PHYS 621 – Legendre Polynomials $P_l(x)$ – Useful formulas

Definition range: $(a, b) = (-1, 1)$.

Standard normalization: $P_l(1) = 1$.

Rodriguez formula:

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l.$$ 

Differential equation:

$$(1 - x^2)P''_l(x) - 2x P'_l(x) + l(l + 1)P_l(x) = 0,$$

$$[(1 - x^2)P'_l(x)]' + l(l + 1)P_l(x) = 0.$$ 

Recurrence relation:

$$(l + 1)P_{l+1}(x) = (2l + 1)xP_l(x) - lP_{l-1}(x).$$

Derivative:

$$\big(1 - x^2\big)P'_l(x) = l\big[P_{l-1}(x) - xP_l(x)\big] = (l + 1)\big[xP_l(x) - P_{l+1}(x)\big].$$

Other relations with derivatives:

$$xP'_l(x) - P'_{l-1}(x) = lP_l(x),$$

$$P'_{l+1}(x) - xP'_l(x) = (l + 1)P_l(x),$$

$$(2l + 1)P_l(x) = P'_{l+1}(x) - P'_{l-1}(x).$$

Generating function:

$$\sum_{l=0}^{\infty} P_l(x) z^l = \frac{1}{\sqrt{1 - 2xz + z^2}}, \quad -1 < x < 1, \quad |z| < 1.$$ 

Series representation:

$$P_l(x) = 2^{-l} \sum_{k=0}^{[l/2]} (-1)^k \binom{l}{k} \binom{2l - 2k}{l} x^{l - 2k},$$

$$P_l(\cos \theta) = \sum_{k=0}^{l} g_k g_{l-k} \cos[(l - 2k)\theta],$$
where \( \lfloor l/2 \rfloor = \) maximum integer smaller than \( l/2 \), and
\[
g_k = \frac{(2k - 1)!!}{2^k k!}.
\]

Symmetry property:
\[
P_l(-x) = (-1)^l P_l(x).
\]

Special values:
\[
\begin{align*}
P_l(\pm 1) &= (\pm 1)^l, \\
P_{2l}(0) &= (-1)^l g_l, \\
P_{2l+1}(0) &= 0, \\
P'_{2l}(0) &= 0, \\
P'_{2l+1}(0) &= (-1)^l (2l + 1) g_l.
\end{align*}
\]

First polynomials:
\[
\begin{align*}
P_0(x) &= 1, \\
P_1(x) &= x, \\
P_2(x) &= \frac{1}{2} (3x^2 - 1), \\
P_3(x) &= \frac{1}{2} (5x^3 - 3x), \\
P_4(x) &= \frac{1}{8} (35x^4 - 30x^2 + 3).
\end{align*}
\]

Orthonormality:
\[
\int_{-1}^{1} dx \, P_l(x) P_{\nu}(x) = \frac{2}{2l + 1} \delta_{l,\nu}.
\]

Other useful relations:
\[
\begin{align*}
\int_{-1}^{1} dx \, x P_l(x) P_{\nu}(x) &= \begin{cases} \\
\frac{2(l + 1)}{(2l + 1)(2l + 3)} \delta_{l,\nu} \delta_{l+1,\nu+1} \\
\frac{2l}{(2l - 1)(2l + 1)} \delta_{l,\nu+1} \\
\end{cases}, \\
\int_{-1}^{1} dx \, x^2 P_l(x) P_{\nu}(x) &= \begin{cases} \\
\frac{2(l + 1)(l + 2)}{(2l + 1)(2l + 3)(2l + 5)} \delta_{l,\nu} \delta_{l+2,\nu+2} \\
\frac{2(2l^2 + 2l - 1)}{(2l - 1)(2l + 1)(2l + 3)} \delta_{l\nu} \\
\end{cases}.
\end{align*}
\]