nu(z) \sim -iH_\nu^{(1)}(z) \sim iH_\nu^{(2)}(z) \sim -(1/\pi)\Gamma(\nu)(z/2)^{-\nu}\,, \qquad \mathrm{Re}\,(\nu) > 0\,,$$

$$Y_0(z) \sim -iH_0^{(1)}(z) \sim iH_0^{(2)}(z) \sim (2/\pi)\ln z\,, \qquad \nu = 0\,.$$

Asymptotic expansion ($|z| \to \infty$):

$$J_\nu(z) = \sqrt{2/(\pi z)}\,\cos(z - \pi\nu/2 - \pi/4) + \ldots\,,$$

$$Y_\nu(z) = \sqrt{2/(\pi z)}\,\sin(z - \pi\nu/2 - \pi/4) + \ldots\,,$$

$$H_\nu^{(1)}(z) = \sqrt{2/(\pi z)}\,e^{i(z-\pi\nu/2-\pi/4)} + \ldots\,,$$

$$H_\nu^{(2)}(z) = \sqrt{2/(\pi z)}\,e^{-i(z-\pi\nu/2-\pi/4)} + \ldots\,.$$

Expansion for large orders ($\nu \to \infty$):

$$J_\nu(z) \sim \frac{1}{\sqrt{2\pi\nu}}\left(\frac{ez}{2\nu}\right)^\nu\,,$$

$$Y_\nu(z) \sim -\sqrt{\frac{2}{\pi\nu}}\left(\frac{ez}{2\nu}\right)^{-\nu}\,.$$

Recurrence relations ($\Omega_\nu =$ all Bessel/Hankel):

$$\Omega_{\nu-1}(z) + \Omega_{\nu+1}(z) = \frac{2\nu}{z}\,\Omega_\nu(z)\,,$$

$$\Omega_{\nu-1}(z) - \Omega_{\nu+1}(z) = 2\,\Omega_\nu'(z)\,,$$

$$\Omega_\nu'(z) = \Omega_{\nu-1}(z) - \frac{\nu}{z}\,\Omega_\nu(z)\,,$$

$$\Omega_\nu'(z) = -\Omega_{\nu+1}(z) + \frac{\nu}{z}\,\Omega_\nu(z)$$

MODIFIED BESSEL FUNCTIONS

Differential equation:

$$\left[z^2\frac{d^2}{dz^2}+z\frac{d}{dz}-(z^2+\nu^2)\right]u_\nu(z)=0\,,$$

Independent solutions:

$$I_\nu(z)\,,\qquad \text{and} \qquad I_{-\nu}(z)\,.$$

Usually, instead of $I_{-\nu}(z)$ the following function is defined:

$$K_\nu(z)=\frac{\pi}{2\sin(\nu\pi)}[I_{-\nu}(z)-I_\nu(z)]\,.$$

J_ν and Y_ν are called Bessel functions of first and second kind.

Series representation:

$$I_\nu(z)=\left(\frac{z}{2}\right)^\nu\sum_{k=0}^{\infty}\frac{(z^2/4)^k}{k!\Gamma(\nu+k+1)}$$

If $\nu=n$ is an integer:

$$I_{-n}(z)=I_n(z)\,,\qquad K_{-\nu}(z)=K_\nu(z)\,.$$

$$\begin{aligned}K_n(z)=&\frac{1}{2}\left(\frac{z}{2}\right)^{-n}\sum_{k=0}^{\infty}\frac{(n-k-1)!}{k!}(-z^2/4)^k-(-1)^n\ln(z/2)I_n(z)+\\&+(-1)^n\frac{1}{2}\left(\frac{z}{2}\right)^n\sum_{k=0}^{\infty}[\psi(k+1)+\psi(n+k+1)]\frac{(z^2/4)^k}{k!(n+k)!}\,.\end{aligned}$$

Integral representation:

$$I_n(z)=\frac{1}{\pi}\int_0^\pi d\theta\, e^{z\cos\theta}\cos(n\theta)\,.$$

Small arguments:

$$I_\nu(z)\sim\frac{(z/2)^\nu}{\Gamma(\nu+1)}\,,\qquad \nu\neq-1,-2,-3\ldots$$

$$K_\nu(z)\sim(1/2)\Gamma(\nu)(z/2)^{-\nu}\,,\qquad \mathrm{Re}\,(\nu)>0\,,$$

$$K_0(z)\sim-\ln z\,,\qquad \nu=0\,.$$

Asymptotic expansion ($|z| \to \infty$):

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}} \left[1 - (4\nu^2 - 1)/(8z) + \ldots\right],$$

$$K_\nu(z) = \sqrt{\pi/(2z)} e^{-z} \left[1 + (4\nu^2 - 1)/(8z) + \ldots\right].$$

Recurrence relations ($\Omega_\nu =$ all Modified Bessel):

$$\Omega_{\nu-1}(z) - \Omega_{\nu+1}(z) = \frac{2\nu}{z} \, \Omega_\nu(z),$$

$$\Omega_{\nu-1}(z) + \Omega_{\nu+1}(z) = 2 \, \Omega'_\nu(z),$$

$$\Omega'_\nu(z) = \Omega_{\nu-1}(z) - \frac{\nu}{z} \, \Omega_\nu(z),$$

$$\Omega'_\nu(z) = \Omega_{\nu+1}(z) + \frac{\nu}{z} \, \Omega_\nu(z)$$