1. **2 points:** Find out whether and where in the complex plane the following function is analytic:

$$f(x,y) = \frac{y - ix}{x^2 + y^2}\,.$$

2. **2 points:** Show that the following function $f(x,y)$ is harmonic, and find the function $F(z)$ of which the function is the real part, i.e. $\mathrm{Re}[F(z)] = f(x,y)$. Show that $\mathrm{Im}[F(z)]$ is also harmonic.

$$f(x,y) = e^x \cos y\,.$$

3. **3 points:** Compute the integral

$$I = \oint_C \frac{e^{3z}}{z - \ln 2}\,dz\,,$$

where C is the square with vertices $\pm 1 \pm i$.

4. **2 points each:** Find the residue of the following functions at the indicated points:

$$f_1(z) = \frac{z-2}{4z^2-1}\,, \qquad \text{at } z = \frac{1}{2}$$

$$f_2(z) = \frac{e^{iz}}{(z^2+4)^2}\,, \qquad \text{at } z = 2i$$

Bonus (3 points):

$$f_3(z) = \frac{e^{2z}}{1+e^z}\,, \qquad \text{at } z = i\pi$$

5. **3 points each:** Evaluate the following integrals

$$I_1 = \int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta}\,,$$

$$I_2 = \int_{-\infty}^{\infty} \frac{dx}{x^2+4x+5}\,,$$

$$I_3 = \int_0^{\infty} \frac{dx}{(4x^2+1)^3}\,.$$

Bonus (5 points):

$$I_4 = \int_0^{\infty} \frac{\sqrt{x}}{1+x^2}\,dx\,.$$

(1) Solutions

$$f(x,y) = \frac{y-ix}{x^2+y^2}$$

Set $z = x+iy$, therefore

$$f(x,y) = \frac{-i(x+iy)}{x^2+y^2} = -\frac{iz}{zz^*} = -\frac{i}{z^*}$$

the function is nowhere analytic.

(2) $f(x,y) = e^x \cos y$

Remember $e^{iy} = \cos y + i \sin y$, therefore

$$f(x,y) = e^x(e^{iy} - i\sin y)$$

$$= e^{x+iy} - ie^x \sin y$$

$$= e^z - ie^x \sin y$$

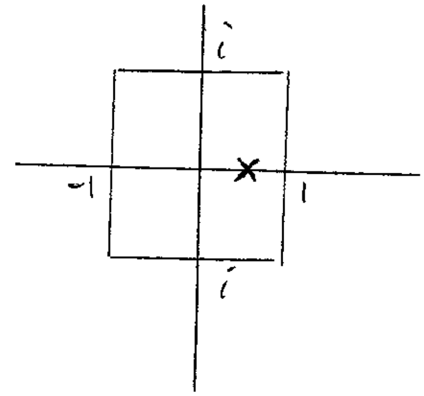
therefore: $F(z) = e^z$

$Re[F(z)] = e^x \cos y$

$Im[F(z)] = e^x \sin y$

Since $F(z)$ is analytic both the Real and Imaginary parts are harmonic.
(One can add a constant to $Im[F(z)]$ if so wishes ...)

(3) $I = \oint_c \frac{e^{3z}}{z-\ln 2}\, dz$ on



the integrand has a pole of order one @ $z = \ln 2$

Using Cauchy's Integral formula:

$$\oint_c \frac{e^{3z}}{z-\ln 2} = 2\pi i\, e^{3(\ln 2)} = 16\pi i$$

(4) Res $f_1(z)$ at $z = 1/2$

$$f_1 = \frac{z-2}{4(z-\frac{1}{2})(z+\frac{1}{2})} \longrightarrow \text{Res } f_1 \Big|_{z=1/2} = \frac{\frac{1}{2}-2}{4\cdot 1} = -3/8$$

Res $f_2(z)$ at $z = 2i$

$$f_2 = \frac{e^{iz}}{(z+2i)^2(z-2i)^2}$$ pole of 2nd order at $z = 2i$

for example setting $z - 2i = \omega$

$$f_2 = \frac{e^{i(\omega+2i)}}{(\omega+4i)^2\omega^2} = e^{-2}\frac{1}{\omega^2}\frac{e^{i\omega}}{(\omega+4i)^2}$$

Expanding around $\omega = 0$

$$f_2 = e^{-2} \frac{1}{(-16)\omega^2} \left(1 + i\omega + \cdots \right)\left(1 + \frac{\omega i}{2} + \cdots \right)$$

term of order $1/\omega$: $-\frac{e^{-2}}{16}\left(\frac{i}{2} + i \right) \frac{1}{\omega}$

Residue is coefficient : $-\frac{3i}{32} e^{-2}$

Other methods for solving this are possible

<u>Bonus</u> Residue at $z = i\pi$ of $f_3 = \frac{e^{2z}}{1+e^{z}}$

At $z = i\pi$ Denominator $\to 0$ singularity.

Need to expand around $z = i\pi$.

Set $\omega = z - i\pi$:

$$f_3 = \frac{e^{2(\omega + i\pi)}}{1 + e^{\omega + i\pi}} = \frac{e^{2\omega}}{1 - e^{\omega}}$$

Expand $\omega \sim 0$:

$$f_3 \sim \frac{(1 + 2\omega + \cdots)}{\cancel{1} - \cancel{1} - \omega - \frac{\omega^2}{2!} + \cdots} = -\frac{1}{\omega} \frac{(1 + 2\omega + \cdots)}{1 + \frac{\omega}{2} + \cdots}$$

$$\sim -\frac{1}{\omega}\left(1 + 2\omega + \cdots \right)\left(1 - \frac{\omega}{2} + \cdots \right)$$

Residue is coefficient of $1/\omega$: -1

(5) Compute I_1 :

$$I_1 = \int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta}$$

set $z = e^{i\theta}$:

$$I_1 = \int_{|z|=1} \frac{dz}{iz} \frac{1}{5 - \frac{3}{2z}(z^2+1)}$$

$$= +2i \int_{|z|=1} \frac{dz}{3z^2 - 10z + 3}$$

Result is sum of residues inside circle $|z|=1$ $\times 2\pi i$:

Poles : $3z^2 - 10z + 3 = 0 \longrightarrow z_1 = 1/3$
$z_2 = 3$

Only z_1 is inside $|z|=1$
therefore

$$I_1 = 2\pi i \,(2i)\, \operatorname{Res}_{z=1/3} \frac{1}{3z^2 - 10z + 3}$$

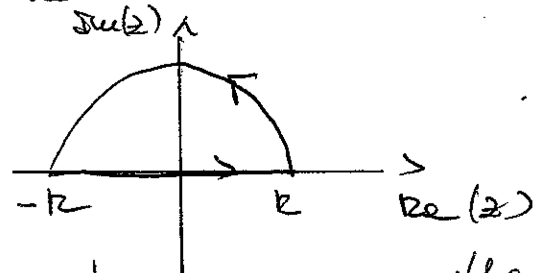
$$= -\frac{4\pi}{3} \operatorname{Res}_{z_1=1/3} \frac{1}{(z-1/3)(z-3)}$$

$$= -\frac{4\pi}{3} \frac{1}{-8/3} = \pi/2$$

Compute I_2:

$$I_2 = \int_{-\infty}^{+\infty} \frac{dx}{x^2+4x+5}$$

Consider the complex plane and this contour:



Sending $R \to \infty$ the integral on the semicircle vanishes, therefore

$$I_2 = \oint \frac{dz}{z^2+4z+5}$$

the integrand has poles at:

$$z^2+4z+5 = 0 \qquad z = -2 \pm i$$

therefore

$$I_2 = 2\pi i \operatorname{Res}_{z=-2+i} \frac{1}{z^2+4z+5}$$

$$= 2\pi i \operatorname{Res}_{z=-2+i} \frac{1}{(z+2-i)(z+2+i)}$$

$$= 2\pi i \frac{1}{2i} = \pi$$

Compute I_3

$$I_3 = \int_0^{\infty} \frac{dx}{(4x^2+1)^3}$$

the integrand is an even function, therefore

$$I_3 = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{dx}{(4x^2+1)^3}$$

Proceeding as before (see I_2) :

$$I_3 = \frac{1}{2}\oint \frac{dz}{(4z^2+1)^3}$$

there are 2 poles of 3rd order at :

$$z = \pm i/2$$

to find the residue at $z = i/2$, for example:

$$z - i/2 = \omega$$

$$\frac{1}{(4z^2+1)^3} = \frac{1}{(2z+i)^3(2z-i)^3} = \frac{1}{64\left(z+\frac{i}{2}\right)^3\left(z-\frac{i}{2}\right)^3}$$

$$= \frac{1}{64w^3}\cdot\frac{1}{(\omega+i)^3} = \frac{i}{64w^3}\,\frac{1}{(1-i\omega)^3}$$

Expanding the last factor around $\omega = 0$:

$$\frac{1}{(4z^2+1)^3} \sim \frac{i}{64w^3}\left(1 + 3i\omega - \frac{12\omega^2}{2!} + \cdots\right)$$

the residue of $\frac{1}{(4z^2+1)^3}$ is the coefficient of $1/w$: $-\frac{3i}{32}$

therefore : $I_3 = 2\pi i \cdot \frac{1}{2}\left(-\frac{3i}{32}\right) = \frac{3\pi}{32}$

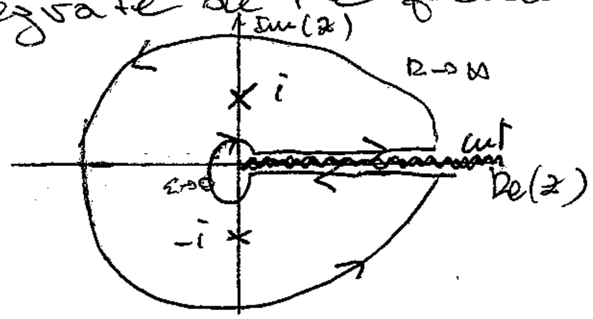
(Alternatively , use $f^{n}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}}$)

Bonus : Compute I_4

$$I_4 = \int_0^{\infty} \frac{\sqrt{x}}{1+x^2}\, dx$$

the integrand has a branching point at $z=0$ and two 1st-order poles at $z = \pm i$.

Integrate on the following path :



therefore :

$$\oint_C = \int_{\substack{\text{circle}\\ R\to\infty}} + \int_{\xrightarrow{x}} + \int_{\substack{\text{circle}\\ R\to 0}} + \int_{\xleftarrow{x}}$$

the first and third integrals are zero (proof follows):

(*) $\displaystyle\int_{\substack{\text{circle}\\ R\to\infty}} \frac{\sqrt{z}}{z^2+1}\,dz$ set $z = Re^{i\theta} \to dz = i\,d\theta\, z$

$$\int_{\substack{\text{circle}\\ R\to\infty}} \frac{d\theta\, i\, z^{3/2}}{z^2+1} = i\int \frac{d\theta\, R^{3/2} e^{i3\theta/2}}{R^2 e^{2i\theta}+1} \xrightarrow[R\to\infty]{} 0$$

(\$) $\displaystyle\int_{\substack{\text{circle}\\ \varepsilon\to 0}} \frac{\sqrt{z}\,dz}{z^2+1}$ set $z = \varepsilon e^{i\theta}$ $dz = i\,d\theta\, z$

$$\int_{\substack{\text{circle}\\ \varepsilon\to 0}} \frac{d\theta\, i\, z^{3/2}}{z^2+1} = i\int d\theta \frac{\varepsilon^{3/2} e^{i3\theta/2}}{\varepsilon^2 e^{2i\theta}+1} \xrightarrow[\varepsilon\to 0]{} 0$$

therefore:

$$\oint_C \frac{\sqrt{z}}{z^2+1}\,dz = \int_0^\infty \frac{\sqrt{x}\,dx}{x^2+1} + \int_\infty^0 \frac{(-\sqrt{x})\,dx}{x^2+1}$$

$$= 2\int_0^\infty \frac{\sqrt{|x|}\,dx}{x^2+1}$$

$$\Rightarrow \int_0^\infty \frac{\sqrt{x}\,dx}{x^2+1} = \frac{1}{2}\oint_C \frac{\sqrt{z}}{z^2+1}\,dz$$

where the last integral can be evaluated with the residue theorem:

$$\frac{1}{2}\oint \frac{\sqrt{z}}{z^2+1}\,dz = \frac{1}{2}\,2\pi i\left\{ \operatorname*{Res}_{z=i}\left(\frac{\sqrt{z}}{z^2+1}\right) + \operatorname*{Res}_{z=-i}\left(-\frac{\sqrt{z}}{z^2+1}\right)\right\}$$

Therefore:

$$\int_0^\infty \frac{\sqrt{x}\,dx}{x^2+1} = \pi i \left\{ \frac{\sqrt{i}}{2i} - \frac{\sqrt{-i}}{-2i} \right\}$$

$$= \frac{\pi}{2}\left(\sqrt{i}+\sqrt{-i}\right) = \frac{\pi}{2}\left(e^{i\pi/4}+e^{-\pi i/4}\right)$$

$$= \frac{\pi}{2}\left(\cos\frac{\pi}{4} \cancel{+ i\sin\frac{\pi}{4}} + \cos\left(\frac{\pi}{4}\right) \cancel{- i\sin\left(\frac{\pi}{4}\right)}\right)$$

$$= \frac{\pi}{2}\left(\frac{2}{\sqrt{2}}\right)$$

$$= \pi/\sqrt{2}$$